ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM, BASED ON A PROBLEM OF NONLINEAR VIBRATION OF AN ELASTIC STRING

PETYA BRAYNOVA, O. CHRISTOV

In this paper we study the problem for non-integrability of a Hamiltonian system, based on the nonlinear vibrations of an elastic string. We have the following hamiltonian:

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2(t) + \frac{c_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2(t) - \frac{c_2}{2}\sum_{k=1}^{N} {q_k}^2(t) +$$
$$+\frac{h1}{8}\left(\sum_{k=1}^{N} k^2 {q_k}^2(t)\right)^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2(t)\right)^2 = const$$

The main result is that the responding Hamiltonian system is non-integrable, except in the cases $N > 2$ and $h_1 = 0$ and $N = 2$ and $h_1 = 0$ or $h_2 = 4h_1$. In the proof we use the Morales - Ramis theorem based on Differential Galois Theory.

Keywords: Nonlinear elastic string, Hamiltonian system, Morales-Ramis theory

2000 MSC: 70J50, 70H08

1. INTRODUCTION

Free lateral "finite" vibrations of uniform beam with the ends restrained can be described by the equation

$$\rho h \frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = \left(P_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial w}{\partial x})^2 dx\right)\frac{\partial^2 w}{\partial x^2}, \tag{1.1}$$

where $w(t,x)$ is the lateral deflection of the string, E – the Young's modulus, EI – the flexural rigidity, ρ – the mass density, h is the thickness of a beam of unit width, L – the string's length, P_0 is the initial axial tension. Suppose the following initial and boundary conditions

$$w(0,x) = w_0(x), \qquad \frac{\partial w}{\partial t}(0,x) = w_1(x)$$

$$w(t,0) = \frac{\partial^2 w}{\partial x^2}(t,0) = w(t,L) = \frac{\partial^2 w}{\partial x^2}(t,L) = 0.$$

In 1971 Nishida [1] examined the problem of the elastic string's vibration, in the case there is no resistance ($EI = 0$) and the equation (1.1) changes into

$$\rho h \frac{\partial^2 w}{\partial t^2} = \left(P_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial w}{\partial x})^2 dx\right)\frac{\partial^2 w}{\partial x^2}$$

If there is such a natural number N, that the initial and boundary conditions look like

$$w_0(x) = \sum_{k=1}^{N} a_k \sin(k\frac{\pi}{L}x), \quad w_1(x) = \sum_{k=1}^{N} b_k \sin(k\frac{\pi}{L}x),$$

where $a_k, b_k, k = 1,\ldots,N$ are real constants, then there exists a solution

$$w(t,x) = \sum_{k=1}^{N} u_k(t)\sin(k\frac{\pi}{L}x), \tag{1.2}$$

which is unique in a certain class of functions. Having put (1.2) in (1.1), Nishida got a Hamiltonian system of differential equations for $u_k(t), k = 1,\ldots,N$ and proved that conditional-periodic motions are preserved around equilibrium using the KAM theorem.

Another kind of problems on the vibrations of the nonlinear string were studied by Dickey [2].

In 1994 Iliev [3] studied a more general integro-differential equation

$$\frac{\partial^2 \omega}{\partial t^2} - \left(c_1 + h_1\int_0^{\pi} (\frac{\partial \omega}{\partial x})^2 dx\right)\frac{\partial^2 \omega}{\partial x^2} = \left(c_2 + h_2\int_0^{\pi} \omega^2 dx\right)\omega \tag{1.3}$$

and under the same assumptions as Nishida, he brought it to a Hamiltonian system with N degrees of freedom, namely

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2(t) + \frac{c_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2(t) - \frac{c_2}{2}\sum_{k=1}^{N} {q_k}^2(t) +$$

$$+\frac{h_1}{8}\left(\sum_{k=1}^{N} k^2 {q_k}^2(t)\right)^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2(t)\right)^2 = const \tag{1.4}$$

Then Iliev focused himself on the integrability problem in analytic functions in the case $N = 2$. Using the Ziglin's theory, he has proved the following result:

Theorem 1. *The Hamiltonian system with Hamiltonian (1.4) is not integrable for $N = 2$. if we have*

$$\frac{c_2 - 4c_1}{c_2 - c_1} < 0 \qquad \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}} \textit{ is not odd.}$$

In 2003 Yagasaki [4] studied the same model of unforced and undamped beam as equation (1.1) with $EI = 1$. He proved non-integrability of the corresponding Hamiltonian system after the same truncation as the solution (1.2) using Differential Galois Theory for Hamiltonian systems.

