ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 107

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 107

REVIEW OF CONTINUUM MECHANICS AND ITS HISTORY PART I. DEFORMATION AND STRESS. CONSERVATION LAWS. CONSTITUTIVE EQUATIONS

BOGDANA A. GEORGIEVA

This is a review of continuum mechanics and its history, citing its original sources. It “bridges” the contributions of Bernoulli, Euler, Lagrange, Cauchy, Helmholtz, St. Venant, Stokes, Fresnel, Cesaro, and others, written in a period of two centuries in 5 languages, in a coherent and historically accurate presentation in the contemporary notation. The only prerequisite knowledge to understand the paper is advanced calculus and elementary differential equations. Some valuable, but little known, results are reviewed in detail, like the exact solution of Cesaro to the system of differential equations which every continuous medium obeys, as well as his derivation of the conditions of St. Venant for compatibility of the deformations. The last section presents the contemporary applications of continuum mechanics. The review continues with Part II. The Mechanics of Thermoelastic Media. Perfect Fluids, reference [45]. It discusses the consequences of Navier’s system of linear elasticity and approaches for its solution. It also gives a perspective of how waves propagate in continuous media. Reviewed are perfect fluids and linearly viscous fluids. At the end, Part II discusses the conditions for compatibility of the stresses.

Keywords: Mechanics of continuous media, continuum mechanics, history of continuum mechanics, elasticity, theory of elasticity.

2020 Math. Subject Classification: 74-02, 74B-05.

1. INTRODUCTION

Mechanics of continuous media is one of the classical branches of applied mathematics, which was built by several of the most prominent mathematicians of the 18th, 19th and the early 20th centuries. In addition to being a discipline of its own, it is the heart of several modern branches of applied mathematics: fluid mechanics, gas dynamics, theory of elasticity, theory of deformable solids and others. It's applications penetrate almost every aspect of contemporary applied mathematics and mathematical physics. Over the centuries so much material accumulated in this subject, that at present only a few mathematicians know what is a fundamental notion in it and what is an application or a consequence of its core results. It is important that the mathematicians of today do know continuum mechanics not only for this knowledge itself, but also for the correct vision and proper sight of Mathematics and Science that it gives. It will help them size their own gauge to the contemporary needs of their profession. In addition to its powerful applications, continuum mechanics is precious for its esthetics - it is a part of the most elegant and sophisticated classical mathematics and reading it gives a pleasure and a professional growth.

The first attempt to discuss local features of the motion of a continuous medium in more than one dimension occurs in an isolated passage by D. Bernoulli from 1738 ([1], §11, paragraph 4). We are surrounded by matter in the form of continuous media – deformable solids, liquids and gasses. Let us begin at the moment of time $t = 0$ with a continuous medium, like a gallon of water, which we can easily imagine fills the volume V, with a shape specified by our imagination. Atomic structure is not considered. If the water is not held in a vessel, when we "unfreeze" time, it will move under the law of gravity and the laws of conservation of mass, momentum and energy, in a perfectly deterministic manner, continuously changing its shape, and eventually splash on the floor. This is a simple example of a motion of a continuous medium and is suitable to demonstrate what is meant by "material coordinates" and by "spatial coordinates". **Material coordinates**, also called **Lagrangian coordinates**, are denoted by (X_1, X_2, X_3) and are the coordinates of the material points of the continuous medium at time $t = 0$. Lagrange introduced them in 1788 in [54], part II, section II. **Spatial coordinates**, also known as **Eulerian coordinates**, are denoted by (x_1, x_2, x_3) and are the coordinates of the points of 3-dimensional space (in which we observe the medium) occupied by the medium at time $t > 0$. Since the material coordinates are the coordinates of the material points at an arbitrary initial time $t = 0$, they can serve for all time as names for the *particles* of the material. The spatial coordinates, on the other hand, we think of as assigned once and for all to a point in the Euclidean space. They are the names of *places*. The motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ chronicles the places $\mathbf{x}$ occupied by the particle $\mathbf{X}$ in the course of time. Under external influences - forces and heating - the continuous body deforms. *The goal of Mechanics of Continua is to find the family of transformations*

$$x_i = x_i(X_1, X_2, X_3, t)\ , \quad i = 1, 2, 3, \tag{1}$$

giving the Eulerian coordinates as functions of the Lagrangian coordinates for $t \geq 0$.

This motion is perfectly deterministic, obeying only the natural laws, that we will present. We will arrive at a system of 20 partial differential equations for 20 unknown functions. This system is one of the finest triumphs of the symbiosis between mathematics and physics. We sketch the solution to this system and present the conditions for its existence and uniqueness. We give credit to the mathematicians and physicists who built this discipline by citing the date, name and the historical reference where the result was published for the first time.

The general theory of the motion of a continuous medium, which is understood of as a family of deformations continuously varying in time, is almost exclusively due to Euler, published in the period 1745 – 1766 in [25] – [41], and Cauchy, published in the period 1815 – 1841 in [3] – [18]. Important special results were added by D'Alembert in 1749 in [22], Green in 1839 in [46], Stokes in 1845 in [61], Helmholtz in 1858 in [48] and Cesaro in 1906 in [19].

2. STRAIN

The change in length and relative direction occasioned by the transformation is called **strain**. The term is due to Rankine [56] in 1851. Let us begin its study by defining the displacement vector $\mathbf{u}$, with components $u_i = x_i - X_i$, where $x_i = x_i(X_1, X_2, X_3, t)$ $i = 1, 2, 3$. The components u_i can be expressed in Lagrangian or in Eulerian coordinates, depending on need. Let P_0 be an arbitrary point of the continuous medium at time $t = 0$ and let Q_0 be a neighboring point, such that in a fixed Cartesian coordinate system $O_{e_1e_2e_3}$ P_0 has coordinates (X_1, X_2, X_3), i.e. the radius vectors to P_0 is $\mathbf{X}$ and to the point Q_0 is $\mathbf{X} + d\mathbf{X}$. At time $t > 0$ the material point P_0 occupies new geometric point P with coordinates (x_1, x_2, x_3), i.e. P has radius vector $\mathbf{x}$ and hence the new geometric location of the material point Q_0 is Q with a radius vector $\mathbf{x} + d\mathbf{x}$. To study the deformation that has occurred, we need to see how much has the distance between the two neighboring points P_0 and Q_0 changed. For that we calculate

$$(d\mathbf{x})^2 - (d\mathbf{X})^2 = \Big(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\Big)\, dX_i\, dX_j = \Big(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\Big)\, dx_i\, dx_j\,. \qquad (2)$$

Here and throughout the paper each index takes the values 1, 2 and 3 and the summation convention on repeated indexes is assumed. We see that all the information about the deformation is contained in the coefficients of $dX_i\, dX_j$ and respectively of $dx_i\, dx_j$ in (2). These sets of coefficients

$$E_{ij} \equiv \frac{1}{2}\Big(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\Big) \quad \text{and} \quad e_{ij} \equiv \frac{1}{2}\Big(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\Big)$$

satisfy the transformation laws for tensors of rang 2 and are called the Lagrangian and the Eulerian **tensors of finite deformations** or **finite strain tensors**. The difference $(d\mathbf{x})^2 - (d\mathbf{X})^2$ is a measure for the size of the deformation in the vicinity of P_0. Because dX_i and dx_i are arbitrary, the necessary and sufficient condition this difference to be 0 is $E_{ij} = 0$ or equivalently $e_{ij} = 0$. In that case the deformation near that point is 0 and the motion is that of a rigid body. Written in terms of the gradients $\partial u_i/\partial X_j$ or $\partial u_i/\partial x_j$ of the displacement vector $\mathbf{u}$, E_{ij} and e_{ij} are

$$E_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\Big) \quad \text{and} \quad e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\Big).$$

We will now make a crucial assumption – that the deformations which we will study are small. This means that the gradients $\partial u_i/\partial X_j$ and $\partial u_i/\partial x_j$ of the displacement $\mathbf{u}$ are small in comparison to 1, and hence the products of these gradients may be ignored in the presence of the gradients themselves. In this manner we obtain the tensors $\bar{E}_{ij}$ and $\bar{e}_{ij}$. A calculation based on the same assumption shows that they are equal and we give them the common name ε_{ij}. This is the **tensor of (infinitesimal) deformations** or the (infinitesimal) **strain tensor**

$$\varepsilon_{ij} \equiv \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

The strain tensor ε_{ij} was introduced by Green in 1841 in [47] and by St. Venant in 1844 in [59]. It is the most popular strain measure even today. The vanishing of ε_{ij} is necessary and sufficient for a rigid displacement. The general deformation $dX \to dx$ as well as the displacement gradients $\partial u_i/\partial X_j$ and $\partial u_i/\partial x_j$ as measures of local changes of length and angle are due to Lagrange 1762, [53] §XLIV and 1788 [54] Part II, Sect. 11. The fully general spatial description is due to Euler, dates 1752, and was first published in 1757 in [31] and then in 1761 in [33]. The theory of finite strain is the creation of Cauchy published in 1823 [4], in 1827 [7] and in 1841 [18]. The theory of infinitesimal strain was first developed by Euler. It was fully elaborated by Cauchy, who obtained it by specialization from his general theory of finite strain.

