ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

SOME THEOREMS ON THE CONVERGENCE OF SERIES IN BESSEL-MAITLAND FUNCTIONS [1]

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Some important properties of the power series in complex domain are given by the classical Cauchy-Hadamard, Abel and Tauber theorems. In this paper we prove same type theorems for series in the Bessel-Maitland functions.

Keywords: Cauchy-Hadamard, Abel and Tauber theorems, Bessel-Maitland function
2000 MSC: 30B10, 30B30; 33C10, 33C20

1. INTRODUCTION

Some important properties of the power series $\sum_{n=0}^{\infty} a_n z^n$ in a complex domain are given by the classical Cauchy-Hadamard, Abel and Tauber theorems.

In general, by the classical Abel theorem, from the convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ at a point z_0, follows the existence of the limit $\lim_{z \to z_0} f(z) = f(z_0)$, when z belongs to a suitable angle domain with a vertex at a point z_0. The geometrical series [6, p.92]: $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$ at $z_0 = 1$ gives an example that, in general, the inverse proposition is not true, i.e. the existence of this limit

[1] This work is partially supported by National Science Research Fund - Bulgarian Ministry of Education and Science, under Grant MM1305/2003

does not imply the convergence of the series $\sum_{n=0}^{\infty} a_n z_0^n$ without additional conditions on the growth of the coefficients.

The corresponding classical result is given by the following theorem.

Theorem (Tauber). *If the coefficients of the power series satisfy the condition* $\lim_{n\to\infty} na_n = 0$ *and if* $\lim_{z\to 1} f(z) = S$ *($z \to 1$ radially), then the series* $\sum a_n$ *is convergent and* $\sum_{n=0}^{\infty} a_n = S$.

It turns out that Abel's theorem fails even for series of the type $\sum_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \dots, n_k, \dots)$ is a suitable permutation of nonnegative integers [6, p.92]. Therefore, it is interesting to know if for series in a given sequence of holomorphic functions, a statement like Abel's theorem is available. A positive answer to this question for series in Laguerre and Hermite polynomials is given in [5, §11.3], [1], and for Bessel functions - in [4].

Let $J_\nu^\mu(z)$ be the so-called Bessel-Maitland function, see [2, p.336, 352], [3, p.110]:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(1+\nu+\mu k)}, \quad z \in \mathbb{C}, \quad \mu > -1.$$

Let us consider series of the form

$$\sum_{n=0}^{\infty} a_n z^n J_n^\mu(z), \quad z \in \mathbb{C}, \quad \mu > 0. \tag{1}$$

We prove in this paper the corresponding Cauchy-Hadamard, Abel and Tauber type theorems for series in Bessel-Maitland functions of form (1).

2. A CAUCHY-HADAMARD TYPE THEOREM

Denote for shortness:

$$\widetilde{J}_n^\mu(z) = z^n J_n^\mu(z), \quad n = 0, 1, 2, \dots.$$

The following asymptotic formula can be easily verified for the Bessel-Maitland functions:

$$\widetilde{J}_n^\mu(z) = z^n (1 + \theta_n^\mu(z)) \,/\, \Gamma(n+1), \;\; z \in \mathbb{C}, \quad \mu > 0, \tag{2}$$
$$\theta_n^\mu(z) \to 0 \;\text{ as }\; n \to \infty \quad (n \in \mathbb{N}).$$

Theorem 1 (Cauchy-Hadamard type). *The domain of convergence of the series* (1) *is the circle domain* $|z| < R$ *with a radius of convergence* $R = 1/\Lambda$, *where*

$$\Lambda = \limsup_{n\to\infty} (\ |a_n| \ / \ \Gamma(n+1)\ )^{1/n}. \tag{3}$$

The cases $\Lambda = 0$ *and* $\Lambda = \infty$ *are incorporated in the common case, if* $1/\Lambda$ *means* ∞, *respectively 0.*

Proof. Let us denote

$$u_n(z) = a_n \widetilde{J}_n^{\mu}(z), \quad b_n = (\ |a_n| \ / \ \Gamma(n+1)\ )^{1/n}.$$

Using the asymptotic formula (2), we get

$$u_n(z) = a_n z^n (1 + \theta_n^{\mu}(z)) \ / \ \Gamma(n+1).$$

The proof goes in three cases.