One should note that considering the model (1.3) without resistance ($EI = 0$) there is no lost of generality. Having in mind the concrete form of the solution (1.2), the contribution of the forth derivative with respect to x will change the coefficients of the Hamiltonian (1.4).

Here we study the Hamiltonian system

$$\left| \begin{aligned} \dot{q_j} &= \frac{\partial H}{\partial p_j} = p_j \\ \dot{p_j} &= -\frac{\partial H}{\partial q_j} = -\left(c_1 + \frac{h_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2\right) j^2 q_j + \left(c_2 + \frac{h_2}{2}\sum_{k=1}^{N} {q_k}^2\right) q_j, \end{aligned} \right. \tag{1.5}$$

$$j = 1, \ldots, N$$

with Hamiltonian (1.4) for N degrees of freedom and generalize the result of the Theorem 1 as follows. Consider the complexified system (1.5) on the phase space $M := \{(q(t), p(t)) \in \mathbb{C}^{2N}\}$ with standard symplectic structure, $t \in \mathbb{C}$. We are interested in the question at which values of the parameters c_1, c_2, h_1, h_2, the system (1.5) is integrable (of course the case $N = 1$ is trivial).

Theorem 2. *The Hamiltonian system with Hamiltonian (1.4) is non-integrable. excluding the following two cases*

a) $N > 2$ and $h_1 = 0$.

b) $N = 2$ and $h_1 = 0$ or $h_2 = 4h_1$.

Remark. 1) If $N > 1$ and $h_1 = 0$, we have

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2 + \frac{c_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2 - \frac{c_2}{2}\sum_{k=1}^{N} {q_k}^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2\right)^2;$$

whence the starting Hamiltonian system is equivalent to so-called 'anharmonic oscillator' with Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} p_k{}^2 + \sum_{k=1}^{N} a_k q_k{}^2 + \left(\sum_{k=1}^{N} q_k{}^2\right)^2,$$

which is integrable in Liouville sense [5].

2) If $N = 2$ and $h_2 = 4h_1$, the variables in the system with Hamiltonian

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}(p_1{}^2 + p_2{}^2) + \frac{(c_1 - c_2)}{2}q_1{}^2 + \frac{(4c_1 - c_2)}{2}q_2{}^2 - \frac{3}{8}h_1 q_1{}^4 + \frac{3}{2}h_1 q_2{}^4$$

can be separated, hence in this case it is integrable.

Comment. The proof of Theorem 2 is based on "Differential Galois Theory", which gives a *necessary* condition for integrability. Moreover, in a view of the last remark, it follows this condition is also a *sufficient* one. So Theorem 2 gives a complete answer, when the system is integrable and when it is not.

The paper is organized as follows. In section 2 we summarize the theoretical results about Ziglin's and Morales-Ramis's theories. The proof of the Theorem 2 is given in section 3. In the last section some numerical experiment, confirming the theoretical results, are presented.

2. THEORY

In this section we summarize briefly some results on integrability of Hamiltonian systems. For more detailed description on Differential Galois theory see [6], [7].

Let (M^{2n}, ω) be a complex symplectic manifold. H is an analytic function over M^{2n} and the respective Hamiltonian system is

$$\dot{x} = X_H(x).$$

A Hamiltonian system is integrable in *Liouville sense* if there exist n independent first integrals $F_1 = H$, F_2,.., F_n in involution, namely $\{F_i, F_j\} = 0$ for all i and j, where $\{,\}$ is the Poisson's bracket [9]. Let $z = z(t)$ is a solution (not equilibrium) of the Hamiltonian system and $\Gamma := \{z = z(t)\}$ is its integral curve. The *variational equations* (VE) responding to $z = z(t)$ are

$$\dot{\eta} = \frac{\partial X_H}{\partial x}(z(t))\eta.$$

Reducing (VE) by the first integral dH, we get so called *normal variational equations* (NVE)

$$\dot{\xi} = A(t)\xi \qquad \text{with dimention } 2(n-1).$$

One of the first, who gave a criterion for having non-integrability, based on (VE) was Poincaré. Let M^{2n} be real and $z = z(t)$ be a periodic solution of the Hamiltonian system. Poincaré has studied the monodromy matrix, corresponding to (VE) [10]and he has proved that if the Hamiltonian system has k first integrals, then k characteristic exponents must be zero.

