In this paper we consider the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = N$, where $N$ is a sufficiently large integer and prove that if $\eta$ is quadratic irrational number and $0 < \lambda < \frac{1}{10}$, then it has a solution in almost-prime numbers $x_1, \ldots, x_4$, such that $\{\eta x_i\} < N - \lambda$ for $i = 1, \ldots, 4$.

**Keywords:** Lagrange’s equation, almost-primes, quadratic irrational numbers.

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### 1. INTRODUCTION AND STATEMENT OF THE RESULT

In 1770 Lagrange proved that for any positive integer $N$ the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N \quad (1.1)$$

has a solution in integer numbers $x_1, \ldots, x_4$. Later Jacobi found an exact formula for the number of the solutions (see [8, Ch. 20]). A lot of researchers studied the equation (1.1) for solvability in integers satisfying additional conditions. There is a hypothesis stating that if $N$ is sufficiently large and $N \equiv 4 \pmod{24}$ then (1.1) has a solution in primes. This hypothesis has not been proved so far, but several approximations to it have been established.
In 1994 J. Brüdern and E. Fouvry [1] proved that for any large \( N \equiv 4 \pmod{24} \), the equation (1.1) has a solution in \( x_1, \ldots, x_4 \in \mathcal{P}_{34} \). (We say that integer \( n \) is an almost-prime of order \( r \) if \( n \) has at most \( r \) prime factors, counted with their multiplicities. We denote by \( \mathcal{P}_r \) the set of all almost-primes of order \( r \).) This result was improved by D. R. Heath-Brown and D. I. Tolev [9]. They showed that, under the same restrictions for \( N \), the equation (1.1) has a solution in prime \( x_1 \) and almost-prime \( x_2, x_3, x_4 \in \mathcal{P}_{101} \). In their paper they also proved that the equation has a solution in \( x_1, \ldots, x_4 \in \mathcal{P}_{25} \). In 2020 Tak Wing Ching [2] improved this result with three of them being in \( \mathcal{P}_3 \) and the other in \( \mathcal{P}_4 \).

On the other hand, let us consider a subset of the set of integers having the form
\[
\mathcal{A} = \{ n \mid a < \{ \eta n \} < b \},
\]
where \( \eta \) is a fixed quadratic irrational number, and \( a, b \in [0, 1] \).

Denote by \( I(N) \) the number of solutions of (1.1) in arbitrary integers and by \( J(N) \) the number of solutions of (1.1) in integers from the set \( \mathcal{A} \).

In 2011 S. A. Gritsenko and N. N. Motkina [6] proved that for any positive small \( \varepsilon \), the following formula holds
\[
J(N) = (b - a)^4 I(N) + O \left( N^{0.9+3\varepsilon} \right).
\]

S. A. Gritsenko and N. N. Motkina consider many others additive problem in which variables are in special set of numbers similar to \( \mathcal{A} \). (See [4] – [5] and [7].) In 2013 A. V. Shutov [12] considered solvability of diophantine equation in integer numbers from \( \mathcal{A} \). Further research in this area was made by A. V. Shutov and A. A. Zhukova [13].

We consider the equation (1.1), where \( x_i \) are almost-prime numbers and belong to a set similar to \( \mathcal{A} \). Our result is

**Theorem 1.1.** Let \( \eta \) be a quadratic irrational number, \( 0 < \lambda < \frac{1}{10} \) and \( k = \left\lfloor \frac{54}{1 - 10\lambda} \right\rfloor \). Then for every sufficiently large integer \( N \), the equation (1.1) has a solution in almost-prime numbers \( x_1, \ldots, x_4 \in \mathcal{P}_k \), such that \( \{ \eta x_i \} < N^{-\lambda} \), \( i = 1, 2, 3, 4 \).

In the present paper we use the following notations.

We denote by \( N \) a sufficiently large odd integer and \( P = N^{\frac{1}{2}} \). Letters \( a, b, k, l, m, n, q, p \) always stand for integers. By \( (n_1, \ldots, n_k) \) we denote the greatest common divisor of \( n_1, \ldots, n_k \). Let \( ||t|| \) denote the distance from \( t \) to the nearest integer. We denote by \( \vec{n} \) four dimensional vectors and let
\[
||\vec{n}|| = \max(|n_1|, \ldots, |n_4|).
\]

As usual, \( \mu(q) \) is the Möbius function and \( \tau(q) \) is the number of positive divisors of \( q \). Sometimes we write \( a \equiv b \pmod{q} \) as an abbreviation of \( a \equiv b \pmod{q} \).
We write $\sum_{x \equiv (q)}$ for a sum over a complete system of residues modulo $q$ and respectively $\sum_{x \equiv (q)}^*$ is a sum over a reduced system of residues modulo $q$. We also denote $e(t) = e^{2\pi it}$.

