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ON THE REGULARITY OF CERTAIN THREE-ROW ALMOST HERMITIAN INCIDENCE MATRICES

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In this note we present a new proof of the regularity of a class of three-row almost Hermitian matrices, based on some properties of Legendre polynomials.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

Throughout this paper, the notation π_n will stand for the set of algebraic polynomials of degree not exceeding n. In 1906 G. D. Birkhoff [2] formulated a general problem on interpolation by algebraic polynomials, which includes as particular cases the Lagrange and Hermite interpolation problems. Before formulating the Birkhoff interpolation problem (BIP), we need the following:

Definition 1. An *incidence matrix* $E=\{e_{ij}\}_{i=1,\,j=0}^{n,\ \ r}$ is a matrix with elements $e_{ij}\in\{0,1\}$. The number of 1-entries in E is denoted by $|E|$, and we shall assume always that E is a *normal incidence matrix*, i.e., $|E|=r+1$.

The Birkhoff interpolation problem (BIP). Given an incidence matrix $E=\{e_{ij}\}_{i=1,\,j=0}^{n,\ \ r}$, a vector of interpolation nodes $\mathbf{X}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$,

$x_1 < x_2 < \cdots < x_n$, and a data set $\{\gamma_{ij} \in \mathbb{C} : e_{ij} = 1\}$, find a polynomial $p \in \pi_{|E|-1}$ such that

$$p^{(j)}(x_i) = \gamma_{ij}, \qquad \{i,j\} : \ \ e_{ij} = 1\,. \tag{1.1}$$

It should be pointed out that, unlike the Lagrange and Hermite interpolation problems, which are known to have a unique solution, the general BIP is not always solvable.

Definition 2. An incidence matrix $E = \{e_{ij}\}_{i=1,\,j=0}^{n\ \ \,r}$ is said to be (*order*) *regular*, if for every vector of interpolation nodes $\mathbf{X} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x_1 < x_2 < \cdots < x_n$, and a data set $\{\gamma_{ij} \in \mathbb{C} : e_{ij} = 1\}$, the BIP (1.1) has a unique solution.

Surprisingly enough, despite the efforts of many mathematicians, the problem of complete characterization of the regular incidence matrices remains open. A simple necessary condition for regularity was found by Pólya.

Pólya condition. A necessary condition for $E = \{e_{ij}\}_{i=1,\,j=0}^{n\ \ \,r}$ to be regular is

$$\sum_{i=1}^{n}\sum_{j=0}^{k} e_{ij} \geq k+1\,, \qquad k = 0, \ldots, |E|-1\,. \tag{1.2}$$

In 1969 Atkinson and Sharma [1] found a simple sufficient condition for regularity. We need another definition before formulating their result.

Definition 3. *A block* is called any maximal sequence of 1-entries in a row of E. A block $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\ell-1} = 1$ is called *even*, resp. *odd*, if its length ℓ is even, resp. odd number. The smallest column index j of 1-entry in a block defines its *level*. *Hermitian block* is a block with level 0.

A row $\mathbf{e}_i = (e_{i,0}, e_{i,1}, \ldots, e_{ir})$ of E is called *Hermitian row of length k* if it contains a single block which is Hermitian with length k.

A block $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\ell-1} = 1$ in an interior row $\mathbf{e}_i$, $1 < i < n$, is called *supported*, if there are 1-entries in rows i_1 and i_2, $i_1 < i < i_2$ with column indices $j_1, j_2 < j$.

Atkinson–Sharma Theorem. *Every incidence matrix $E = \{e_{ij}\}_{i=1,\,j=0}^{n\ \ \,r}$ which satisfies the Pólya condition* (1.2) *and does not contain supported odd blocks is regular.*

Note that the incidence matrices corresponding to Lagrange's and Hermite's interpolation problems fulfill the assumptions of the Atkinson–Sharma Theorem. Indeed, their incidence matrices contain only Hermitian rows (with length one in the Lagrange case), therefore obviously satisfy the Pólya condition and, as their rows contain only blocks with level 0, these blocks are not be supported.

Atkinson and Sharma also conjectured that all matrices that contain odd supported blocks are not regular. However, Lorentz and Zeller [11] found a counterexample to this conjecture, showing that the three-row incidence matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3}$$

is regular, despite having two odd supported blocks.

Since the problem of characterizing the regularity of general incidence matrices turns out to be a very difficult one, some authors [3, 4, 5, 6, 9, 10] have studied the special class of almost Hermitian matrices, which are incidence matrices which have only one (interior) non-Hermitian row. Special attention has been paid to the three–row almost Hermitian matrices. Particular reason for the interest in three–row matrices is that, by applying technique of splitting (de-coalescence) of rows, singularity of such matrices can imply singularity of incidence matrices with more rows, see e.g. [8].