We will now explain the geometry of the process of deformation. The component ε_{11} of the strain tensor is the relative elongation of a linear element in the direction of the unit coordinate vector $\mathbf{e_1}$, and similarly for ε_{22} and ε_{33}. The component ε_{23} is half of the change (as a result of the deformation) of the angle between two lines, that initially had the directions of the unit coordinate vectors $\mathbf{e_2}$ and $\mathbf{e_3}$. Even more surprising is the fact that, at each point inside the deforming medium, the deformations can not take an arbitrary shape. Instead, they form quadratic surfaces only, called **surfaces of Cauchy**. This is not hard to see and is worth the effort. Let us denote by ε the relative elongation in direction of the vector $d\mathbf{X}$, with length dX

$$\varepsilon \equiv \frac{dx - dX}{dX}.$$

Consider the difference

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2\varepsilon_{ij}\, dX_i\, dX_j, \tag{3}$$

and observe the smallness of the deformations, i.e. that $dx \approx dX$. Then by dividing both sides of (3) by $dX\, dX$ we see that

$$\varepsilon = \varepsilon_{ij}\,\frac{dX_i}{dX}\,\frac{dX_j}{dX}. \tag{4}$$

Hence for any vector with components (ξ_1, ξ_2, ξ_3) and magnitude ξ, the last formula (4) gives $\xi^2\varepsilon = \varepsilon_{ij}\,\xi_i\,\xi_j$. For each direction we can select ξ in such a way that $\xi^2\varepsilon = \pm k^2$, where k is a positive constant and the sign is chosen so that the square of the length of vector ξ to be positive. It follows that at any point of the deforming medium the strain takes the shape of the **quadratic surface**

$$\varepsilon_{ij}\,\xi_i\,\xi_j = \pm k^2$$

called **surface of deformations of Cauchy at the point** P_0. From this geometric picture it is clear that the elongation ε in the direction of the vector ξ is inversely proportional to the square of the distance from the center of the surface (the point P_0) to the intersection of the vector ξ with that surface.

Because the vector $(\varepsilon_{1j}\,\xi_j, \varepsilon_{2j}\,\xi_j, \varepsilon_{3j}\,\xi_j)$ is normal to the quadratic surface of Cauchy, we see that the relative displacement at P_0 due to the pure deformation is in the direction of the normal to that surface at the point of intersection of the surface with this vector.

After these observations, it is plausible to seek lines through P_0 with directions that do not change under pure deformation. Of course, these are the lines along the eigenvectors of the strain tensor ε_{ij}. It is symmetric and hence has 3 real eigenvalues, called **main deformations** or **Cauchy principal stretches**, ε_I, ε_{II}, and ε_{III}. To each of them corresponds an eigenvector, called **main direction** or **main axis of the strain tensor**. Cauchy published these results first in 1823 [4] and again in 1827 [7]. To different main deformations correspond main directions that are orthogonal. We can select the axes of the coordinate system to coincide with the main axes of the tensor of deformations and, as a result, obtain the simplest form of the quadratic surface of Cauchy

$$\varepsilon_I\,\xi_1^2 + \varepsilon_{II}\,\xi_2^2 + \varepsilon_{III}\,\xi_3^2 = \pm\,k^2.$$

The invariants of the tensors **E** and **e** were first published by Cauchy in 1827 in [7].

3. CONDITIONS FOR COMPATABILITY OF THE DEFORMATIONS

Common sense tells us that the deformations that take place in a medium are not independent of each other. If we stretch an elastic membrane with a rectangular shape along one of its diagonals, the other diagonal will shrink. St.Venant proved in 1860 that in order for the six functions $\varepsilon_{ij}(x_1, x_2, x_3)$ to adequately define the components of the tensor of deformations ε_{ij}, so that the 6 partial differential equations

$$u_{i,j} + u_{j,i} = 2\varepsilon_{ij} \tag{5}$$

have a unique solution $\mathbf{u}(x_1, x_2, x_3)$, they must satisfy the system of 6 PDEs

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \tag{6}$$

The notation $,j$ denotes partial differentiation with respect to x_j. The 6 restrictions (6) on the components ε_{ij} of the tensor of deformations are called **conditions for compatability of the deformations** and their fulfillment is a necessary and sufficient condition for the existence of the solution vector $\mathbf{u}$ to the system (5), which of course, has the physical meaning of the displacement vector $\mathbf{u} \equiv \mathbf{x} - \mathbf{X}$ in the process (1) of the deformation of the continuous medium. The derivation of the compatability conditions is exceptionally original. On the way of deriving the compatability conditions, an analytic formula for the displacement $\mathbf{u}$ itself is derived, thus obtaining a result of even greater significance. Due to lack of space, this derivation is not presented here, but it is sketched. This method of obtaining the displacement $\mathbf{u}$ is due to E. Cesaro [19], who published it in 1906. Volterra presents it in [62], citing Cesaro. Contemporary references on it are Ivanov [49] and Sokolnikoff [58]. The solution to (5) has components

$$u_j = u_j^0 + \omega_{jk}^0(x_k - x_k^0) + \int_{P_0}^{P} \big(\varepsilon_{jl} + (x_k - y_k)(\varepsilon_{jl,k} - \varepsilon_{kl,j})\big)\, dy_l \;, \quad j = 1,2,3. \tag{7}$$

Here

$$\omega_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big)$$

is the tensor of small rotations, introduced by Euler in 1761, §§46-47. In the components of the exact solution (7) of Cesaro u_j^0 are the components of the translation and ω_{jk}^0 are those of the tensor of rotation in an arbitrary point P_0 of the deforming body, and are assumed known. The first term in the solution (7) for u_j represents the translation and the second term represents the rotation of the continuous medium as a rigid body. The third term in u_j represents its deformation. Because the displacement $\mathbf{u}$ is unique, its components u_j must not depend on the path of integration, so the integrands of the 3 integrals must be total differentials. Demanding this, yields the 6 equations (6) of St. Venant for compatability of the deformations.

The compatability conditions were first published by Kirchhoff in 1859 in [52], but without a statement of their meaning, which was first explained by St. Venant in his memoir [60]. St. Venant obtained these conditions in a different way, than the one presented in this section. Submitted them to Scocièté Philomathique in 1860, who published them in 1864.

4. STRESS

The notion of stress arose in special case studies of theories of flexible, elastic and fluid bodies. Galileo (1638), Pardies (1673), James Bernoulli (1691–1704), Hermann (1716), Coulomb (1776), John Bernoulli (1743), and Euler (1749–1752) published studies on this notion. The general concept and mathematical theory are due to Cauchy, published in 1823 [4] and in 1827 [7]. Cauchy achieved the general theory of stress by adopting the common features and discarding the special aspects of the foregoing theories. The term **stress** was introduced by Rankine in 1856 in [57].

The field of stress vectors is not an ordinary vector field. Rather, since the stress vectors across two different surfaces through the same point are generally different, at any given time, the stress vectors $\sigma(\mathbf{x}, t, \mathbf{n})$ depend both on the position vector $\mathbf{x}$ and on the direction $\mathbf{n}$ of the normal to the surface. We wish to extract all the information about the stress at a point of the body into a single mathematical object, and separate it from the information about the direction. This is accomplished by the **stress tensor** σ_{ij}.

To derive the components of that tensor we take a tetrahedron having 3 edges coming out of an arbitrarily fixed point P, parallel to the coordinate axes. The force acting on the medium occupying the volume V of the tetrahedron is $\int_V \rho\, \mathbf{f}\, dV$, where $\rho(\mathbf{x}, t)$ is the mass density and $\mathbf{f}(\mathbf{x}, t)$ is the mass force acting on $\rho\, dV$. Examples of mass forces are gravity and the centrifugal force in a rotating body. Surface forces act on every surface inside the medium or on its surface. Those forces are modeled with the stress vector $\sigma(\mathbf{x}, t, \mathbf{n})$. The force acting on a portion S of a surface is $\int_S \sigma\, dS$. The orientation of S is given by the outward unit normal $\mathbf{n}(\mathbf{x}, t) = n_i(\mathbf{x}, t)\, \mathbf{e}_i$ to the surface at that point. (The dimension of the vector σ is pressure.)

We assume that all forces acting on the tetrahedron ballance out

$$\sum_{j=1}^{3} \int_{\triangle S_j} \sigma(\mathbf{x}, t, -\mathbf{e}_j)\, dS + \int_{\triangle S} \sigma(\mathbf{x}, t, \mathbf{n})\, dS + \int_{\triangle V} \rho\, \mathbf{f}\, dV = \mathbf{0}, \tag{8}$$

where $\triangle S_j$ is the face perpendicular to $\mathbf{e}_j$, $\triangle S$ is the forth face and $\triangle V$ is the part of 3-space occupied by the tetrahedron. We make use of the mean-value theorem in equation (8). Denote the radius-vector to the point P by $\mathbf{x}$, make use of $\triangle S_j = \triangle S \cos(\mathbf{n}, \mathbf{e}_j) = \triangle S \mathbf{n}_j$, $\triangle V = h \triangle S/3$, and let the altitude h from P approach 0. We get

$$\sigma(\mathbf{x}, t, -\mathbf{e}_j)\, n_j + \sigma(\mathbf{x}, t, \mathbf{n}) = \mathbf{0}. \tag{9}$$

If we now denote by $\sigma_{ij}(\mathbf{x}, t)$ the components of the stress vector with a normal $\mathbf{e}_j$, $\sigma_{ij}(\mathbf{x}, t) = \sigma_i(\mathbf{x}, t, \mathbf{e}_j)$, from the last vector equation (9) we get

$$\sigma_i(\mathbf{x}, t, \mathbf{n}) = \sigma_{ij}(\mathbf{x}, t)\, n_j.$$

This important result is **Cauchy's fundamental theorem** and expresses the relationship between the components of the stress vector and the components of the stress tensor. All the information about the stress at a point is "extracted" in the stress tensor itself, and is "separated" from the orientation $\mathbf{n}$ of the surface. Cauchy published this formula in 1823 [4] and in 1827 [6].