We use Vinogradov’s notation $A \ll B$, which is equivalent to $A = O(B)$. By $\varepsilon$ we denote an arbitrarily small positive number, which is not the same in different occurrences. The constants in the $O$-terms and $\ll$-symbols are absolute or depend on $\varepsilon$.

2. AUXILIARY RESULTS

Now we introduce some lemmas, which shall be used later.

Lemma 2.1. Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^{\pm}(d)$ (called Rosser’s functions of level $D$) with the following properties:

1. For any positive integer $d$ we have
   \[ |\lambda^{\pm}(d)| \leq 1, \quad \lambda^{\pm}(d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0. \]

2. If $n \in \mathbb{N}$ then
   \[ \sum_{d|n} \lambda^{-}(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^{+}(d). \]

3. If $z \in \mathbb{R}$ is such that $z^2 \leq D$ and if
   \[ P(z) = \prod_{2 < p < z} p, \quad B = \prod_{2 < p < z} \left(1 - \frac{1}{p-1}\right), \quad N^{\pm} = \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \quad (2.1) \]
   then we have
   \[ B \leq N^{+} \leq B \left(F(s_0) + O \left((\log D)^{-\frac{1}{3}}\right)\right), \quad (2.2) \]
   \[ B \geq N^{-} \geq B \left(f(s_0) + O \left((\log D)^{-\frac{1}{3}}\right)\right), \quad (2.3) \]

where $F(s)$ and $f(s)$ satisfy
\begin{align*}
F(s) &= 2e^{\gamma}s^{-1}, & \text{if } & 2 \leq s \leq 3, \\
f(s) &= 2e^{\gamma}s^{-1}\log(s-1), & \text{if } & 2 \leq s \leq 3, \\
(sF(s))' &= f(s-1), & \text{if } & s > 3, \\
(sf(s))' &= F(s-1), & \text{if } & s > 2.
\end{align*}

Here $\gamma$ is Euler’s constant.

Lemma 2.2. Suppose that \( \Lambda_i, \Lambda_i^\pm \) are real numbers satisfying \( \Lambda_i = 0 \) or \( 1 \), \( \Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+ \), \( i = 1, 2, 3, 4 \). Then
\[
\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^- + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^- - 3 \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ .
\]  
(2.4)

Proof. The proof is similar to the proof of [1, Lemma 13].

Let
\[
w_0(t) = \begin{cases} e^{\frac{t}{t^2 - \frac{1}{5}}} & \text{if } t \in \left(-\frac{4}{5}, \frac{4}{5}\right) , \\ 0 & \text{if } t \not\in \left(-\frac{4}{5}, \frac{4}{5}\right) \end{cases}
\]
and
\[
w(x) = w_0 \left( \frac{x}{P} - \frac{1}{2} \right) .
\]  
(2.5)

Lemma 2.3. Let \( u, \beta \in \mathbb{R} \) and
\[
J(\beta, u) = \int_{-\infty}^{+\infty} w_0 \left( x - \frac{1}{2} \right) e(\beta x^2 + ux) dx .
\]  
(2.6)

Then:

1. For every \( k \in \mathbb{N} \) and \( u \neq 0 \) we have
\[
J(\beta, u) \ll_k \frac{1 + |\beta|^k}{|u|^k} .
\]

2. The following inequality hold
\[
J(\beta, u) \ll \min \left( 1, |\beta|^{-\frac{1}{2}} \right) .
\]

Proof. See [9, Lemma 9].

Lemma 2.4. Suppose that \( \vec{u} \in \mathbb{Z}^4 \) and
\[
J(\beta, \vec{u}) = \prod_{i=1}^{4} J(\beta, u_i) .
\]

Then we have
\[
\int_{-\infty}^{+\infty} |J(\beta, \vec{u})| d\gamma \ll |\vec{u}|^{-1+\epsilon} .
\]
Proof. Proof can be found in [9, Lemma 10].