In 1982 Ziglin [11]proved the following result for integrability of a complex-analytical Hamiltonian systems:

Theorem 3. *Let a Hamiltonian system have n first integrals. independent around Γ. but not necessary on Γ. Suppose that there is a nonresonant element g in the monodromy group of (NVE). Then every other element g′ of the monodromy group transforms the set of eigendirections of g into itself.*

Let us remind of $g \in Sp(2n, \mathbb{C})$ (the monodromy group is a subgroup of the symplectic group) is a resonant if $l_1^{r_1} \dots l_n^{r_n} = 1$, where r_i are nonzero integers and l_i are the eigenvalues of g.

Note that in the Ziglin's result, there is no assumption that the integrals are in involution, in addition it refers to the case $n = 2$, because in higher dimensions there are resonanses.

Another method for proving non-integrability is based on the *Galois group* of (VE). In result of the efforts of Ramis, Morales-Ruiz, Simo, Chirchil and Rod, the following result has appeared in the end of the last century [6]:

Theorem 4. *Let a Hamiltonian system has n meromorphic first integrals in involution around Γ. but not necessary on Γ. Then the identity component G^0 of the Galois group G of (VE) with respect to Γ is abelian.*

In applications is used the next algorithm:

1) to find out a solution $z = z(t)$ of the hamiltonian system

2) to write the variational equations (VE) and (NVE), corresponding to $z = z(t)$

3) to check for commutativity of the Galois group of (VE), (NVE)

If once is proved, that G^0 is not abelian, than the respective system is non-integrable in Liouville sense. But the fact that G^0 is abelian doesn't imply integrability.

3. PROOF OF THEOREM 2

The proof of Theorem 2 is divided in several lemmas.

Lemma 1. *The system (1.5) has a particular solution*

$$\left|\begin{array}{l} \widetilde{q_r} = \sqrt{\lambda_1} sn\left(\dfrac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \kappa\right) \\ \widetilde{p_r} = \dot{\widetilde{q_r}} \\ \widetilde{q_j} = 0 \\ \widetilde{p_j} = 0 \end{array}\right. \qquad j = 1, \ldots, N, j \neq r, \tag{3.1}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, $|\lambda_1| < |\lambda_2|$ *and* λ_1 *and* λ_2 *are roots of the equation*

$$\frac{h_2 - r^4 h_1}{4}\lambda^2 + \frac{c_2 - r^2 c_1}{2}\lambda + 2f = 0.$$

and $\kappa = \sqrt{\frac{\lambda_1}{\lambda_2}}$.

Proof. There exists r, such that $h_2 - r^4 h_1 \neq 0$. Putting in the Hamiltonian system (1.5) $(\widetilde{q}, \widetilde{p}) = (0, .., 0, \widetilde{q_r}, 0, .., 0, \widetilde{p_r}, 0, .., 0)$ we get

$$\left|\begin{array}{l} \dot{\widetilde{q_r}} = \widetilde{p_r} \\ \dot{\widetilde{p_r}} = -\widetilde{q_r}\left(r^2 c_1 - c_2 + \dfrac{r^4 h_1 - h_2}{2}(\widetilde{q_r})^2\right) \end{array}\right.$$

The corresponding to this system Hamiltonian H is obtained from (1.4) after putting $(\widetilde{q}, \widetilde{p}) = (0, .., 0, \widetilde{q_r}, 0, .., 0, \widetilde{p_r}, 0, .., 0)$

$$\tilde{p}_r^2 = \frac{h_2 - r^4 h_1}{4}\tilde{q}_r^4 + (c_2 - r^2 c_1)\tilde{q}_r^2 + 2f,$$

Taking into account that $p_r = \dot{\widetilde{q_r}}$ we obtain the family of curves

$$\Gamma(f) : (\dot{\widetilde{q_r}})^2 = \frac{h_2 - r^4 h_1}{4}(\widetilde{q_r})^4 + (c_2 - r^2 c_1)(\widetilde{q_r})^2 + 2f$$

from where after some transformations we reach

$$\left(\frac{\dot{\widetilde{q_r}}}{\sqrt{\lambda_1}}\right)^2 = \frac{h_2 - r^4 h_1}{4}\lambda_2\left(1 - \left(\frac{\widetilde{q_r}}{\sqrt{\lambda_1}}\right)^2\right)\left(1 - \left(\sqrt{\frac{\lambda_1}{\lambda_2}}\frac{\widetilde{q_r}}{\sqrt{\lambda_1}}\right)^2\right)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $|\lambda_1| < |\lambda_2|$. This is precisely the definition of Jacobi's elliptic sn [14], so we get the particular solution as the lemma states. □