The geometry of the stress at a point of a deforming medium is also that of quadratic surfaces. Consider the stress vector σ, acting on a surface element with a unit normal $\mathbf{n}$ at a fixed point P of the body. Its components are $\sigma_i = \sigma_{ij}\, n_j$. Let us denote by σ_N the magnitude of its projection on $\mathbf{n}$. σ_N is called **normal stress** and can be expressed as

$$\sigma_N = \sigma_i\, n_i = \sigma_{ij}\, n_i\, n_j\, .$$

If ξ is a vector having the direction of the unit normal $\mathbf{n}$ and size ξ, then from the last equation follows that $\xi^2 \sigma_N = \sigma_{ij}\, \xi_i\, \xi_j$, where ξ_i are the components of ξ. Select the size ξ of the vector ξ in such a way that $\xi^2 \sigma_N = \pm\, k^2$, where k is a fixed positive constant and the sign is chosen so that the length of ξ defined with this equation be positive. Then the "tip" of an *arbitrary* vector ξ with base at P, and magnitude ξ satisfying $\xi^2 \sigma_N = \pm\, k^2$, lies on the surface

$$\sigma_{ij}\, \xi_i\, \xi_j = \pm\, k^2$$

called **quadratic surface of the stress tensor** or **surface of Cauchy of the stress** at the point P. The stress tensor is symmetric and hence has 3 real eigenvalues, called **main stresses**. The corresponding eigenvectors are called **main directions** or **main axes**. If we choose a coordinate system with coordinate axes along the main axes of the stress tensor, the quadratic surface of the stress at the point acquires the form

$$\sigma_I\, \xi_1^2 + \sigma_{II}\, \xi_2^2 + \sigma_{III}\, \xi_3^2 = \pm\, k^2,$$

where $\sigma_I, \sigma_{II}\, , \sigma_{III}$ are the main stresses of σ_{ij} at that point. At a surface element with a normal $\mathbf{n}$ along a main axes of the stress tensor, the stress vector σ has the direction of the normal.

5. CONSERVATION OF MASS, MOMENTUM AND MOMENT OF MOMENTUM

In contemporary mathematics and mathematical physics conservation laws are a main goal of study. Researchers obtain them from variational principles via the famous first theorem of Emmy Noether. In Mechanics of Continua, however, history went differently. All the laws of conservation, namely the conservation of mass, energy, momentum, and moment of momentum, were discovered by judicious guessing and verification with the physical experiment. They are all empirical laws. Much later they were derived from deliberately calculated for this purpose Lagrangians.

The law of conservation of mass is the statement that the mass, contained in any portion of the body with volume V, does not change during the deformation

$$\frac{d}{dt} \int_V \rho\, dV = 0.$$

This can be rewritten as $\int_V \partial\rho/\partial t\, dV + \int_S v_n\, \rho\, dS = 0$, where $v_n = \mathbf{v} \cdot \mathbf{n}$ is the component of the velocity of the points on the surface S of V along the outward unit normal $\mathbf{n}$ to S. Thus, $\int_V (\frac{\partial\rho}{\partial t} + (\rho\, v_i)_{,i})\, dV = 0$ where $v_i(\mathbf{x}, t)$ are the components of the velocity. If the integrand is continuous, we obtain the differential form of the **law of conservation of mass**

$$\frac{\partial\rho}{\partial t} + (\rho\, v_i)_{,i} = 0. \tag{10}$$

The law of conservation of mass was first discovered by Euler in 1757, reference [31], §§16–17.

In mechanics of continua the so-called **equations of motion** play the same role as do the equations of Newton in mechanics of rigid bodies. These equations of motion of a continuous medium follow from the law of **conservation of momentum**, which states that "The total time derivative of the momentum of an arbitrarily fixed portion of the deforming body is equal to the sum of all forces (mass forces $\mathbf{f}(\mathbf{x}, t)$ and surface forces $\sigma(\mathbf{x}, t)$) that act on it"

$$\frac{d}{dt}\int_V \rho\, v_i\, dV = \int_V \rho\, f_i\, dV + \int_S \sigma_i\, dS, \qquad i = 1, 2, 3. \tag{11}$$

A simple calculation shows that, if mass is conserved, for any continuously differentiable function $g(x, t)$ it is true that

$$\frac{d}{dt}\int_V \rho\, g(x, t)\, dV = \int_V \rho\, \frac{dg}{dt}\, dV\, .$$

With $g = v_i$ this formula simplifies the law of conservation of momentum (11) to

$$\int_V \rho\, \frac{dv_i}{dt}\, dV = \int_V \rho\, f_i\, dV + \int_S \sigma_{ij}\, n_j\, dS. \tag{12}$$

Applying Gauss' theorem to the surface integral in (11), combining the resulting 2 integrals, and assuming continuity, we obtain the **equations of motion of a continuous medium**

$$\sigma_{ij,j} + \rho\, f_i = \rho \frac{dv_i}{dt}\ , \qquad i = 1, 2, 3. \tag{13}$$

These equations were first published by Cauchy in 1827 in [9], and also in 1827 in [11].

The **law of conservation of moment of momentum** asserts that "the time rate of change of the moment of momentum is equal to the sum of the moments of the mass forces and the surface forces that act on the body", i.e.,

$$\frac{d}{dt}\int_V \rho\, e_{ijk}\, x_j\, v_k\, dV = \int_V \rho\, e_{ijk}\, x_j\, f_k\, dV + \int_S e_{ijk}\, x_j\, \sigma_k\, dS,$$

where the moments are written with respect to the origin of the coordinate system.

The laws of conservation of momentum and of moment of momentum are both due to Euler and were introduced by him in 1775, [43], §§26–28. While the memoire is about rigid bodies, these two laws are expressly stated to hold for any continuous medium.

The law of conservation of moment of momentum is fully equivalent to the symmetry of the stress tensor

$$\sigma_{ij} = \sigma_{ji} \, .$$

This important result is known as **Cauchy's fundamental theorem**, and was published by him in 1827, [6]. It was discovered (but not published) by Fresnel in 1822, who published it in 1868, [44].

6. CONSERVATION OF ENERGY

In mechanics of rigid bodies thermal effects and thermal consequences of the motion are either considered separately from the equations of motion or completely ignored, if they do not affect the motion in consideration. For example, we ignore the heat generated during the friction between the surface of a cube sliding on a plane and that plane. In Mechanics of Continua heat generation and thermal effects can not be ignored or even considered separately from the equations of motion. The reason is that when a deformation takes place, heat is generated/lost throughout the entire volume where the deformation occurs. This thermal energy affects significantly the motion and the deformation. It becomes a cycle: the deformation generates heat and that heat in turn affects the distance between the particles of the continuous medium, thus causing deformation. The dynamics of a continuous medium and the thermal laws are intertwined and must be studies simultaneously.

That heat is a mode of motion was widely believed in the 18th century. Both Daniel Bernoulli [1] in 1738 and Euler [35] in 1765 constructed kinetic molecular models in which temperature may be identified with the kinetic energy of the molecules. The general and phenomenological principle, independent of molecular interpretation, was known to Carnot by 1824, as proved by his memoir [2]. The first clear statement of the interconvertibility of heat and mechanical work, that any equation of energy ballance should contain terms that represent non-mechanical transfer of energy, are those of Joule [50], [51] from 1843 and 1845 and of Waterston [64] from 1843.

Let us now consider the **law of conservation of energy**. It states that "The total time derivative of the sum of the kinetic energy and the internal energy is equal to the sum of the power of the external forces and the in-flow of all other kinds of energies per unit of time"

$$\frac{dK}{dt} + \frac{dE}{dt} = W + Q, \tag{14}$$

where $K = \int_V \rho \, v_i \, v_i / 2 \, dV$ is the kinetic energy, $W = \int_V \rho \, f_i \, v_i \, dV + \int_S \sigma_i \, v_i \, dS$ is the power of the external forces, $Q = -\int_S q_i \, n_i \, dS + \int_V \rho \, r \, dV$ is the in-flow of heat per unit of time. Here $\mathbf{q} = q_i(\mathbf{x},t)\, e_i$ is the vector of heat flow and $r(\mathbf{x},t)$ is the specific heat source. For simplicity, we assume that there is only in-flow of thermal energy. We also assume the existence of a function $\epsilon(\mathbf{x},t)$ called **specific internal energy** such that

$$\int_{V(t)} \rho\, \epsilon\, dV = E,$$

where E is the total internal energy of the part of the body with volume V at time t. The general law of conservation of energy (when heat effects are included), i.e. equation (14), is called "**the first law of thermodynamics**". The first one to formulate this important law was Duchem [24], Chapter III, §3, in 1892 .