1. $\Lambda = 0$, then $\lim\limits_{n\to\infty} b_n = \limsup\limits_{n\to\infty} b_n = 0$. Let us fix $z \neq 0$. Obviously, there exists a number N_1 such that for every $n > N_1$: $|1 + \theta_n^{\mu}(z)| < 2$ and $2b_n < 1/|z|$ which is equivalent to $|u_n(z)| = b_n^n |z|^n |1 + \theta_n^{\mu}(z)| < 2^{1-n}$. The absolute convergence of (1) follows immediately from this inequality.

2. $0 < \Lambda < \infty$. First, let z be inside the domain $|z| < R$ ($z \in \mathbb{C}$), i.e. $|z|/R < 1$. Then $\limsup\limits_{n\to\infty} |z| b_n < 1$. Therefore, there exists a number $q < 1$ such that $\limsup\limits_{n\to\infty} |z| b_n \leq q$, whence $|z|^n b_n^n \leq q^n$. Using the asymptotic formula for the common member $u_n(z)$ of the series (1), we obtain $|u_n(z)| = b_n^n |z|^n |1 + \theta_n^{\mu}(z)| \leq q^n |1+\theta_n^{\mu}(z)|$. Since $\lim\limits_{n\to\infty} \theta_n^{\mu}(z) = 0$, there exists N_2: for every $n > N_2$ $|1+\theta_n^{\mu}(z)| < 2$ and hence $|u_n(z)| \leq 2q^n$. Since the series $\sum\limits_{n=0}^{\infty} 2q^n$ is convergent, the series (1) is also convergent, even absolutely.

Now, let z lie outside this domain. Then $|z|/R > 1$ and $\limsup\limits_{n\to\infty} |z| b_n > 1$. Therefore there exist infinite number of values n_k of n: $|z|^{n_k} b_{n_k}^{n_k} > 1$. Because $\lim\limits_{n\to\infty} \theta_n^{\mu}(z) = 0$, there exists N_3 so that for $n_k > N_3$; $|1 + \theta_{n_k}^{\mu}(z)| \geq 1/2$, i.e. $|u_{n_k}(z)| \geq 1/2$ for infinite number of values of n. The necessary condition for convergence is not satisfied. Therefore the series (1) is divergent.

3. $\Lambda = \infty$. Let $z \in \mathbb{C}\backslash\{0\}$. Then $b_{n_k} > 1/|z|$ for infinite number of values n_k of n. But, from here $|u_{n_k}(z)| = |z|^{n_k}\ b_{n_k}^{n_k}\ |1 + \theta_{n_k}^{\mu}(z)| \geq 1/2$ and the necessary condition for the convergence of the series (1) is not satisfied and we conclude that the series (1) is divergent for every $z \neq 0$. □

3. AN ABEL TYPE THEOREM

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angle domain with size $2\varphi < \pi$ and vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0. The following theorem is valid:

Theorem 2 (Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be defined by* (3). $0 < \Lambda < \infty$. *Let $K = \{|z| < R, R = 1/\Lambda\}$. If $f(z)$ is the sum of the series* (1) *on the domain K and this series is convergent at the point z_0 of the boundary of K. then $\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0)$. for $|z| < R$ and $z \in g_\varphi$, i.e.*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0), \quad z \in g_\varphi. \tag{4}$$

Proof. Consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)), \tag{5}$$

representing it in the form

$$\Delta(z) = \sum_{n=0}^{k} a_n (\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) + \sum_{n=k+1}^{\infty} a_n (\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)).$$