The function $sn(\tau, \kappa)$ is double periodic meromorphic function with periods $4K(\kappa)$ and $i2K'(\kappa)$. In the parallelogram of the periods $sn(\tau, \kappa)$ has two simple poles $iK'(\kappa)$ and $2K(\kappa) + iK'(\kappa)$ [14].

Therefore $sn\left(\dfrac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \kappa\right)$ has periods $T_1 = \dfrac{8K(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$, $T_2 = \dfrac{i4K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$ and poles $t_1 = \dfrac{i2K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$, $t_2 = \dfrac{4K(\kappa) + i2K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$.

Geometrically, $\Gamma(f)$ are tori with two points removed.

Next, in order to reduce the domain of the solution (3.1) consider the involution

$$R : (q_1, \ldots, q_r, \ldots, q_n, p_1, \ldots, p_r, \ldots, p_n) \to (q_1, \ldots, -q_r, \ldots, q_n, p_1, \ldots, -p_r, \ldots, p_n)$$

The involution R leaves the Hamiltonian system invariant and changes the places of the two missing points. Let us denote with F_R the set of the fixed points of the involution R,

$$F_R := \{(q_1, \ldots, q_{r-1}, 0, q_{r+1}, \ldots, q_n, p_1, \ldots, p_{r-1}, 0, p_{r+1}, \ldots, p_n).$$

Then factoring $M\backslash F_R$ in R we get the smooth symplectic manifold $\hat{M} = (M\backslash F_R)/R$. The Hamiltonian H is transformed to the Hamiltonian $\hat{H}$ for the same Hamiltonian system (1.5) defined on $\hat{M}$. It is clear that if the system (1.5) has enough independent first integrals they will be transformed into independent first integrals on $\hat{M}$. Then factorizing $\hat{\Gamma}(f) = \Gamma(f)/R$ and having in mind that

$$sn(\tau + 2K(\kappa)) = -sn(\tau), \tau \in \mathbb{C}.$$

we obtain that the domain of the family of the curves is mapped as tori with one point removed.

Lemma 2. *The normal variational equations (NVE) of the system with hamiltonian (1.4) around the particular solution (3.1) are*

$$\left| \begin{array}{l} \dot{\xi}_j = \eta_j \\ \dot{\eta}_j = \xi_j \left((c_2 - j^2 c_1) + \dfrac{(h_2 - j^2 r^2 h_1)}{2} (\widetilde{q_1})^2 \right) \end{array} \right. \quad j = 1, \ldots, N, j \neq r. \tag{3.2}$$

The proof is straightforward and therefore is omitted.

In view of Lemma 2, (NVE) breaks into $N-1$ separate systems, as each of them consists of two first-order linear differential equations. So each of these $N-1$ systems can be written as a second-ordered linear differential equation denoted with $(NVE_j), j = 1, \ldots, N, j \neq r$, namely

$$\ddot{\xi}_j + \left((j^2 c_1 - c_2) + \frac{(j^2 r^2 h_1 - h_2)}{2} \lambda_1 sn^2 \left(\frac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \sqrt{\frac{\lambda_1}{\lambda_2}} \right) \right) \xi_j = 0. \tag{3.3}$$

In our problem, because of the specific kind of (NVE), the Galois group G looks like a direct product

$$G = G_1 \otimes G_2 \otimes \ldots \otimes G_{r-1} \otimes G_{r+1} \otimes \ldots \otimes G_N,$$

where the missing part G_r corresponds to the tangent equations.

Therefore, in order to prove non-integrability, it is sufficient one part G_j – corresponding to the equation (NVE_j) to be nonabelian.

The equation (3.3) is Fuchsian one. It is known that in this case the monodromy group M topologically generates the Galois group G [8], [6].The monodromy group M has the same specific structure as G.

$$M = M_1 \otimes M_2 \otimes \ldots \otimes M_{r-1} \otimes M_{r+1} \otimes \ldots \otimes M_N, \tag{3.4}$$

Again if one M_j (corresponding to the equation (NVE_j)) is nonabelian, then this will imply that G_j is non-abelian and therefore due to Morales-Ramis theorem (. Theorem 4) we have non-integrability.