In the special case $Q = 0$ the first law of thermodynamics reduces to the **law of conservation of mechanical energy**

$$\frac{dK}{dt} + \int_V \sigma_{ij}\, d_{ij}\, dV = W,$$

where

$$d_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}) = d_{ji}$$

is the tensor of rate of deformations, introduced by Euler [41], §§ 9–12, in 1769. By a simple, but tedious calculation, substituting dK/dt, E and Q into the general law of conservation of energy (14), transforming the surface integral into a volume integral, and assuming continuity, we obtain the **differential form of the general law of conservation of energy**

$$\rho\, \frac{d\epsilon}{dt} = \sigma_{ij}\, d_{ij} - q_{i,i} + \rho\, r\, . \tag{15}$$

That use of a differential equation expressing balance of energy is necessary, except in specially simple circumstance, was first emphasized by Duhem [23], Vol. I, Livre II, Chapter III, in 1891. In 1769 Euler [41], §13, showed that the vanishing of all components of the tensor of rate of deformations is the criterion for a rigid motion.

7. ENTROPY

In the present section we define and explain the concept of entropy and the second law of thermodynamics.

Let us begin with some history. During the Industrial Revolution in Western Europe, it was observed that the steam engines of locomotives and other engines that transform thermal energy into mechanical energy can not achieve efficiency of 100%. In 1865 Rudolf Clausius [21], §14, introduced the concept of entropy for the lost thermal energy in steam engines, i.e., the heat which remained unconverted into mechanical energy. **Entropy** is defined by

$$d\eta = c\, \frac{d\theta}{\theta}$$

where $\eta(\mathbf{x}, t)$ is the entropy for unit mass, c is the specific heat and $\theta(\mathbf{x}, t)$ is the absolute temperature of the body.

The **inequality of Clausius - Duhem** is

$$\frac{d}{dt}\int_V \rho\eta\; dV \geq -\int_S \frac{q_i}{\theta}\, n_i\; dS + \int_V \rho\,\frac{r}{\theta}\; dV,$$

where $r(\mathbf{x}, t)$ is the specific heat source, and $\mathbf{q} = q_i\,\mathbf{e}_i$ is the vector of heat flow. It has the direction of motion of heat. The normal $\mathbf{n}$ is outward to the surface S. The first integral in the right hand side is the flow of entropy per unit time through the surface S of the volume V and the second integral is the creation of entropy inside V by outside sources per unit time. This inequality is one of the fundamental empirical laws of thermodynamics – the second law of thermodynamics. It is due to Clausius [20] (1854). The meaning of the second law of thermodynamics is can be explained as follows. It is known from experience that a substance at uniform temperature and free fro sources of heat may consume mechanical work, but can not give it out. That is, whatever work is not recoverable is lost, not created. Also, in a body at rest and subject to no sources of heat, the flow of heat is from the hotter to the colder parts, not vice versa.

Using the well known formula

$$\frac{d}{dt}\int_V \rho\, f\; dV = \int_V \rho\,\frac{df}{dt}\; dV,$$

where ρ is the mass density, which holds for any continuously differentiable function $f(\mathbf{x}, t)$, we obtain the differential form of the inequality of Clausius - Duhem:

$$\rho\,\frac{d\eta}{dt} + \left(\frac{q_i}{\theta}\right)_{,i} - \rho\,\frac{r}{\theta} \geq 0. \tag{16}$$

8. CONSTITUTIVE EQUATIONS

We consider the differential forms of: the law of conservation of mass (10), the law of conservation of energy (15), the equations of motion of a continuous medium (13), and the inequality of Clausius-Duhem (16) as a system. These are 5 scalar differential equations and 1 inequality for the 16 unknown functions u_i, ρ, σ_{ij}, ϵ, η and θ. We take in consideration the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$, the definition of $d_{ij} = (v_{i,j} + v_{j,i})/2$, and assume that $\mathbf{f}$ and r are given. It is remarkable, but not surprising, that physics provides the additional equations necessary to solve this system. These are the so called **constitutive equations** and contain information about the specific material of the medium. An elastic is very different from water, which is very different from an oil or a gas. The constitutive equations characterize the mechanical and thermal properties of the medium.

In experiments and observations, the motion of the material particles of the continuous medium and its temperature can be observed and measured, so from mathematical stand point the components u_i of the displacement vector, the temperature θ, as well as their derivatives, will be the independent variables in the constitutive equations, which we are trying to build. All of the rest of the variables will dependent on these ones and will be dependent variables. These are: σ_{ij}, ϵ, q_i and η, a total of 11 such variables. The mass density ρ is also a dependent variable. For it we already have a differential equation, relating it to the rest of the variables, namely the law of conservation of mass.

Because the constitutive equations characterize the properties of the materials, they must remain invariant under a rotation or a translation. This requirement is met if the variables (both independent and dependent), which those equations relate, are themselves independent of such transformations. It is easy to show that such variables are:

$$\Sigma_{kl} = \sigma_{ij} \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_l}, \qquad Q_j = q_i \frac{\partial x_i}{\partial X_j}$$

as well as the scalar functions ϵ and η. Thus, in the constitutive equations which we are trying to construct, it will be reasonable to regard as independent variables the temperature θ, the coordinates X_i, the gradient $\delta\theta/\delta X_i$ of the temperature and the tensor of deformations E_{ij}. Hence for a thermoelastic medium the constitutive equations are :

$$\Sigma_{ij} = \Sigma_{ij}(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X}), \ \mathbf{Q_i} = \mathbf{Q_i}(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X}), \ \epsilon = \epsilon(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X}), \ \eta = \eta(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X}),$$

where $\mathbf{G}$ denotes the gradient of the temperature with respect to the Lagrangian coordinates X_i, $i = 1, 2, 3$. Using

$$(\partial x_i/\partial X_k)(\partial X_k/\partial x_j) = \delta_{ij}\ , \quad (\partial X_i/\partial x_k)(\partial x_k/\partial X_j) = \delta_{ij}\ ,$$

we invert the equations for Σ_{kl} and Q_j to obtain

$$\begin{aligned} &\sigma_{ij} = \Sigma_{kl}(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X})\, X_{k,i}\, X_{l,j}\,, \quad q_i = Q_j(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X})\, X_{j,i}\,, \\ &\epsilon = \epsilon(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X})\,, \qquad \eta = \eta(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X})\,. \end{aligned}$$

Using the inequality of Clausius-Duhem we will be able to see the form of the constitutive equations in more detail. For this, a new function, **free energy**, is introduced:

$$\psi \equiv \epsilon - \eta\, \theta.$$

Obviously $\psi = \psi(\mathbf{E}, \theta, \mathbf{G}, \mathbf{X})$ and we assume that it is symmetric with respect to E_{ij} and E_{ji}. (This is possible, because $E_{ij} = E_{ji}$ and so we can replace E_{ij} and E_{ji} in ψ with $(E_{ij} + E_{ji})/2$.) By elementary mathematical manipulations we eliminate r from the inequality of Clausius-Duhem (16) to obtain

$$-\rho \frac{d\psi}{dt} - \rho\, \eta \frac{d\theta}{dt} + \sigma_{ij}\, d_{ij} - \frac{q_i\, \theta_{,i}}{\theta} \geq 0. \tag{17}$$

Substituting ψ in inequality (17), we get

$$-\rho\,\frac{\partial\psi}{\partial E_{ij}}\,\frac{\partial E_{ij}}{\partial t} - \rho\,\frac{\partial\psi}{\partial\theta}\,\frac{d\theta}{dt} - \rho\,\frac{\partial\psi}{\partial G_i}\,\frac{dG_i}{dt} - \rho\,\eta\,\frac{d\theta}{dt} + \sigma_{ij}\,d_{ij} - \frac{q_i\,\theta_{,i}}{\theta} \geq 0. \qquad (18)$$

Let us now do the calculation

$$\begin{aligned}\frac{\partial E_{ij}}{\partial t} &= \frac{1}{2}\,\frac{\partial}{\partial t}\Big(\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_k}{\partial X_j} - \delta_{ij}\Big) = \frac{1}{2}\Big(\frac{\partial v_l}{\partial X_i}\,\frac{\partial x_l}{\partial X_j} + \frac{\partial x_k}{\partial X_i}\,\frac{\partial v_k}{\partial X_j}\Big)\\ &= \frac{1}{2}\Big(\frac{\partial v_l}{\partial x_k}\,\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_l}{\partial X_j} + \frac{\partial x_k}{\partial X_i}\,\frac{\partial v_k}{\partial x_l}\,\frac{\partial x_l}{\partial X_j}\Big) = d_{kl}\,\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_l}{\partial X_j} = \frac{d\varepsilon_{kl}}{dt}\,\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_l}{\partial X_j}.\end{aligned}$$

We substitute this result in the last inequality (18) to obtain

$$\Big(\sigma_{kl} - \rho\,\frac{\partial\psi}{\partial E_{ij}}\,\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_l}{\partial X_j}\Big)\frac{d\varepsilon_{kl}}{dt} - \rho\Big(\eta + \frac{\partial\psi}{\partial\theta}\Big)\frac{d\theta}{dt} - \rho\,\frac{\partial\psi}{\partial G_i}\,\frac{dG_i}{dt} - \frac{q_i\,\theta_{,i}}{\theta} \geq 0. \quad (19)$$

The inequality (19) is linear with respect to the three variables $d\varepsilon_{kl}/dt$, $d\theta/dt$ and dG_i/dt with coefficients which do not depend on them. Because $d\varepsilon_{kl}/dt$, $d\theta/dt$ and dG_i/dt are independent of each other (since u, θ and their gradients at an arbitrary point are independent variables), it follows that a necessary and sufficient condition for inequality (19) to hold is that the coefficients of these three variables are zeros. Thus,

$$\sigma_{kl} = \rho\,\frac{\partial\psi}{\partial E_{ij}}\,\frac{\partial x_k}{\partial X_i}\,\frac{\partial x_l}{\partial X_j}\;, \qquad \eta = -\frac{\partial\psi}{\partial\theta}\;, \qquad \frac{\partial\psi}{\partial G_i} = 0\;, \qquad q_i\,\theta_{,i} \leq 0.$$