Let $p > 0$. By using the notations

$$\beta_m = \sum_{n=k+1}^{m} a_n \widetilde{J}_n^\mu(z_0), \quad m > k, \quad \beta_k = 0,$$

$$\gamma_n(z) = 1 - \widetilde{J}_n^\mu(z)/\widetilde{J}_n^\mu(z_0),$$

and the Abel transformation [1], we obtain consequently:

$$\sum_{n=k+1}^{k+p} a_n (\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1}) \gamma_n(z)$$

$$= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n (\gamma_{n+1}(z) - \gamma_n(z)),$$

i.e.

$$\sum_{n=k+1}^{k+p} a_n (\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) = (1 - \widetilde{J}_{k+p}^\mu(z)/\widetilde{J}_{k+p}^\mu(z_0)) \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n^\mu(z_0)$$

$$
-\sum_{n=k+1}^{k+p-1}(\sum_{s=k+1}^{n} a_s\widetilde{J}_s^{\mu}(z_0))(\frac{\widetilde{J}_n^{\mu}(z))}{\widetilde{J}_n^{\mu}(z_0))}-\frac{\widetilde{J}_{n+1}^{\mu}(z))}{\widetilde{J}_{n+1}^{\mu}(z_0))}).
$$

From the asymptotic formula (2), it follows that there exists a natural number M such that $\widetilde{J}_n^{\mu}(z_0)\neq 0$ for $n>M$. Let $k>M$. Then, for every natural $n>k$:

$$
\widetilde{J}_n^{\mu}(z)/\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_{n+1}^{\mu}(z)/J_{n+1}^{\mu}(z_0)=(z/z_0)^n \tag{6}
$$

$$
\times\frac{(1+\theta_n^{\mu}(z))(1+\theta_{n+1}^{\mu}(z_0))-(z/z_0)(1+\theta_{n+1}^{\mu}(z))(1+\theta_n^{\mu}(z_0))}{(1+\theta_n^{\mu}(z_0))(1+\theta_{n+1}^{\mu}(z_0))}.
$$

For the right hand side of (6) we apply the Schvartz lemma. Then we get that there exists a constant C:

$$
|\widetilde{J}_n^{\mu}(z)/\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_{n+1}^{\mu}(z)/J_{n+1}^{\mu}(z_0)|\leq C|z-z_0||z/z_0|^n.
$$

Analogously there exists a constant B:

$$
|1-\widetilde{J}_{k+p}^{\mu}(z)/\widetilde{J}_{k+p}^{\mu}(z_0)|\leq B|z-z_0|\leq 2B|z_0|.
$$

Let ε be an arbitrary positive number and choose $N(\varepsilon)$ so large that for $k>N(\varepsilon)$ the inequality

$$
|\sum_{s=k+1}^{n} a_s\widetilde{J}_s^{\mu}(z_0)|<\min(\varepsilon\cos\varphi/(12B|z_0|),\varepsilon\cos\varphi/(6C|z_0|))
$$

holds for every natural $n>k$. Therefore, for $k>\max(M,N(\varepsilon))$:

$$
|\sum_{s=k+1}^{\infty} a_sJ_s^{\mu}(z_0)|\leq\min(\varepsilon\cos\varphi/(12B|z_0|),\varepsilon\cos\varphi/(6C|z_0|),
$$

and

$$
|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z))|\leq(\varepsilon\cos\varphi/6)(1+\sum_{n=k+1}^{\infty}|z_0|^{-1}|z-z_0||z/z_0|^n)
$$

$$
\leq(\varepsilon\cos\varphi/6)(1+|z-z_0|/(|z_0|-|z|)).
$$

But near the vertex of the angle domain g_φ in the part d_φ closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle we have $|z-z_0|/(|z_0|-|z|)<2/\cos\varphi$, i.e. $|z-z_0|\cos\varphi<2(|z_0|-|z|)$. That is why the inequality