Now we shall study the monodromy group M_j for the equation (NVE_j).

Let g_1 and g_2 are the generators of the monodromy group M_j. The element g_1 is associated with a path along the parallel of the torus $\hat{\Gamma}$, which corresponds to adding the period $\frac{T_1}{2}$. Similarly, g_2 is associated with a path along the meridian of $\hat{\Gamma}$ or adding the period T_2 of the function $sn^2(\tau)$.

Lemma 3. *The commutator* $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ *has the following eigenvalues*

$$\exp\left(\pi i\left(1 \pm \sqrt{1 + 8\frac{j^2 r^2 h_1 - h_2}{r^4 h_1 - h_2}}\right)\right).$$

Proof. The commutator corresponds to one winding around the regular singular point $t_1 = \dfrac{i2K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$ of the equation (3.3).

It is known that for a linear differential equation [12]

$$\ddot{\xi_j} + \frac{P(t)}{(t - t_1)}\dot{\xi_j} + \frac{Q(t)}{(t - t_1)^2}\xi_j = 0$$

where $P(t)$ and $Q(t)$ are holomorphic in a neighborhood of $t = t_1$, then the eigenvalues of the monodromy transformation, corresponding to one circle around the regular singular point $t = t_1$, are $\exp(2\pi i \rho_{1,2})$, where $\rho_{1,2}$ are the roots of the indicial equation

$$\rho(\rho - 1) + P(t_1)\rho + Q(t_1) = 0 \tag{3.5}$$

The analytical theory of the differential equations is described in details in [13].Hence we have

$$sn\left(\frac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2}t, \sqrt{\frac{\lambda_1}{\lambda_2}}\right) = -\frac{2}{\sqrt{(h_2 - r^4 h_1)\lambda_1}}\frac{1}{(t - t_1)} + O(1),$$

so

$$Q(t_1) = -2\frac{(j^2r^2h_1 - h_2)}{(r^4h_1 - h_2)}, \qquad P(t) \equiv 0$$

and the roots of the quadratic equation (3.5) are exactly

$$\rho_{1.2} = \frac{1}{2}\left(1 \pm \sqrt{1 + 8\frac{j^2r^2h_1 - h_2}{r^4h_1 - h_2}}\right). \qquad \square$$

Taking into account Lemma 3 we conclude that if the eigenvalues of the j-th commutator are not units, than M_j is not abelian. Let us denote

$$\mu_j := 1 + 8\frac{j^2r^2h_1 - h_2}{r^4h_1 - h_2}.$$

Then the sufficient condition for non-integrability is the existence of j, such that the number μ_j is not equal to a square of some odd integer.

Lemma 4. *The monodromy group (3.4) is not abelian for $N > 2$ and $h_1 \neq 0$.*

Proof. Suppose that there exists $j \neq r$ such that $\rho_1 \in \mathbb{Z}$, so $\mu_j = (2k-1)^2$ for some $k \in \mathbb{Z}$. Hence when $h_2 \neq 0$ we get

$$\frac{h_1}{h_2} = \frac{1 - s_j}{j^2r^2 - s_jr^4}, \tag{3.6}$$

where $k(k-1) = 2s_j, s_j \in \mathbb{Z}$. We notice that for the numbers $s_j, 1 \leq j \leq N, j \neq r$ we have $s_j \geq 1$ or $s_j = 0$. From (3.6), if some $s_j = 0$, that can happen for only one j, namely j: $j^2r^2h_1 = h_2$.

The aim is to show, there exists a number l, such that $\mu_l \neq (2p-1)^2$ for all $p \in \mathbb{Z}$. For the purpose we examine cases according to r.

Case 1. *Let $1 < r < N$, $j < r$ and none of the numbers s_j is zero.*

Then there exists $l = r + 1$ and we assume $\mu_l = \mu_{r+1} = (2p-1)^2$ for some $p \in \mathbb{Z}$. Again we express

$$\frac{h_1}{h_2} = \frac{1 - s_{r+1}}{(r+1)^2r^2 - s_{r+1}r^4}, \tag{3.7}$$

where $p(p-1) = 2s_{r+1}, s_{r+1} \in \mathbb{Z}$. From both (3.6) and (3.7), after some computations we get

$$(s_j - 1)(2r+1) + (s_{r+1} - 1)(r-j)(r+j) = 0,$$

which is true if and only if $s_j = s_{r+1} = 1$, exactly when $h_1 = 0$ – the integrable case called "anharmonic oscillator" [5]. The last is in contradiction with the current lemma, so we have proved that the monodromy group M_{r+1} is not abelian.