Hence ψ does not depend on G_i, i.e. $\psi = \psi(\mathbf{E},\theta,\mathbf{X})$. Traditionally, the left hand side of the inequality $q_i\,\theta_{,i} \leq 0$ is written as

$$q_i\,\theta_{,i} = Q_j\,X_{j,i}\,\frac{\partial\theta}{\partial X_k}\,X_{k,i} = Q_j(\mathbf{E},\theta,\mathbf{G},\mathbf{X})\,G_k\,X_{j,i}\,X_{k,i}.$$

Let us summarize what we have accomplished in this section. To the original system of 5 differential equations for the 16 unknown functions, stated in the beginning of the section, we added 7 new unknowns (E_{ij} and ψ) and their defining equations

$$E_{ij} \equiv \frac{1}{2}\Big(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\,\frac{\partial u_k}{\partial X_j}\Big), \qquad \psi \equiv \epsilon - \eta\,\theta,$$

and also added 7 equations – for σ_{ij} and η. So we have a total of 19 equations for 23 unknowns and the inequality $q_i\,\theta_{,i} \leq 0$. Thus, we need 4 more equations. These are the equations that specify the nature of the free energy $\psi = \psi(\mathbf{E},\theta,\mathbf{X})$ and that of the heat flow $\mathbf{q} = \mathbf{q}(\mathbf{E},\theta,\mathbf{G},\mathbf{X})$.

For historical references on the constitutive equations of continuous media we refer the reader to Truesdell and Toupin [63].

9. VISCOELASTIC MEDIUM

We are interested in deriving the equations of motion of a viscoelastic medium. We use a “dot” above a letter to denote the time derivative of the variable.

Let us assume that the continuous medium we consider has a constant density ρ, constant temperature θ and constant entropy η. Let us also assume that the stresses depend not only on the deformations, but also on the time derivatives of the deformations, namely that

$$\Sigma_{ij} = \Sigma_{ij}(\mathbf{E}, \dot{\mathbf{E}}, \mathbf{X}).$$

The specific internal energy ϵ depends on the same variables.

Assuming that the deformations are small, the formula

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial E_{kl}} \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_l}$$

derived above, which is valid for any continuous medium even in the case of large deformations and with no restrictions on the form that the free energy ψ, acquires the form

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}}.$$

Let us assume that the free energy ψ is a quadratic function of the deformations and their time derivatives, namely,

$$\rho\psi = a + \alpha_{ij}\varepsilon_{ij} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \beta_{ij}\dot{\varepsilon}_{ij} + \frac{1}{2}\beta_{ijkl}\varepsilon_{ij}\dot{\varepsilon}_{kl} + \frac{1}{2}\gamma_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl}.$$

Thus we arrive at the system of equations which an elastic medium with viscosity, a constant density ρ, constant temperature θ and constant entropy η obeys:

$$\begin{aligned} 2\varepsilon_{ij} &= u_{i,j} + u_{j,i} \\ \sigma_{ij,j} + \rho f_i &= \rho \ddot{u}_i \,, \qquad i = 1, 2, 3 \\ \sigma_{ij} &= \rho \frac{\partial \psi}{\partial \varepsilon_{ij}}. \end{aligned}$$

Let us now calculate σ_{ij} by differentiating ψ with respect to the deformations. We obtain

$$\sigma_{ij} = \alpha_{ij} + c_{ijkl}\varepsilon_{kl} + \beta_{ijkl}\dot{\varepsilon}_{kl}.$$

If there are no stresses in a nondeformed state, $\alpha_{ij} = 0$, so

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} + \beta_{ijkl}\dot{\varepsilon}_{kl}.$$

Then,

$$\sigma_{ij,j} = c_{ijkl}\frac{\partial \varepsilon_{kl}}{\partial x_j} + \beta_{ijkl}\frac{\partial \dot{\varepsilon}_{kl}}{\partial x_j}$$

$$= c_{ijkl}\frac{1}{2}\frac{\partial}{\partial x_j}(u_{k,l} + u_{l,k}) + \beta_{ijkl}\frac{1}{2}\frac{\partial}{\partial x_j}(\dot{u}_{k,l} + \dot{u}_{l,k})$$

$$= c_{ijkl}\frac{1}{2}(u_{k,lj} + u_{l,kj}) + \beta_{ijkl}\frac{1}{2}(\dot{u}_{k,lj} + \dot{u}_{l,kj}).$$

Thus, the equations of motion of a viscoelastic medium with a constant density, constant temperature and constant entropy are:

$$c_{ijkl}\frac{1}{2}(u_{k,lj} + u_{l,kj}) + \beta_{ijkl}\frac{1}{2}(\dot{u}_{k,lj} + \dot{u}_{l,kj}) + \rho f_i = \rho \ddot{u}_i \qquad i = 1, 2, 3.$$

In the one-dimensional case these equations become the single equations for the displacement $u = u(x, t)$

$$c\, u_{xx} + \beta\, \dot{u}_{xx} + \rho f = \rho\, \ddot{u}\ .$$

This equation can also be written as

$$u_{tt} - \frac{\beta}{\rho}\, u_{xxt} - \frac{c}{\rho}\, u_{xx} - f = 0,$$

where $f = f(x, t)$ is given and ρ, β and c are known constants.

10. LINEAR THERMOELASTIC MEDIUM

In this section we will reach our ultimate goal – to derive the system of 20 PDEs, for 20 unknown functions, that governs the motion of a continuous medium.

Let us get started by rewriting the general law of conservation of energy (15) in a simpler form. For this, substitute in it $\epsilon = \psi + \eta\,\theta$ and use $\psi = \psi(\mathbf{E}, \theta, \mathbf{X})$. Then the law acquires the form

$$\rho\left(\frac{\partial\psi}{\partial E_{kl}}\,\frac{\partial E_{kl}}{\partial t} + \frac{\partial\psi}{\partial\theta}\,\frac{d\theta}{dt} + \frac{d\eta}{dt}\theta + \eta\,\frac{d\theta}{dt}\right) = \sigma_{ij}\, d_{ij} - q_{i,i} + \rho\, r$$

and with the help of

$$\frac{\partial E_{ij}}{\partial t} = d_{kl}\,\frac{\partial x_k}{\partial X_i}\frac{\partial x_l}{\partial X_j}$$

it becomes

$$\rho\left(\frac{\partial\psi}{\partial E_{kl}}\,\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_l}\, d_{ij} + \frac{\partial\psi}{\partial\theta}\,\frac{d\theta}{dt} + \frac{d\eta}{dt}\,\theta + \eta\,\frac{d\theta}{dt}\right) = \sigma_{ij}\, d_{ij} - q_{i,i} + \rho\, r\ . \tag{20}$$

Now substitute σ_{ij} and η with

$$\sigma_{ij} = \rho\frac{\partial\psi}{\partial E_{kl}}\,\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_l}\ , \qquad \eta = -\frac{\partial\psi}{\partial\theta}$$

and 4 terms in the above law (20) cancel out. The law of conservation of energy becomes

$$\rho\,\theta\,\frac{d\eta}{dt} + q_{i,i} = \rho\, r\,. \tag{21}$$

A linear thermoelastic homogeneous medium is one for which the following assumptions hold:

1. The deformations are small, so the product of the gradients of the displacement are ignored. Also we substitute E_{ij} with ε_{ij};

2. The mass density ρ does not change during the deformation process;

3. The free energy ψ is a quadratic function of the components ε_{ij} and of the temperature change $T = \theta - T_0$. Also $|T|/T_0$ is small with respect to 1, thus $\theta \approx T_0$.

4. The components of the heat flow $\mathbf{q}$ are linear functions of ε_{ij}, T and $T_{,i}$.

With these assumptions the gradient of the temperature becomes

$$G_i = \frac{\partial\theta}{\partial X_i} = \frac{\partial T}{\partial X_i} = \frac{\partial T}{\partial x_j}\,\frac{\partial x_j}{\partial X_i} = \frac{\partial T}{\partial x_j}\left(\delta_{ij} + \frac{\partial u_j}{\partial X_i}\right) = \frac{\partial T}{\partial x_i} + \frac{\partial T}{\partial x_j}\,\frac{\partial u_j}{\partial X_i}\,,$$

where $u_j = x_j – X_j$. We ignore the product of the gradients, and obtain $G_i = \partial T/\partial x_i$. In the calculations that follow we will substitute Q_i with q_i, because $q_i = Q_j\,X_{j,i} = (\delta_{ij} - u_{j,i})\,Q_j = Q_i - Q_j\,u_{j,i}$ and we ignore $Q_j\,u_{j,i}$ in the presence of Q_i.