Lemma 2.5. There exists a function $\sigma(v, q, \gamma)$ defined for $-\frac{q}{2} < v \leq \frac{q}{2}, q \leq P, |\gamma| \leq \frac{P}{q}$, integrable with respect to $\gamma$, satisfying

$$|\sigma(v, q, \gamma)| \leq \frac{1}{1+|v|}$$

and also for every $a \in \mathbb{Z}, (a, q) = 1$ we have

$$\sum_{-\frac{q}{2} < v \leq \frac{q}{2}} e\left(\frac{\pi v}{q}\right) \sigma(v, q, \gamma) = \begin{cases} 1 & \text{if } \gamma \in \mathcal{N}(a, q), \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\mathcal{N}(a, q) = \left( -\frac{P^2}{q(q+q')}, \frac{P^2}{q(q+q'')} \right)$$

and

$$P < q + q', q + q'' \leq P + q, \quad aq' \equiv 1(\text{mod } q), \quad aq'' \equiv -1(\text{mod } q). \quad (2.7)$$

Proof. See [15, Lemma 45].

For $q \in \mathbb{N}$ and $m, n \in \mathbb{Z}$, the Gauss sum is defined by

$$G(q, m, n) = \sum_{x(q)} e\left(\frac{mx^2 + nx}{q}\right). \quad (2.8)$$

For $\vec{d} = \langle d_1, \ldots, d_4 \rangle \in \mathbb{Z}^4$ and $\vec{n} = \langle n_1, \ldots, n_4 \rangle \in \mathbb{Z}^4$ we denote

$$G(q, a\vec{d}, \vec{n}) = \prod_{i=1}^{4} G(q, ad_i^2, n_i).$$

We need to estimate an exponential sum of the form

$$V_q = V_q(N, \vec{d}, v, \vec{n}) = \sum_{a(q)}^* e\left(\frac{\pi v - Na}{q}\right) G(q, a\vec{d}, \vec{n}). \quad (2.9)$$

To estimate $V_q$ we use the properties of the Gauss sum and the Kloosterman sum.

Lemma 2.6. Suppose that $N, q \in \mathbb{N}, v \in \mathbb{Z}$ and $\vec{d}, \vec{n} \in \mathbb{Z}^4$. Then we have

$$V_q(N, \vec{d}, v, \vec{n}) \ll q^\frac{5}{4} \tau(q)(q, N)^\frac{1}{2}(q, d_1)(q, d_2)(q, d_3)(q, d_4).$$

Moreover, if some of the conditions

$$(q, d_i)|n_i, \quad i = 1, \ldots, 4$$

do not hold, then $V_q(N, \vec{d}, v, \vec{n}) = 0$. 

Proof. This result is analogous to this one in [1, Lemma 1].

**Lemma 2.7.** (Liouville) If $\eta$ is an irrational number which is the root of a polynomial $f$ of degree 2 with integer coefficients, then there exists a real number $A > 0$ such that, for all integers $p, q$, with $q > 0$,

$$|\eta - \frac{p}{q}| \geq \frac{A}{q^2}.$$

**Proof.** See [11, Theorem 1A].

3. PROOF OF THE THEOREM

3.1. BEGINNING OF THE PROOF

Let $N$ be a sufficiently large integer. We denote

$$z = N^\alpha, \quad P(z) = \prod_{p < z} p, \quad \delta = N^{-\lambda}.$$

We apply the well-known Vinogradov’s “little cups” lemma (see [10, Chapter 1, Lemma A]) with parameters

$$\alpha_1 = \frac{\delta}{4}, \quad \beta_1 = \frac{3\delta}{4}, \quad \Delta = \frac{\delta}{2}, \quad r = [\log N]$$

and construct a function $\theta(t)$ which is periodic with period 1 and has the following properties:

$$\theta \left( \frac{\delta}{2} \right) = 1; \quad 0 < \theta(t) < 1 \quad \text{for} \quad 0 < t < \frac{\delta}{2} \quad \text{or} \quad \frac{\delta}{2} < t < \delta;$$

$$\theta(t) = 0 \quad \text{for} \quad \delta \leq t \leq 1.$$

Furthermore, from the Fourier series of $\theta(t)$ we find

$$\theta(t) = \frac{\delta}{2} + \sum_{0 < |m| \leq H} c(m) e(mt) + O(P^{-A}), \quad (3.1)$$

with

$$|c(m)| \leq \min \left( \frac{\delta}{2}, \frac{1}{|m|} \left( \frac{[\log N]}{\delta \pi |m|} \right)^{[\log N]} \right).$$
where $A$ is arbitrary large constant and
\[
H = \left[\frac{\log N}{\delta}\right]^2.
\] (3.2)