$$
|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z))|<(\varepsilon\cos\varphi)/6+\varepsilon/3\leq\varepsilon/2 \tag{7}
$$

holds for $z \in d_\varphi$ and $k > \max(M, N(\varepsilon))$. Fix some $k > \max(M, N(\varepsilon))$ and after that choose $\delta(\varepsilon)$ such that if $|z - z_0| < \delta(\varepsilon)$ then the inequality

$$|\sum_{n=0}^{k} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z))| < \varepsilon/2 \tag{8}$$

holds inside d_φ. We get

$$|\Delta(z)| = |\sum_{n=0}^{\infty} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z))|$$

for the module of the difference (5). From (7) and (8) it follows that the equality (4) is satisfied. □

4. A TAUBER TYPE THEOREM

Let us consider the series $\sum_{n=0}^{\infty} a_n,\ a_n \in \mathbb{C}$. Let
$z_0 \in \mathbb{C},\quad |z_0| = R,\quad 0 < R < \infty,\quad J_n^\mu(z_0) \neq 0\quad$ for $n = 0, 1, 2, \ldots$.
For shortness, denote

$$J_{n,\mu}^*(z; z_0) = \frac{\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}.$$

Let the series $\sum_{n=0}^{\infty} a_n J_{n,\mu}^*(z; z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J_{n,\mu}^*(z; z_0), \quad |z| < R.$$

Theorem 3 (Tauber type). *If $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with*

$$\lim\{na_n\} = 0, \tag{9}$$

and there exists

$$\lim_{z \to z_0} F(z) = S \quad (|z| < R, z \to z_0 \text{ radially}),$$

then the series $\sum_{n=0}^{\infty} a_n$ is convergent and

$$\sum_{n=0}^{\infty} a_n = S.$$

Proof. For a point z of the segment $[0, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J_{n,\mu}^*(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}_n^{\mu}(z_0)}{\widetilde{J}_n^{\mu}(z_0)} - \sum_{n=0}^{\infty} a_n \frac{\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} - \sum_{n=k+1}^{\infty} a_n J_{n,\mu}^*(z; z_0)$$

and therefore

$$|\sum_{n=0}^{k} a_n - F(z)| \le \sum_{n=0}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^*(z; z_0) \right| . \tag{10}$$

By using the asymptotic formula (2) for the Bessel-Maitland functions, we obtain:

$$a_n J_{n,\mu}^*(z; z_0) = a_n \left(\frac{z}{z_0} \right)^n \frac{1 + \theta_n^{\mu}(z)}{1 + \theta_n^{\mu}(z_0)} = a_n \left(\frac{z}{z_0} \right)^n \left(1 + \widetilde{\theta}_{n,\mu}(z; z_0) \right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_{k,\mu}(z; z_0)| < 2, |ka_k| < \frac{\varepsilon}{6}$ hold as $k \ge N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (10) the following estimate is valid:

$$\sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^*(z; z_0) \right| = \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1 + \widetilde{\theta}_{n,\mu}(z; z_0)| \tag{11}$$

$$\le 2 \left| \frac{z}{z_0} \right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^{n-k-1} \le 2 \sum_{n=0}^{\infty} |a_{n+k+1}| \left| \frac{z}{z_0} \right|^n$$

$$= 2 \sum_{n=0}^{\infty} \frac{|(n+k+1) a_{n+k+1}|}{n+k+1} \left| \frac{z}{z_0} \right|^n < 2 \sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left| \frac{z}{z_0} \right|^n$$

$$< \frac{2}{k} \frac{\varepsilon}{6} \frac{1}{1 - |z/z_0|} = \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0| - |z|} .$$