To find the form of the functions ψ and q_i we develop them in Taylor series around their undeformed values, which are 0's. In the series for ψ we will keep terms up to and including second order, and in the series for q_i we will keep only the linear terms:

$$\rho\,\psi = a - \rho\,\eta_0\,T - \frac{c_\varepsilon}{2T_0}\,T^2 + \alpha_{ij}\,\varepsilon_{ij} - \chi_{ij}\,\varepsilon_{ij}\,T + \frac{1}{2}c_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}\,,$$

$$q_i = a_i + b_i\,T - k_{ij}\,T_{,j} + d_{ijk}\,\varepsilon_{jk}\,.$$

In these Taylor expansions the constants will be determined by the calculations that follow. Because of the requirement that ψ is symmetric with respect to the components ε_{ij} and ε_{ji} of the strain tensor, we have the following relations among the constants in its Taylor polynomial: $\alpha_{ij} = \alpha_{ji},\ \chi_{ij} = \chi_{ji},\ c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$.

A short calculation shows that in the theory of small deformations

$$\sigma_{ij} = \rho\,\partial\psi/\partial\varepsilon_{ij}\,. \tag{22}$$

We also remember from the previous section that $\eta = -\partial\psi/\partial\theta$. So

$$\eta = -\partial\psi/\partial\theta = -(\partial\psi/\partial T)(\partial T/\partial\theta) = -\partial\psi/\partial T. \tag{23}$$

Substitute the Taylor expansion for ψ in the last formulae for σ_{ij} and η to get

$$\sigma_{ij} = \rho\,\frac{\partial\psi}{\partial\varepsilon_{ij}} = \alpha_{ij} - \chi_{ij}\,T + c_{ijkl}\,\varepsilon_{kl}\,,$$

$$\rho\,\eta = -\frac{\partial(\rho\,\psi)}{\partial T} = \rho\,\eta_0 + \frac{c_\varepsilon}{T_0}\,T + \chi_{ij}\,\varepsilon_{ij}.$$

We assume that when there is no deformation, i.e. $\varepsilon_{ij} = 0,\ T = 0$, there are no stresses, so $\sigma_{ij} = \alpha_{ij} = 0$. Also, $Q_j(\mathbf{E}, \theta, \mathbf{0}, \mathbf{X}) = 0$. Substituting 0 for Q_j in $q_i = Q_i - Q_j\,u_{j,i}$, we get $q_i(\varepsilon_{kl}, T, T_{,k})\big|_{T_{,1}=T_{,2}=T_{,3}=0} = 0$. Thus, when there are no deformations, $T = 0$ and $q_i = 0$, and we obtain the following equation which relates the constants in the Taylor expansion for q_i, namely $0 = a_i + b_i\,T + d_{ijk}\,\varepsilon_{jk}$. But 1, T and ε_{jk} are linearly independent functions, so from this equation we conclude that the coefficients of these three linearly independent functions are zeros, i.e. $a_i = b_i = d_{ijk} = 0$. Substituting these constants in the Taylor expansion for q_i, we get $q_i = -k_{ij}\,T_{,j}$. With this expression for q_i the inequality $q_i\,\theta_{,i} \le 0$ becomes $k_{ij}\,T_{,j}\,T_{,i} \ge 0$.

Thus, we arrive at the system of partial differential equations that every (linear) **continuous medium** obeys:

$\sigma_{ij,j} + \rho\,f_i = \rho\,\ddot{u}_i$	equations of motion
$\rho\,T_0\,\dfrac{\partial\eta}{\partial t} + q_{i,i} = \rho\,r$	law of conservation of energy
$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - \chi_{ij}\,T$	constitutive equations for the stress tensor
$\rho\,\eta = \rho\,\eta_0 + \dfrac{c_\varepsilon}{T_0}\,T + \chi_{ij}\,\varepsilon_{ij}$	constitutive equation for the entropy
$q_i = -k_{ij}\,T_{,j}$	constitutive equation for the heat flow
$\varepsilon_{ij} = \dfrac{1}{2}\Big(u_{i,j} + u_{j,i}\Big)$	equations of strain.

These are 20 equations for the 20 unknown functions $\sigma_{ij}\ , u_i,\ q_i,\ \varepsilon_{ij}\ , T,\ \eta$. The mass density ρ does not change during the deformation process, so ρ coincides with the initial mass density which we consider known. If we substitute the expressions for $\sigma_{ij}\ , q_i,\ \varepsilon_{ij}, \eta$ from the last 4 lines of this system in the first two lines - the equations of motion and the law of conservation of energy, we obtain the equations

$c_{ijkl}\,u_{k,jl} - \chi_{ij}\,T_{,j} + \rho\,f_i = \rho\,\ddot{u}_i\ , \quad i = 1,2,3$	equations of motion
$k_{ij}\,T_{,ij} - c_\varepsilon\,\dfrac{\partial T}{\partial t} - \chi_{ij}\,T_0\,\dfrac{\partial u_{i,j}}{\partial t} + \rho\,r = 0$	equation of thermocondactivity

for the unknown functions $u_i,\ T$. These 4 equations are valid for any **thermoelastic anisotropic medium**, that is a medium with different mechanical and thermal properties in different directions. Some crystals are examples of such media. For isotropic media the constants in the constitutive equations remain unchanged under rotation of the body. Hence for such a medium $\chi_{ij} = \chi\,\delta_{ij}$, $k_{ij} = k\,\delta_{ij}$, $c_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,\delta_{ik}\delta_{jl} + \nu\,\delta_{il}\delta_{jk}$. From the symmetries $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ it is clear that $\mu = \nu$, and consequently $c_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$.

The constants λ and μ are called constants of Lamè. Thus, the equations of a **thermoelastic isotropic medium** are

$$(\lambda+\mu)\, u_{j,ji} + \mu\, u_{i,jj} - \chi\, T_{,i} + \rho\, f_i = \rho\, \ddot{u}_i\,,$$
$$k\, T_{,ii} - c_\varepsilon \frac{\partial T}{\partial t} - \chi\, T_0 \frac{\partial u_{i,i}}{\partial t} + \rho\, r = 0\,.$$

The system of the general equations of linear elasticity in the case of absence of thermal effects was first derived by Navier [55] in 1821.

The system of 20 differential equations above or equivalently the system of 4 equations for thermoelastic anisotropic medium can be solved with suitable initial and boundary conditions. If the system of PDEs in question has a solution (u_1, u_2, u_3), it is given by the formula (7) of Cesaro. This solution is unique, provided that $c_\varepsilon > 0$ and the quadratic form $c_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}$ is positive definite. The fact that the solution (7) of Cesaro satisfies the whole system is demonstrated by a direct substitution in the equations. The proof of uniqueness uses an identity, relating the variables involved in the system of PDEs. It is delightfully elegant and surprisingly short, see Ivanov [49] or Sokolnikoff [58].

11. TWO PROBLEMS

In this section we consider a couple of concrete problems.

Problem 1. Let us first consider an elastic body undergoing spherically symmetric deformation. Then the displacement vector is of the form

$$\mathbf{u} = u(r)\mathbf{e}_r, \qquad r \neq 0$$

where $\mathbf{e}_r$ is the unit vector along the radial direction. For such a displacement, compute (i) the corresponding stress components, (ii) the normal stress on a spherical surface $r = constant$ and (iii) the normal stress on a radial plane. Then determine $u(r)$ so that Navier's equation of equilibrium with zero body force is satisfied.

Solution. (i) The given form of the displacement vector can be rewritten as

$$\mathbf{u} = u(r)\frac{1}{r}\mathbf{x} = \phi(r)\mathbf{x}\,,$$

where

$$\phi(r) = \frac{1}{r}u(r).$$

From this we find that $u_i = \phi(r)x_i$, so that

$$u_{i,j} = \phi(r)\delta_{ij} + \phi'(r)\Big(\frac{1}{r}x_j\Big)x_i = u_{j,i}.$$

Hence

$$u_{k,k} = 3\phi(r) + r\phi'(r).$$

Let us now substitute these last two results in the stress–displacement relation

$$\sigma = \lambda(\text{div}\mathbf{u})\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

and make use of the fact that $\phi(r) = (1/r)u(r)$. We obtain the following expression for the stresses associated with the given displacement field:

$$\sigma_{ij} = 2\Big((\lambda+\mu)\delta_{ij} - 2\mu\frac{1}{r}x_i x_j\Big)\frac{1}{r}u(r) + \Big(\lambda\delta_{ij} + 2\mu\frac{1}{r^2}x_i x_j\Big)u'(r).$$

(ii) For a spherical surface $r = constant$, we have $\mathbf{n} = \mathbf{e}_r$, so that $n_i = x_i/r$. Hence, by the formula

$$\sigma_N = \sigma_{ik} n_i n_k\,,$$

enabling us to determine the normal stress σ_N directly from the stress components σ_{ik}, the normal stress σ_r on this surface is given by $\sigma_r = \sigma_{ij} n_i n_j = (\sigma_{ij} x_i x_j)/r^2$. Using the expression for σ_{ij} obtained in part (i) of this problem, we get

$$\sigma_r = 2\lambda\frac{1}{r}u(r) + (\lambda + 2\mu)u'(r).$$

This normal stress is the radial stress.

(iii) If $\mathbf{n}$ is the unit normal to a radial plane, we have $\mathbf{n}\cdot\mathbf{e}_r = 0$, and the normal stress σ_N on the plane is given by $\sigma_N = \sigma_{ij} n_i n_j$. Another use of the expression for σ_{ij} obtained in part (i), we arrive at the following expression for the normal stress:

$$\sigma_\varphi = 2(\lambda+\mu)\frac{1}{r}u(r) + \lambda u'(r).$$

This normal stress is the peripheral stress.