Let us denote
\[
\theta(\eta\vec{x}) = \theta(\eta x_1)\theta(\eta x_2)\theta(\eta x_3)\theta(\eta x_4)
\]
and
\[
w(\vec{x}) = w(x_1)w(x_2)w(x_3)w(x_4).
\]
We consider the sum
\[
\Gamma = \sum_{x_1^2 + x_2^2 + x_3^2 + x_4^2 = N} \theta(\eta\vec{x})w(\vec{x}).
\]
From the condition $(x_i, P(z)) = 1$ it follows that any prime factor of $x_i$ is greater than or equal to $z$. Suppose that $x_i$ has $l$ prime factors, counted with their multiplicities. Then we have
\[
N^{\frac{1}{2}} \geq x_i \geq z^l = N^{\alpha l}
\]
and hence $l \leq \frac{1}{2\alpha}$. This implies that if $\Gamma > 0$ then equation (1.1) has a solution in almost-prime numbers $x_1, \ldots, x_4$ with at most $\left[\frac{1}{2\alpha}\right]$ prime factors, such that $\{\eta x_i\} < N^{-\lambda}$, $i = 1, \ldots, 4$.

For $i = 1, 2, 3, 4$ we define
\[
\Lambda_i = \sum_{d|(x_i, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (x_i, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)
\]
Then we find that
\[
\Gamma = \sum_{x_1^2 + x_2^2 + x_3^2 + x_4^2 = N} \Lambda_1\Lambda_2\Lambda_3\Lambda_4 \theta(\eta\vec{x})w(\vec{x}).
\]
We can write $\Gamma$ as
\[
\Gamma = \sum_{x_i \in \mathbb{Z}} \Lambda_1\Lambda_2\Lambda_3\Lambda_4 \theta(\eta\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N)) d\alpha.
\]
Suppose that $\lambda^{\pm}(d)$ are the Rosser functions of level $D$ (see Lemma 2.1). Let also denote
\[
\Lambda_i^{\pm} = \sum_{d|(x_i, P(z))} \lambda^{\pm}(d), \quad i = 1, 2, 3, 4. \quad (3.4)
\]
Then from Lemma 2.1, (3.3) and (3.4) we find that
\[
\Lambda_i^{-} \leq \Lambda_i \leq \Lambda_i^{+}.
\]
We use Lemma 2.2 and find that
\[
\Gamma \geq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - 3\Gamma_5,
\]
where \(\Gamma_1, \ldots, \Gamma_5\) are the contributions coming from the consecutive terms of the right side of (2.4). We have \(\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4\) and
\[
\Gamma_1 = \sum_{x_i \in \mathbb{Z}} A^+_1 A^+_2 A^+_3 A^+_4 \theta(\eta \vec{x}) w(\vec{x}) \int_0^1 e(\alpha (x_1^2 + x_2^2 + x_3^2 + x_4^2 - N)) d\alpha,
\]
\[
\Gamma_5 = \sum_{x_i \in \mathbb{Z}} A^+_1 A^+_2 A^+_3 A^+_4 \theta(\eta \vec{x}) w(\vec{x}) \int_0^1 e(\alpha (x_1^2 + x_2^2 + x_3^2 + x_4^2 - N)) d\alpha.
\]
Hence, we get
\[
\Gamma \geq 4\Gamma_1 - 3\Gamma_5. \tag{3.5}
\]

3.2. ASYMPTOTIC FORMULA FOR \(\Gamma_1\)

We shall find an asymptotic formula for the integral \(\Gamma_1\). We have
\[
\Gamma_1 = \sum_{d_i \mid P(z)} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4) \sum_{x_i \equiv 0 (d_i)} \theta(\eta \vec{x}) w(\vec{x}) \times \int_0^1 e(\alpha (x_1^2 + \cdots + x_4^2 - N)) d\alpha
\]
\[
= \sum_{d_i \mid P(z)} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4) \times \int_0^1 \prod_{1 \leq i \leq 4} \left( \sum_{x \equiv 0 (d_i)} \theta(\eta x) w(x) e(\alpha x^2) \right) e(-N\alpha) d\alpha.
\]
Let
\[
S(\alpha, d, m) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 (d)}} w(x) e(\alpha x^2 + m\eta x). \tag{3.6}
\]
Then using the Fourier series of \(\theta(t)\) (see (3.1)), we find
\[
\sum_{x \equiv 0 (d)} \theta(\eta x) w(x) e(\alpha x^2) = \sum_{|m| \leq H} c(m) \sum_{x \equiv 0 (d)} w(x) e(\alpha x^2 + m\eta x) + O(P^{-A}).
\]
Denoting
\[
S(\alpha, \vec{d}, \vec{m}) = S(\alpha, d_1, m_1) S(\alpha, d_2, m_2) S(\alpha, d_3, m_3) S(\alpha, d_4, m_4) \tag{3.7}
\]
and
\[
\lambda(\vec{d}) = \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4), \tag{3.8}
\]
we find that