Now let us consider the first summand in (10). We have:

$$\sum_{n=0}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right|$$

$$= \sum_{n=0}^{m} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| + \sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| .$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}\right| < C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m}|a_n|\left|\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}\right| \le C\,|z-z_0|\,k\,\frac{\sum\limits_{n=0}^{m}|a_n|}{k} \tag{12}$$

$$< C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC} = |z-z_0|\,k\,\frac{\varepsilon}{3R}.$$

holds as $k > N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k}|a_n|\left|\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}\right|.$$

To this end, using asymptotic formula (2) for the Bessel-Maitland functions, we find consequently:

$$\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)} = \frac{(z_0)^n(1+\theta_n^\mu(z_0))-z^n(1+\theta_n^\mu(z))}{z_0^n(1+\theta_n^\mu(z_0))} = 1-\left(\frac{z}{z_0}\right)^n\frac{1+\theta_n^\mu(z)}{1+\theta_n^\mu(z_0)}$$

$$= 1-\left(\frac{z}{z_0}\right)^n\left[1+\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right] = 1-\left(\frac{z}{z_0}\right)^n-\left(\frac{z}{z_0}\right)^n\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}.$$

Therefore,

$$\left|\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}\right| \le \left|1-\left(\frac{z}{z_0}\right)^n\right| + \left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right|. \tag{13}$$

We obtain the following inequalities

$$\left|1-\left(\frac{z}{z_0}\right)^n\right| = \left|1-\frac{z}{z_0}\right|\left|1+\frac{z}{z_0}+\left(\frac{z}{z_0}\right)^2+\cdots+\left(\frac{z}{z_0}\right)^{n-1}\right| \le n\left|1-\frac{z}{z_0}\right|$$

for the first summand of (13). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right| \le 1 \qquad \text{as} \qquad |z-z_0|<\rho.$$

Then, for such $|z|$, we obtain for the second summand of (14):

$$\left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right| \le \left|\frac{z}{z_0}\right|^n|z-z_0|.$$

From (9) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} n|a_n|}{k} = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} |a_n|}{k} = 0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{and} \quad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{as} \quad k > N_3.$$

Therefore,

$$\sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| \le \sum_{n=m+1}^{k} n|a_n| \left|1 - \frac{z}{z_0}\right| \tag{14}$$

$$+ \sum_{n=m+1}^{k} |a_n| \left|\frac{z}{z_0}\right|^n |z - z_0| \le k\, \frac{|z-z_0|}{R}\, \frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z - z_0|\, \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k}$$

$$< k\,|z - z_0|\, \frac{1+R}{R}\, \frac{\varepsilon}{3(1+R)} = k\,|z - z_0|\, \frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \le \sum_{n=0}^{m} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right|$$

$$+ \sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^{*}(z; z_0) \right| .$$

Let $N = \max(N_1, N_2, N_3), k > N$ and $|z - z_0| < \rho$. Then by using (11),(12),(14), we can conclude that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| < |z - z_0|\, k\, \frac{\varepsilon}{3R} + k\,|z - z_0|\, \frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\, \frac{1}{k}\, \frac{|z_0|}{|z_0| - |z|}$$

$$= \frac{\varepsilon}{3} \left[\frac{2k}{R}\, |z - z_0| + \frac{1}{k}\, \frac{|z_0|}{|z_0| - |z|} \right] .$$

If we substitute z by $z_0(1 - \frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1 - \frac{1}{k})\right)\right| < \frac{\varepsilon}{3}\, 3 = \varepsilon.$$

This proves that $\lim_{k\to\infty} \sum_{n=0}^{k} a_n$ exists and equals $\lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right) = S.$$

Thus the theorem is proved. □

Remark. Putting $\mu = 1$ in above considerations leads to the corresponding results (see [4]) for series in the Bessel functions $J_\nu(z) = (z/2)^\nu J_\nu^1(z^2/4)$, namely for

$$\sum_{n=0}^{\infty} a_n J_n(z), \quad z \in \mathbb{C}.$$

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Received on June 6, 2006

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