Case 2. *Let $1 < r < N$, $r < j$ and none of the numbers s_j is zero.*

Then we take $l = r - 1$ and we obtain the following equation

$$(s_j - 1)(2r - 1)(-1) + (s_{r-1} - 1)(r - j)(r + j) = 0,$$

and similarly as in the previous case, we obtain that M_{r-1} is not abelian.

Case 3. *Let* $N = 3$.

The cases $\{r = 1, j = 2, l = 3\}$ and $\{r = 1, l = 2, j = 3\}$ are equivalent, if one transposes j and l and again from (3.6) and (3.7) we have

$$\frac{1 - s_2}{4 - s_2} = \frac{1 - s_3}{9 - s_3},$$

going back, to that we have noted above, we obtain

$$3(p + 1)(p - 2) = 8(k + 1)(k - 2),$$

and there is an equivalent case too, in transposing p and k. The last is true only when $p, k \in \{-1, 2\}$, which implies the integrable case - $h_1 = 0$. From the another pair of equivalent cases $\{l = 1, j = 2, r = 3\}$ and $\{j = 1, l = 2, r = 3\}$, analogously we get

$$5(p + 1)(p - 2) = 8(k + 1)(k - 2),$$

– the next contradiction with the current lemma. The cases $\{j = 1, r = 2, l = 3\}$ and $\{l = 1, r = 2, j = 3\}$ lead to

$$5(s_1 + 1) = -3(s_3 + 1),$$

which is not possible.

Case 4. *Let* $N > 3$, $1 < r < N$ *and* $s_j = 0$.

Then we can choose another number j_0 instead of j and to fall in case 1) or case 2), because only one s_j can be zero.

Case 5. *Let* $N > 3$, $r = 1$ *or* $r = N$.

If $r = 1$, then there exist at least two equations, the j-th and the l-th for $N > 3$, such that $s_j \neq 0 \neq s_l$. Without lost generality we can focus on the case $j = 2$, $l = 3$, so we obtain

$$3(s_3 - 1) = 8(s_2 - 1),$$

and similarly as the previous cases it leads to a contradiction. For $r = N$, we take the variational equations with numbers $j = N - 1$ and $l = N - 2$, therefore we get

$$4(s_{N-1} - 1)(N - 1) = (s_{N-2} - 1)(2N - 1),$$

which is available only when $h_1 = 0$.

Case 6. *The last case left is* $h_1 \neq 0$ *and* $h_2 = 0$.

Here

$$\mu_j = 1 + 8\frac{j^2}{r^2}.$$

Let first $r > 1$, then there exists an equation with number $j = r - 1$. If $\mu_j = \mu_{r-1}$ is equal to a square of an odd

$$1 + 8\frac{(r-1)^2}{r^2} = (2k-1)^2$$

we get

$$r^2(k+1)(k-2) = 2(1-2r),$$

which is possible only when $k = 0, 1$ and $r = 1$, which is in contradiction with the case. So we conclude that the group M_{r-1} is not abelian. Now let $r = 1$. Then $\mu_j = 1 + 8j^2$, which, for example, for $j = 2$, is not equal to a square of an odd number, so the monodromy group M_2 is not abelian and that proves Lemma 4. □

The first four lemmas prove part a) of Theorem 2. Let us formulate the last two lemmas, proving the second part of the theorem.

Lemma 5. *The system with Hamiltonian (1.4) is non-integrable for $N = 2$, $h_2 \neq 4h_1$ and $h_1 \neq 0$.*

Proof. The monodromy group for $j = 2$, corresponding to variations around the particular solution $(\widetilde{q_1}, \widetilde{p_1})$ is not abelian when

$$1 + 8\frac{4h_1 - h_2}{h_1 - h_2} \neq (2k-1)^2, \qquad k \in \mathbb{Z}. \tag{3.8}$$

First, we write the particular solution of the system $(\widetilde{q_2}, \widetilde{p_2})$, then the respective (NVE) around it and the following condition for non-integrability

$$1 + 8\frac{4h_1 - h_2}{16h_1 - h_2} \neq (2m-1)^2, \qquad m \in \mathbb{Z}. \tag{3.9}$$