$$\Gamma_1 = \sum_{d_i \mid P(z)} \sum_{|m_i| \leq H} \lambda(d_i) \sum_{x_i \equiv 0(d_i)} \frac{c(m_i)}{x_i^2 + x_i^2 + x_i^2} \int_0^1 S(\alpha, \vec{d}, \vec{m}) e(-N\alpha) d\alpha + O(1).$$

We divide $\Gamma_1$ into two parts:

$$\Gamma_1 = \Gamma_0^1 + \Gamma_1^* + O(1),$$

where

$$\Gamma_0^1 = c^4(0) \sum_{d_i \mid P(z)} \lambda(d_i) \sum_{x_i \equiv 0(d_i)} w(\vec{d})$$

and

$$\Gamma_1^* = \sum_{d_i \mid P(z)} \lambda(d_i) \sum_{0 < |m_i| \leq H} c(m_i) \int_0^1 S(\alpha, \vec{d}, \vec{m}) e(-N\alpha) d\alpha.$$  (3.9)

Hence

$$\Gamma \geq 4\Gamma_0^1 - 3\Gamma_0^5 + O(\Gamma_5^*) + O(\Gamma_5^*).$$  (3.10)

According to [1] and [9], for $D \leq P^{1/8-\epsilon}$, $s = \frac{\log D}{\log z} = 3.13$ the estimate

$$4\Gamma_1^0 - 3\Gamma_5^0 \gg \frac{C\delta N}{(\log N)^4} + O(\delta P^{3/2+\epsilon} D^4)$$  (3.11)

with some constant $C$ is obtained. Thus it suffices to evaluate $\Gamma_1^*$ and $\Gamma_5^*$.

### 3.3. ESTIMATION OF $\Gamma_1^*$

In this subsection we find the upper bound for $\Gamma_1^*$ defined in (3.9). The function in the integral in $\Gamma_1^*$ is periodic with period 1, so we can integrate over the interval $I$ defined as

$$I = \left[ \frac{1}{1 + \lfloor P \rfloor}, \frac{1}{1 + \lfloor P \rfloor} + 1 \right].$$

We apply the Kloosterman form of the Hardy-Littlewood circle method. We divide the interval only into large arcs. Using the properties of the Farey fractions, we represent $I$ as an union of disjoint intervals in the following way:

$$I = \bigcup_{q \leq P} \bigcup_{a=1}^q \mathcal{L}(a, q),$$

where

$$\mathcal{L}(a, q) = \left[ \frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q(q+q')} + \frac{1}{q(q+q''')} \right].$$

and where the integers \( q', q'' \) are specified in (2.7). Then

\[
\Gamma^*_1 = \sum_{d_i \mid P(z)} \lambda(\vec{d}) \sum_{0 \leq |m_i| \leq H} c(m_i) \sum_{q \leq P} \sum_{a=1}^{q} \int_{\mathcal{L}(a,q)} S(\alpha, \vec{d}, \vec{m}) e(-N\alpha) \, d\alpha.
\]

We change variable of integration \( \alpha = \frac{a}{q} + \beta \) to get

\[
\Gamma^*_1 = \sum_{d_i \mid P(z)} \lambda(\vec{d}) \sum_{0 \leq |m_i| \leq H} c(m_i) \sum_{q \leq P} \sum_{a=1}^{q} \times
\]

\[
\times \int_{\mathcal{M}(a,q)} S\left( \frac{a}{q} + \beta, \vec{d}, \vec{m} \right) e\left( -N \left( \frac{a}{q} + \beta \right) \right) \, d\beta,
\]

where

\[
\mathcal{M}(a,q) = \left[ -\frac{1}{q(q + q')}, \frac{1}{q(q + q'')} \right].
\]