(iv) Finally, to determine $u(r)$, we return to the expressions for $u_{i,j}$ and $u_{k,k}$ obtained in part (i) of the problem and calculate that

$$u_{i,ij} = u_{k,ki} = \Big(\phi''(r) + \frac{4}{r}\phi'(r)\Big)x_i.$$

Substituting these into Navier's equation of equilibrium

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{k,ki} + f_i = 0,$$

with $f_i = 0$, we see that it is satisfied if $\phi(r)$ obeys the following differential equation:

$$\frac{d^2\phi}{dr^2} + \frac{4}{r}\frac{d\phi}{dr} = 0\,.$$

The general solution of this equation is

$$\phi(r) = \frac{A}{r^3} + B\,,$$

where A and B are arbitrary constants. Thus,

$$u(r) = \frac{A}{r^2} + Br\,,$$

which is the sought solution of Navier's equation of equilibrium with zero body force.

The interested reader is invited to apply the ideas demonstrated in the above problem to solve the following

Problem 2. An elastic body undergoes a deformation, which is symmetric about the x_3 axes. Then the displacement vector is of the form

$$\mathbf{u} = u(R)\mathbf{e}_R, \qquad R \neq 0\,,$$

where $R^2 = x_1^2 + x_2^2$ and $\mathbf{e}_R$ is the unit vector along the radial direction in the cylindrical polar coordinate system with x_3 axis as axis. For this displacement compute (i) the corresponding stress components; (ii) the normal stress on a cylindrical surface $R = constant$; and (iii) the normal stress on a plane containing the x_3 axis. Also, determine $u(R)$ such that the Navier's equation of equilibrium with zero body force is satisfied.

12. THE CONTEMPORARY APPLICATIONS

In many applications the analytic solution (7) of Cesaro, to the system which a continuous medium obeys, can be obtained. Examples of such applications are the elongation, the twisting and the bending of cylindrical elastic beams; the stretching of a beam by its own weight; the twisting of a rectangular beam by two pairs of forces applied at each end of the beam; the twisting of circular cylinder with one base fixed and the other subjected to a pair of forces creating a torque; the displacement of a bended beam; and many others. Some 2-dimensional problems, like the displacement of an elastic membrane, subjected to uniform pressure from one side, have analytic solutions that use harmonic functions. The solution for the twisting of hollow, tube-like, beams also uses harmonic functions. The solution for the twisting of a cylinder by forces applied to its surface, and that for the bending of a tube with a circular or an elliptical cross-section, uses conformal maps. Most of these problems, solved in all detail, can be found in Sokolnikoff [58].

During the mid-1950s and 1960s the computer started to become a major tool for solving problems in continuum mechanics. At first the finite difference methods and the Rayleigh-Ritz method (using the theorem of minimal potential theory), were employed. Both of these methods required the solution of large numbers of simultaneous equations and faced the danger of the system becoming ill-conditioned as the number of equations increased. Finite difference methods have a long history, including contributions by Newton, Laplace, Gauss, Bessel and others. The method of finite differences replaces the defining differential equation with equivalent difference equations. The boundary conditions are satisfied at discrete points by specifying either the function or its derivatives. The result of this analysis are numerical values of the function at discrete points throughout the body.

Computer simulations of exploding stars, the expansion of the early Universe, and the evolution of nebulae are so unbelievably realistic, only because they obey the equations of continuum mechanics. May be less dramatic, but significant from an applied point of view, is the fact that the flow of water or the spilling of oil can be modeled with the system for the motion of that continuous medium, and be presented visually in real time.

Modern cosmological simulations following the evolution of large portions of the Universe use numerical methods from hydrodynamics, more specifically the numerical solutions of the equations of compressible fluids. Simulations of merging clusters of gallaxes are made this way. More specifically, the equations of motion for a compressible fluid are solved using a Lagrangian formulation in which the fluid is partitioned into elements, a subset of which is represented by particles of known mass and specific energy. Continuous fields are represented by interpolating between particles using a smoothing kernel, which is normally defined in terms of a sphere containing a fixed number of neighbors, centered on the particle in question. This method uses an artificial viscosity.

Continuum mechanics has become a fundamental science in investigations in tissue biomechanics. Soft tissue constitutive equations have been developed and the stresses and strains are being calculated for skin, tendon, ligament and bone. As new materials are being developed, they are being modeled as a continuum. Continuum mechanics is also being used in nanotechnology even on that small of a scale.

The most prominent relevant texts in Russian are listed as references [65] – [68].

Making an exhaustive list of the contemporary applications of Continuum Mechanics is impossible, as the subject is vast, vibrant, and multidisciplinary and develops literary every day. New branches of the subject are the nonlinear theory of elasticity, relativistic continuum mechanics and computational fluid dynamics. In recent years it has found connections with biomechanics and nanomechanics. A few of the most recent applications of continuum mechanics are: memory effects, the qualitative studies of the equations of Navier-Stokes, cross-diffusion systems from biology and physics, the decay of acceleration waves, and the fluid animation implementing numerical solutions to the 3D Navier-Stokes equations.

ACKNOWLEDGEMENTS. I am indebted to Professor Tsolo Ivanov, Professor Emeritus at the Department of Mechatronics, Robotics and Mechanics, Sofia University "St. Kliment Ohridski", Bulgaria, for valuable conversations in continuum mechanics.

13. REFERENCES

[1] Bernoulli, D.: Hydrodynamica sive de Viribus. Motibus Fluidorum Commentarii. *Argentorati.* (1738).

[2] Carnot, S.: Reflexions sur la Puissance Motrice du Few et sur les Machines Propres a Developper cette Puissance. Paris (1824) = *Ann. Ècole Norm* (2) **1** (1872), 393–457.

[3] Cauchy, A.-L.: Theorie de la propagation des ondes a la surface d'un fluide pesant d'une profondeur indefinie (1815). *Mem. divers savants* (2) **1** (1816), 3-312 = *Oeuvres* (1) **1**, 5–318.

[4] Cauchy, A.-L.: Recherches sur l'èquilibre et le mouvement intèrieur des corps solides ou fluides, èlastiques ou non èlastiques. *Bull. Soc. Philomath*, 9–13 (1823) = *Oeuvres* (2) **2**, 300–304.

[5] Cauchy, A.-L.: Memoire sur une espece particuliere de mouvement des fluides. *J. Ecole Polytech.* **12** (1823), cahier 19, 204–214 =*Oeuvres* (2) **1**, 264–274.

[6] Cauchy, A.-L.: De la pression ou tension dans un corps solide. *Ex. de math.* **2** (1827), 42–56 = *Oeuvres* (2) **7**, 60–78.

[7] Cauchy, A.-L.: Sur la condensation et la dilatation des corps solides. *Ex. de math.* **2** (1827), 60–69 = *Oeuvres* (2) **7**, 82–83.

[8] Cauchy, A.-L.: Sur les moments d'inertie. *Ex. de math.* **2** (1827), 93–103 = *Oeuvres* (2) **7**, 124–136.

[9] Cauchy, A.-L.: Sur les relations qui existent dans l'etat d'equlibre d'un corps solide ou fluide, entre les pressions ou tensions et les forces acceleratrices. *Ex. de math.* **2** (1827), 108–111 = *Oeuvres* (2) **7**, 141–145.

[10] Cauchy, A.-L.: Sur les centers, les plans principaux et les axes principaux des surfaces du second degre. *Ex. de math.* **3** (1828), 1–22 = *Oeuvres* (2) **8**, 9–35.

[11] Cauchy, A.-L.: Sur les equations qui expriment les conditions d'equlibre, ou les lois du mouvement interieur d'un corps solide, elastique, ou non elastique. *Ex. de math.* **3** (1828), 160–187 = *Oeuvres* (2) **8**, 195–226.

[12] Cauchy, A.-L.: Sur quelques theoremes relatifs a la condensation ou a la dilatation des corps. *Ex. de math.* **3** (1828), 237–244 = *Oeuvres* (2) **8** (1828), 278–287.

[13] Cauchy, A.-L.: Sur les pressions ou tensions supportees en un point donne d'un corps solide par trois plans perpendiculaires entre eux. *Ex. de math.* **4** (1829), 30–40 = *Oeuvres* (2) **9**, 41–52.

[14] Cauchy, A.-L.: Sur la relation qui existe entre les pressions ou tensions supportees par deux plans quelconques en un point donne d'un corps solide. *Ex. de math.* **4** (1829), 41–46 = *Oeuvres* (2) **9**, 53–55.

[15] Cauchy, A.-L.: Sur les corps solides ou fluides dans lesquels la condensation ou dilatation lineaire est la meme en tous sens autour de chaque point. *Ex. de math.* **4** (1829), 214–216 = *Oeuvres* (2) **9**, 254–258.

[16] Cauchy, A.-L.: Sur l'èquilibre et le mouvement interieur des corps consideres comme des masses continues. *Ex. de math.* **4** (1829), 293–319 = *Oeuvres* (2) **9**, 243–369.

[17] Cauchy, A.-L.: Sur les diverses methodes a l'aide desquelles on peut etablir les equations qui representent les lois d'equlibre, ou le mouvement interieur des corps solides ou fluides. *Bull. sci. math. soc. prop. conn.* **13** (1830), 169–176.

[18] Cauchy, A.-L.: Mèmoire sur les dilatations, les condensations et les rotations produits par un changement de forme dans un système de points matèriels. *Ex. d'an. phys. math.* **2**, 302–330 (1841) = *Oeuvres* (2) **12**, 343–377.