Definition 4. A three–row almost Hermitian incidence matrix $E(p, q; k_1, k_2)$ is an incidence matrix with its first and third row Hermitian of length p and q, respectively (with $p \le q$), and single 1-entries (blocks of length one) in the middle row in positions k_1 and k_2, where $1 \le k_1 < k_2 - 1$ (the case $k_1 = k_2 - 1$ is handled by the Atkinson-Sharma theorem).

It follows from the results in [9, 10] that $E(p, q; k_1, k_2)$ is not regular unless one of the following conditions is satisfied (see [13, Theorem 8.5]):

$$p \le k_1 < k_2 - 1 \le q\,, \tag{1.4}$$

$$q + 1 < k_2 \quad \text{and} \quad k_1 + k_2 = p + q + 1\,. \tag{1.5}$$

Only in the second case (called in [13, p. 104] as *the symmetric exterior case*) the regularity is completely characterized. Precisely, in this case $E(p, q; k_1, k_2)$ is regular if and only if $p = q$ (for more details, see [13, Theorem 8.15]). In the present note we present a short proof of the "if part" (the sufficiency). More precisely, we prove the following

Theorem 1. *The almost Hermitian matrix $E(m, m; k, 2m + 1 - k)$ is regular for every $k \in \mathbb{N}$, $1 \le k < m$.*

Notice that the matrix in (1.3) corresponds to the case $m = 2, k = 1$.

Our proof of Theorem 1 makes use of some properties of the Gegenbauer polynomials, in particular of the Legendre polynomials.

2. PROOF OF THEOREM 1

The claim of Theorem 1 is equivalent to the following statement:

Proposition 1. *Let $m, k \in \mathbb{N}$, $1 \le k < m$. Then for every $x \in (-1,1)$ and data set $\{(a_j, b_j),\ j = 0, 1, \dots, m-1;\ c,\ d\}$ there exist a unique algebraic polynomial $Q(x)$ of degree not exceeding $2m+1$ satisfying the interpolation conditions*

$$\begin{aligned} &Q^{(j)}(-1) = a_j, \qquad j = 0, 1, \dots, m-1,\\ &Q^{(j)}(1) = b_j, \qquad\ \ j = 0, 1, \dots, m-1,\\ &Q^{(k)}(x) = c,\\ &Q^{(2m+1-k)}(x) = d\,. \end{aligned} \tag{2.1}$$

The linear system for the coefficients of Q has a unique solution if and only if the corresponding homogeneous system has only trivial solution. The polynomial Q which satisfy the homogeneous system has zeros of multiplicity m at ± 1, therefore is of the form

$$Q(t) = \omega(t)\big[A(t-x) + B\big]\,, \qquad \omega(t) = (x^2-1)^m$$

with constants A and B determined by $Q^{(k)}(x) = Q^{(2m+1-k)}(x) = 0$, i.e., by the linear system

$$\left|\begin{array}{lcll} B\,\omega^{(k)}(x) & + & A\,k\,\omega^{(k-1)}(x) & = \ 0\\ B\,\omega^{(2m+1-k)}(x) & + & A\,(2m+1-k)\,\omega^{(2m-k)}(x) & = \ 0\,. \end{array}\right.$$

To prove Proposition 1, and thereby Theorem 1, we heed to show that the unique solution of this last system is $A = B = 0$, which is equivalent to showing that $\Delta(x) \ne 0$ for every $x \in (-1,1)$, where

$$\Delta(x) = k\omega^{(k-1)}(x)\omega^{(2m+1-k)}(x) - (2m+1-k)\omega^{(k)}(x)\omega^{(2m-k)}(x)\,. \tag{2.2}$$

For the proof of (2.2) we shall use some properties of the Legendre polynomials, the orthogonal polynomials in $[-1,1]$ with respect to the constant weight function. Recall that the n-th Legendre polynomial P_n is defined by

$$P_n(x) = \frac{1}{2^n\, n!}\Big(\frac{d}{dx}\Big)^n\big\{(x^2-1)^n\big\}\,.$$

For $j = 1, 2, \dots, m$, we define recursively the j-fold anti-derivative $S_j(x)$ of P_m by

$$S_j(x) = \int_{-1}^{x} S_{j-1}(t)\,dt, \qquad S_0(x) = P_m(x)\,.$$

In view of the definition of Legendre polynomials, we have

$$S_j(x) = \frac{1}{2^m\, m!}\,\omega^{(m-j)}(x)\,, \qquad j = 0, 1, \dots, m\,. \tag{2.3}$$

For the proof of (2.2) we shall need the following lemma.