From (2.7) we find that

\[
\left[ -\frac{1}{2qP}, \frac{1}{2qP} \right] \subset \mathcal{M}(a,q) \subset \left[ -\frac{1}{qP}, \frac{1}{qP} \right]
\]

and hence

\[
|\beta| \leq \frac{1}{qP} \quad \text{for} \quad \beta \in \mathcal{M}(a,q).
\]

Now we consider the sum \( S(\alpha, d_i, m_i) \) defined in (3.6). As \( \eta \) is irrational number, \( ||s\eta|| \neq 0 \) for all \( s \in \mathbb{Z} \). Using that fact and working as in the proof of [9, Lemma 12], we find that for \( \beta \in \mathcal{M}(a,q) \) we have

\[
S\left( \frac{a}{q} + \beta, d_i, m_i \right) = \frac{P}{d_i q} \sum_{n \in M_i} \left( \beta P^2, (m_i \eta - \frac{n}{d_i q}) P \right) G(q, a d_i^2, n) + O(P^{-B}),
\]

where \( G(q, m, n) \) and \( J(\gamma, u) \) are defined respectively by (2.8) and (2.6), \( B \) is an arbitrarily large constant, \( M_i = d_i P^\varepsilon \), \( \varepsilon > 0 \) is arbitrarily small and the constant in the \( O \)-term depends only on \( B \) and \( \varepsilon \). We leave the verification of the last formula to the reader.

Let

\[
F(P, \vec{d}) = \sum_{0 \leq |m_i| \leq H} c(m_i) \sum_{q \leq P} \sum_{a=1}^{q} e\left( -\frac{aN}{q} \right) \int_{\mathcal{M}(a,q)} S\left( \frac{a}{q} + \beta, \vec{d}, \vec{m} \right) e(-\beta N) d\beta.
\]

It is obvious that

\[
\Gamma^*_1 = \sum_{d_i \mid P(z)} \lambda(\vec{d}) F(P, \vec{d}).
\]
Using (3.13) and Lemma 2.3 we get
\[ F(P, \vec{d}) = F^*(P, \vec{d}) + O(1), \tag{3.15} \]
where
\[
F^*(P, \vec{d}) = \frac{P^4}{d_1d_2d_3d_4} \sum_{0 < |m_i| \leq H \atop 1, 2, 3, 4} c(m_i) \sum_{q \leq P} \frac{1}{q^2} \sum_{a(q)} \epsilon \left( -\frac{aN}{q} \right) \times \\
\times \sum_{|n_i - m_i d_i q \eta| < M_i} G(q, a d_i^2, \vec{n}) \int_{N(a, q)} J \left( \beta P^2, (\vec{m} \eta - \vec{n} \eta \vec{d}/d) P \right) e(-\gamma) d\gamma.
\]

Using Lemma 2.5 and working as in the proof of [14, Lemma 2] we find that
\[ F^*(P, \vec{d}) = F'(P, \vec{d}) + O(P^{3/2 + \varepsilon}), \tag{3.16} \]
where
\[
\begin{aligned}
F'(P, \vec{d}) &= \frac{P^2}{d_1d_2d_3d_4} \sum_{0 < |m_i| \leq H \atop 1, 2, 3, 4} c(m_i) \sum_{q \leq P} \frac{1}{q^2} \sum_{|n_i - m_i d_i q \eta| < M_i \atop (q, d_i)} V_q(N, \vec{d}, 0, \vec{n}) \times \\
&\times \int_{|\gamma| \leq \beta P} J \left( \gamma, (\vec{m} \eta - \vec{n} \eta \vec{d}/d) P \right) e(-\gamma) d\gamma,
\end{aligned}
\]
and \( V_q(N, \vec{d}, 0, \vec{n}) \) is defined by (2.9). We represent the sum \( F'(P, \vec{d}) \) as
\[ F'(P, \vec{d}) = F_1 + F_2, \tag{3.17} \]
where \( F_1 \) is the contribution of these addends with \( q \leq Q \) and \( F_2 \) for addends with \( Q < q \leq P \). Here \( Q \) is parameter, which we choose later. Using Lemma 2.3 (2), Lemma 2.6 and (3.1), we get
\[ F_2 \ll \frac{P^2 \delta^4}{d_1d_2d_3d_4} \sum_{0 < |m_i| \leq H \atop 1, 2, 3, 4} \sum_{Q < q \leq P} \frac{Q^{5/2} T(q, N)^{1/2} (q, d_1) \cdots (q, d_4)}{q^4} \times \\
\times \sum_{|n_i - m_i d_i q \eta| < M_i \atop (q, d_i)} 1.
\]
(3.18)