[19] Cesaro, E.: Rendiconto dell' Accademia delle scienze fisiche e matematiche (Società reale di Napoli) (1906).

[20] Clausius, R.: Ueber eine veranderte Form des zweiten Hauptsatzes der mechanischen Warmetheorie. *Ann. Physik* **93** (1854), 481–506.

[21] Clausius, R.: Uber verschiedene fur die Anwendung bequeme Formen der Hauptgleichungen der mechanischen Warmetheorie. *Vjschr. nat. Ges. Zurich* **10** (1865), 1–59.

[22] D'Alembert, J. L.: *Traite de l'Equilibre et du Mouvement des Fluides pour servir de Suite au Traite de Dynamique.* Paris (1749); 2nd ed., 1770.

[23] Duchem, P.: *Hydrodynamique, Èlactisitè, Acoustique.* Paris. Hermann, Vol. I, II (1891).

[24] Duchem, P.: Commentaire aux principes de la thermodynamique, Première partie. *J. math pures appl.* (4) **8** (1892), 269–330.

[25] Euler, L.: De motu corporum in superficiebus mobilibus. *Ousc. var. arg.* **1** (1746), 1–136 = *Opera omnia* (2) **6**, 75–174.

[26] Euler, L.: Decouverte d'un nouveau principe de mecanique. *Mem. Acad. Sci. Berlin* **6** (1750), 185–217 = *Opera omnia* (2) **5**, 81–108.

[27] Euler, L.: Recherches sur l'effet d'une machine hydraulique proposee par Mr. Segner Professeur a Gottingue. *Mem. Acad. Sci. Berlin* **6** (1750), 311–354 = *Opera omnia* (2) **15**, 1–39.

[28] Euler, L.: Recherche sur une nouvelle maniere d'elever de l'eau proposee par Mr. De Mour. *Mem. Acad. Sci. Berlin* **7** (1751), 305–330 = *Opera omnia* (2) **15**, 134–156.

[29] Euler, L.: Theorie plus complette des machines qui sont mises en mouvement par la reaction de l'eau. *Mem. Acad. Sci. Berlin* **10** (1754), 227–295 = *Opera omnia* (2) **15**, 157–218.

[30] Euler, L.: 1757 Principes gènèraux de l'etat d'equilibre des fluides. *Mèm. Acad. Sci. Berlin* **11**, 217–273 (1755) = *Opera omnia* (2) **12**, 2–53.

[31] Euler, L.: 1757 Principes gènèraux du mouvement des fluides. *Mèm. Acad. Sci. Berlin* **11**, 274–315 (1755) = *Opera omnia* (2) **12**, 54–91.

[32] Euler, L.: Continuation des recherches sur la theorie du mouvement des fluides. *Mèm. Acad. Sci. Berlin* **11** (1755), 316–361 = *Opera omnia* (2) **12**, 92–132.

[33] Euler, L.: 1761 Principia motus fluidorum (1752-1755). *Novi Comm. Acad. Sci. Petrop.* **6** (1756-1757), 271–311 = *Opera omnia* (2) **12**, 133–168.

[34] Euler, L.: Lettre de M. Euler a M. La Grange, Recherches sur la propagation des branlemens dans un milieu elastique. *Misc. Taur.* $\mathbf{2}^2$ (1760-1761), 1–10 = *Opera omnia* (2) **10**, 255–263.

[35] Euler, L.: De motu fluidorum a diverso caloris gradu oriundo. *Novi Comm. Acad. Sci. Petrop.* **11** (1765), 232–267 = *Opera omnia* (2) **12**, 244–271.

[36] Euler, L.: Recherches sur la connaissance mecanique des corps. *Mem. Acad. Sci Berlin* **14** (1758), 131–153.

[37] Euler, L.: Du mouvement de rotation des corps solides autour d'un axe variable. *Mem. Acad. Sci Berlin* **14** (1758), 154–193.

[38] Euler, L.: Supplement aux recherches sur la propagation du son. *Mem. Acad. Sci Berlin* **15** (1759), 210–240 = *Opera omnia* (3) **1**, 452–483.

[39] Euler, L.: Recherche sur le mouvement des rivieres (1751). *Mem. Acad. Sci Berlin* **16]** (1760), 101–118 = *Opera omnia* (2) **12**, 212–288.

[40] Euler, L.: Theoria Motus Corporum Solidorum seu Rigidorum ex Primis nostrae Cognitionis Principiis Stabilita et ad Omnis Motus, qui in hujusmodi Corpora Cadere Possunt, *Accomodata* (1765), Rostock = *Opera omnia* (2) **3, 4**, 3–293.

[41] Euler, L.: Sectio secunda de principiis motus fluidorum. *Novi Comm. Acad. Sci. Petrop.* **14** (1769), 270–386 = *Opera omnia* (2) **13**, 73–153.

[42] Euler, L.: De gemina methodo tam aequilibrium quam motus corporum flexibilium determinandi et utriusque egregio consensu. *Novi Comm. Acad. Sci. Petrop.* **20**, 286–303 (1775).

[43] Euler, L.: Formulae generales pro translatione quacunque corporum rigidorum. *Novi Comm. Acad. Sci. Petrop.* **20** (1775), 189–207.

[44] Fresnel, A.: Second supplèment au mèmoire sur la double rèfraction. *Oeuvres* **2** (1822), 369-442.

[45] Georgieva, B.: Review of continuum mechanics and its history. Part II. The mechanics of thermoelastic media. Perfect fluids. Linearly viscous fluids. *Ann. Sofia Univ., Fac Math and Inf.*, **107** (2020), 57–79.

[46] Green, G.: On the laws of reflection and refraction of light at the common surface of two non-crystalized media. *Trans. Cambridge Phil. Soc.* **7** (1839), 1–24.

[47] Green, G.: On the propagation of light in crystalized media. *Trans. Cambridge Phil. Soc.* **7** (1841), 121–140.

[48] Helmholtz, H.: Uber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math.* **55** (1858), 25–55.

[49] Ivanov, Ts.: *Theory of Elasticity.* Sofia University Publishing House (1995).

[50] Joule, J. P.: On the caloric effects of magneto-electricity, and on the mechanical value of heat. *Phil. Mag.* (3) **23** (1843), 263–276, 347–355, 435–443.

[51] Joule, J. P.: On the existence of an equivalent relation between heat and the ordinary forms of mechanical power. *Phil. Mag.* (3) **27** (1845), 205–207.

[52] Kirchhoff, G.: Uber das Gleichgewicht und die Bewegung eines unendlich dunnen elastischen Stabes. *J. Reine Angew. Math.* **56** (1859), 285–313.

[53] Lagrange, J. L.: Nouvelles recherches sur la nature et la propagation du son. *Misc. Taur.* $\mathbf{2}^2$ (1760-1761), 11–172 = *Oeuvres* **1** (1762), 151–316.

[54] Lagrange, J. L.: *Mechanique Analitique*, Paris, *Oeuvres* **11, 12** (1788).

[55] Navier, C. L. M. H: Sur les lois des mouvements des fluides, en ayant egard a l'adhesion des molequles. *Ann. Chimie* **19** (1821), 244–260.

[56] Rankine, W. J. M.: Laws of the elasticity of solid bodies. *Cambr. Dubl. Math. J.* **6** (1851), 41–80, 178–181.

[57] Rankine, W. J. M.: On axes of elasticity and crystalline forms. *Phil. Trans. Roy. Soc. Lond.* **46** (1856), 261–285.

[58] Sokolnikoff, I. S.: *Mathematical Theory of Elasticity.* McGraw-Hill Book Company Inc., 2nd Ed. (1956).

[59] St. Venant, A.-J.-C. B.: Sur les pressions qui se dèveloppent a l'intèrieur des corps solides lorsque les dèplacements de leurs points, sans altèrer l'èlasticitè, ne peuvent cependant pas être considèrès comme très petits. *Bull. Soc. Philomath.* **5** (1844), 26–28.

[60] St. Venant, A.-J.-C. B.: Theorie de l'elasticite des solides, ou cinematique de leurs deformations. *L'Institut* **32**[1] (1864), 389–390.

[61] Stokes, G. G.: On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Trans. Cambridge Phil. Soc.* **8** (1844–1849), 287–319.

[62] Voltera, V.: L'Equilibre des corps èlastiques. *Annales de lècole normal supèrieure* **24** (1907), 416–417.

[63] Truesdell, C., Toupin, R.: *The Classical Field Theories.* Encyclopedia Of Physics **245** (S. Flugge, Ed.), Vol. III. Principles of Classical Mechanics and Field Theory. Springer-Verlag, Berlin (1960).

[64] Waterston, J. J.: Note on the physical constitution of gaseous fluids and a theory of heat. Appendix to *Thoughts on the Mental Functions.* Edinburgh (1843).

[65] Ландау, Л. Д., Лифшиц, Е. М.: *Теория упругости*, Наука, Москва, 1975.

[66] Лурье, А. И.: *Теория упругости*, Наука, Москва, 1970.

[67] Работнов, Ю. Н., *Механика твердого деформируемого тела*, Наука, Москва, 1988.

[68] Седов, Л. И.: *Механика сплошной среды*, том 1, Наука, Москва, 1970.

Received on November 20, 2020

Bogdana A. Georgieva

Faculty of Mathematics and Informatics
Dept. of Mechatronics, Robotics and Mechanics
Sofia University St. "Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: georgieva@fmi.uni-sofia.bg