Lemma 1. *For $j = 1, 2, \ldots, m$, there holds*

$$S_j(x) = \frac{(m-j)!}{(m+j)!}\,(x^2-1)^j \left(\frac{d}{dx}\right)^j \bigl\{P_m(x)\bigr\}\,. \tag{2.4}$$

Proof. We apply backward induction on j. Since $\left(\frac{d}{dx}\right)^m \bigl\{P_m(x)\bigr\} = \frac{(2m)!}{2^m\,m!}$, (2.3) shows that equality (2.4) is true for $j = m$. Assuming that (2.4) is true for some j, $1 \le j \le m$, we obtain

$$\begin{aligned}
S_{j-1}(x) = S_j'(x) &= \frac{(m-j)!}{(m+j)!}\,\frac{d}{dx}\left\{(x^2-1)^j \left(\frac{d}{dx}\right)^j \bigl\{P_m(x)\bigr\}\right\}\\
&= \frac{(m-j)!}{(m+j)!}(x^2-1)^{j-1}\left\{(x^2-1)\left(\frac{d}{dx}\right)^{j+1}\bigl\{P_m(x)\bigr\}+2j\,x\left(\frac{d}{dx}\right)^{j}\bigl\{P_m(x)\bigr\}\right\}\\
&= \frac{(m-j)!}{(m+j)!}(x^2-1)^{j-1}\Bigl\{(x^2-1)z'' + 2j\,x\,z'\Bigr\},
\end{aligned} \tag{2.5}$$

where

$$z(x) = \left(\frac{d}{dx}\right)^{j-1}\bigl\{P_m(x)\bigr\}\,. \tag{2.6}$$

At this point we exploit some well-known properties of the Gegenbauer polynomials. The Gegenbauer polynomial C_n^λ is the n-th orthogonal polynomial in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$ (and the n-th Legendre polynomials P_n equals $C_n^{1/2}$). The Gegenbauer polynomials satisfy the ordinary differential equation

$$(1-x^2)y'' - (2\lambda+1)x\,y' + n(n+2\lambda)y = 0\,, \qquad y = C_n^\lambda(x) \tag{2.7}$$

and their derivatives satisfy $\frac{d}{dx}\bigl\{C_n^\lambda(x)\bigr\} = 2\lambda\,C_{n-1}^{\lambda+1}(x)$ (see [14, eqns. (4.7.5) and (4.7.14)]). From this last property we observe that, apart from a constant factor, the polynomial $z(x)$ in (2.6) is equal to $C_{m-j+1}^{j-1/2}(x)$. Then, according to (2.7),

$$(x^2-1)z'' + 2j\,xz' = (m-j+1)(m+j)\,z\,,$$

and substituting this expression in (2.5) we obtain

$$S_{j-1}(x) = \frac{(m-j+1)!}{(m+j-1)!}\,(x^2-1)^{j-1}\,z(x)\,.$$

With this the induction step from j to $j-1$ is done. Lemma 1 is proved. □

We proceed with the proof of (2.2). From (2.3) and

$$\left(\frac{d}{dx}\right)^j \bigl\{P_m(x)\bigr\} = \frac{1}{2^m\,m!}\,\omega^{(m+j)}(x)$$

we observe that Lemma 1 is equivalent to the identity

$$\frac{\omega^{(m-j)}(x)}{(m-j)!} = (x^2-1)^j \, \frac{\omega^{(m+j)}(x)}{(m+j)!}\,, \qquad j = 1, \ldots, m\,. \tag{2.8}$$

With $j = m-k+1$ and $j = m-k$ this yields

$$\frac{\omega^{(k-1)}(x)}{(k-1)!} = (x^2-1)^{m-k+1} \, \frac{\omega^{(2m+1-k)}(x)}{(2m+1-k)!}\,,$$

$$\frac{\omega^{(k)}(x)}{(k)!} = (x^2-1)^{m-k} \, \frac{\omega^{(2m-k)}(x)}{(2m-k)!}\,.$$

By expressing $\omega^{(2m+1-k)}$ and $\omega^{(2m-k)}$ and substitution in (2.2) we find that

$$\begin{aligned}\Delta(x) &= k!(2m+1-k)!\Big\{\frac{\omega^{(k-1)}(x)}{(k-1)!}\,\frac{\omega^{(2m+1-k)}(x)}{(2m+1-k)!} - \frac{\omega^{(k)}(x)}{k!}\,\frac{\omega^{(2m-k)}(x)}{(2m-k)!}\Big\} \\ &= \frac{(2m+1-k)!}{k!}(x^2-1)^{k-m-1}\Big\{\big[k\omega^{(k-1)}(x)\big]^2 + (1-x^2)\big[\omega^{(k)}(x)\big]^2\Big\}\,.\end{aligned}$$

Since the zeros of $\omega^{(k-1)}$ and $\omega^{(k)}$ interlace, the sum in the last curl brackets is positive for $x \in (-1,1)$, and consequently $\Delta(x) \neq 0$ for $x \in (-1,1)$. With this the proof of Proposition 1 is complete.

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