It is clear that the sum over \( \vec{n} \) in the expression above is
\[ \ll \prod_{1 \leq i \leq 4} \frac{M_1 M_2 M_3 M_4}{(q, d_1)(q, d_2)(q, d_3)(q, d_4)} \times \]
\[ \ll \frac{P^c d_1d_2d_3d_4}{(q, d_1)(q, d_2)(q, d_3)(q, d_4)}, \]

which, together with (3.18) and (3.2), gives

$$F_2 \ll P^{2+\varepsilon} \sum_{Q < q \leq P} \frac{\tau(q)(q, N)^{1/2}}{q^{3/2}}.$$ 

Now we apply Cauchy’s inequality to get

$$F_2 \ll P^{2+\varepsilon} \left( \sum_{Q < q \leq P} \frac{\tau(q)}{q} \right)^{1/2} \left( \sum_{Q < q \leq P} \frac{(q, N)}{q^2} \right)^{1/2} \ll P^{2+\varepsilon} \frac{1}{Q^{1/2}}.$$  \hspace{1cm} (3.19)

To evaluate $F_1$ we firstly apply Lemma 2.4 to get

$$\int_{|\gamma| \leq \frac{Q}{q}} \left| J \left( \gamma, (m\eta - \frac{\tilde{n}}{dq}) P \right) \right| d\gamma \ll \left( \left| (m\eta - \frac{\tilde{n}}{dq}) P \right| \right)^{-1+\varepsilon}.$$ 

Then using Lemma 2.6 and (3.2) we obtain

$$F_1 \ll \frac{P^2}{d_1 d_2 d_3 d_4} \sum_{Q \leq q} \frac{q^{5/2} \tau(q)(q, N)^{1/2}(q, d_1)(q, d_2)(q, d_3)(q, d_4)}{q^4} \times \sum_{1 \leq i \leq 4} \frac{1}{|(m\eta - \frac{\tilde{n}}{dq}) P|}.$$  \hspace{1cm} (3.20)

It is clear that if $n_i = (q, d_i)t_i$, $d_i = (q, d_i)d_i'$ and

$$\left| (m\eta - \frac{n_i}{dq}) P \right| = \frac{P(q, d_i)}{qd_i'} |t_i - m_i d_i' \eta q|,$$

then the sum over $(m\eta - \frac{n}{dq}) P$ in the expression above is

$$\ll \frac{P}{d_1} \sum_{\substack{|t_i - m_i d_i' \eta q| < \frac{M(t_i, d_i)}{d_i} \\text{for } 1 \leq i \leq 4}} \frac{1}{\max_{1 \leq i \leq 4} (q, d_i)|t_i - m_i d_i' \eta q|/d_i}.$$  \hspace{1cm} (3.21)

Let $t_1^q$ be such that

$$|t_1^q - m_1 d_1' \eta q| = || - m_1 d_1' \eta q|| = ||m_1 d_1' \eta q||.$$ 

As $\eta$ is quadratic irrational number, then $||m_1 d_1' \eta q|| \neq 0$ and for $t_1 \neq t_1^q$ we have $|t_1 - m_1 d_1' \eta q| \geq 1/2$. Hence

$$\max_{1 \leq i \leq 4} \frac{(q, d_i)|t_i - m_i d_i' \eta q|}{d_i} \gg \frac{(q, d_1)}{d_1},$$
which, together with (3.21), gives

\[
\frac{q}{P} \sum_{|t_i - m_i d_i' q| < \frac{M_1}{d_i}} \frac{1}{\max (q, d_i)|t_i - m_i d_i' q|/d_i} 
\ll \frac{q}{P} \left( \frac{d_1 M_1 M_3 M_4}{(q, d_1)^2(q, d_2)(q, d_3)(q, d_4)} + \frac{d_1 M_2 M_3 M_4}{(q, d_1)(q, d_2)(q, d_3)(q, d_4)} ||m_1 d_1' q|| \right) 
\ll \frac{q P^{-1} d_1 d_2 d_3 d_4}{(q, d_1)^2(q, d_2)(q, d_3)(q, d_4)} + \frac{q P^{-1} d_1 d_2 d_3 d_4}{(q, d_1)(q, d_2)(q, d_3)(q, d_4)} ||m_1 d_1' q||. 
\]

(3.22)

As \( \eta \) is quadratic irrationality, it has periodic continued fraction and if \( \frac{a_n}{b_n} \), \( n \in \mathbb{N} \) is the \( n \)-th convergent, then \( b_n \leq c^n \) for some constant \( c > 0 \). Using that \( ||m_1 d_1' q|| \leq \frac{3HDQ}{(d_1, q)} \) and Liouville's inequality for quadratic numbers (see Lemma 2.7), we can find convergent \( \frac{a}{b} \) to \( \eta \) with denominator such that

\[
3\frac{HDQ}{(d_1, q)} < b \ll c \frac{HDQ}{(d_1, q)}. 
\]

(3.23)

Since \( (a, b) = 1 \) we have that \( m_1 d_1' q \frac{a}{b} \not\in \mathbb{Z} \). As \( |\eta - \frac{a}{b}| < \frac{1}{b^2} \) and (3.23) we get

\[
||m_1 d_1' q\eta|| \geq \left| m_1 d_1' q \frac{a}{b} \right| - \left| m_1 d_1' q \left( \eta - \frac{a}{b} \right) \right| \geq \left| m_1 d_1' q \frac{a}{b} \right| - \frac{|m_1 d_1' q|}{b^2} 
\geq \frac{1}{b} - \frac{|m_1 d_1' q|}{3bH_DQ} \geq \frac{1}{b} - \frac{1}{3b} = \frac{2}{3b} 
\geq \frac{(d_1, q)}{HDQ}.
\]

From (3.21) and (3.22) it follows that

\[
\sum_{\left| m_1 - m_i d_i q \right| < M_i \atop (q, d_i)|n_1, ..., 4}} \frac{1}{\left| (\bar{m} \eta - \frac{a}{d_4})P \right|} \ll \frac{q P^{-1} d_1 d_2 d_3 d_4 HDQ}{(q, d_1)^2(q, d_2)(q, d_3)(q, d_4)}. 
\]

Then for \( F_1 \) (see (3.20)) we obtain

\[
F_1 \ll \frac{P^{1+\varepsilon} DQ}{\delta} \sum_{q \leq Q} \frac{\tau(q)(q, N)^{1/2}}{q^{1/2}}. 
\]

(3.24)
Applying Cauchy’s inequality we get

\[ F_1 \ll \frac{P^{1+\varepsilon}DQ}{\delta} \left( \sum_{q \leq Q} \tau^2(q) \right)^{\frac{1}{2}} \left( \sum_{q \leq Q} \frac{(q,N)}{q} \right)^{\frac{1}{2}} \]

\[ \ll \frac{P^{1+\varepsilon}DQ}{\delta} \cdot Q^{3/2} (\log Q)^{3/2} \left( \sum_{t \leq Q} \sum_{q \leq Q} \frac{1}{q} \right)^{\frac{1}{2}} \]

\[ \ll \frac{P^{1+\varepsilon}DQ^{3/2}}{\delta}. \quad (3.25) \]

We choose \( Q = \delta^{1/2} P^{1/2} D^{-1/2} \). Then

\[ F_1, F_2 \ll P^{7/4+\varepsilon} \delta^{-1/4} D^{1/4}. \]

From (3.14), (3.15), (3.16), (3.17) it follows that

\[ \Gamma^*_1 \ll D^{17/4} P^{7/4+\varepsilon} \delta^{-1/4}. \]

The estimate of \( \Gamma^*_5 \) goes along the same lines.

3.4. END OF THE PROOF OF THEOREM 1.1

From (3.10) and (3.11) we get

\[ \Gamma \gg \frac{\delta N}{(\log N)^4} + D^{17/4} P^{7/4+\varepsilon} \delta^{-1/4}. \]

Then for a fixed small \( \varepsilon > 0 \), \( \lambda < \frac{9-8\varepsilon}{10} \), \( D < N^{\frac{1-10\lambda}{10\lambda-8\varepsilon}} \) and \( z = D^{1/3} 13 \) we get \( \Gamma \gg \frac{\delta N}{(\log N)^7} \). So the equation (1.1) have solutions in almost-prime numbers \( x_1, \ldots, x_4 \in \mathcal{P}_k, k = \left\lfloor \frac{53.21}{1-10\lambda-8\varepsilon} \right\rfloor \) such that \( \{ \eta x_i \} < N^{-\lambda}, i = 1, 2, 3, 4. \)

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