Let us assume that for some values of the parameters $h_{1,2}$ such that $h_1 \neq 0$ and $h_2 \neq 4h_1$ there exist integers k and m, that we have equalities in (3.8) and (3.9).

$$1 + 8\frac{4h_1 - h_2}{h_1 - h_2} = (2k-1)^2, \qquad 1 + 8\frac{4h_1 - h_2}{16h_1 - h_2} = (2m-1)^2.$$

After some transformations in the first equality, like in the proof of Lemma 4, we express

$$h_2 = 4h_1\frac{4s-1}{s-1}$$

Here $s \neq 1$ since $s = 1$ implies $h_1 = 0$. Putting this h_2 in the second equality, we obtain

$$\frac{4s}{5s-1} \in \mathbb{Z}$$

which is true only in the case $s = 0$, that is exactly the separable case – $h_2 = 4h_1$.□

The last case we haven't examined yet is $h_1 = h_2$.

Lemma 6. *The system with Hamiltonian (1.4) is non-integrable for $N = 2$, $h_2 = h_1$ and $h_1 \neq 0$.*

Proof. In this case, the hamiltonian is

$$H_0 = \frac{1}{2}({p_1}^2 + {p_2}^2) + \frac{(c_1 - c_2)}{2}{q_1}^2 + \frac{(4c_1 - c_2)}{2}{q_2}^2 + \frac{3}{4}h_1{q_1}^2{q_2}^2 + \frac{15}{8}h_1{q_2}^4$$

and we find a particular solution $(\widetilde{q_2}, \widetilde{p_2})$, of respective Hamiltonian system, namely

$$\left| \begin{array}{l} \widetilde{q_2} = \sqrt{\mu_1}sn\left(\frac{i\sqrt{(15h_1\mu_2)}}{2}t, \sqrt{\frac{\mu_1}{\mu_2}}\right) \\ \widetilde{p_2} = \dot{\widetilde{q_2}} \\ \widetilde{q_1} = 0 \\ \widetilde{p_1} = 0 \end{array} \right. \tag{3.10}$$

where μ_1, $\mu_2 \in \mathbb{C}$ and $|\mu_1| < |\mu_2|$, μ_1 and μ_2 are the roots of the equation

$$-\frac{15h_1}{4}\mu^2 + (c_2 - 4c_1)\mu + 2H_0 = 0.$$

The normal variational equations (NVE) for $j = 1$ around the solution (3.10) is

$$\left| \begin{array}{l} \dot{\xi}_1 = \eta_1 \\ \dot{\eta}_1 = -\xi_1\left((c_1 - c_2) + \frac{3}{2}h_1(\widetilde{q_2})^2\right) \end{array} \right. .$$

Hence we get the second-ordered differential equation

$$\ddot{\xi}_1 + \xi_1\left((c_1 - c_2) + \frac{3}{2}h_1(\widetilde{q_2})^2\right) = 0$$

and having in mind the Laurant's expansion of (3.10), we write the indicial equation

$$\rho^2 - \rho - \frac{2}{5} = 0,$$

whose roots are not integers, therefore the monodromy group is not abelian, which proves nonintegrabitity in the case $h_1 = h_2$. □

This concludes the proof of Theorem 2.

4. NUMERICAL EXPERIMENTS

Practically the integrability of a Hamiltonian system can be examined with so called "Poicaré sections". Let there be a Hamiltonian system in $\mathbb{R}^{2n}$

$$\dot{z} = X_H(z)$$

with Hamiltonian H and a periodic solution – $z = z(t)$, let Γ be the respective phase curve. We build a transversal intersection S to Γ and the solution $z = z(t)$ crosses S in a point z_0. In a sufficiently small neighborhood U of S, containing z_0, we look at those solutions of the Hamiltonian system, which have initial conditions in U. We always take solutions, whose initial conditions lay on the same energy level $H = E$. We draw the consecutive intersection points where these paths cross S. This mapping $P : S \to S$ is called "Poincaré mapping".

If the intersection points form regular curves, then we suppose integrability. If a chaotic picture is obtained, then we conclude that the Hamiltonian system is non-integrable.

In practice we examine two-dimensional intersection S and here are some Poincaré sections for our system, drawn by Maple.

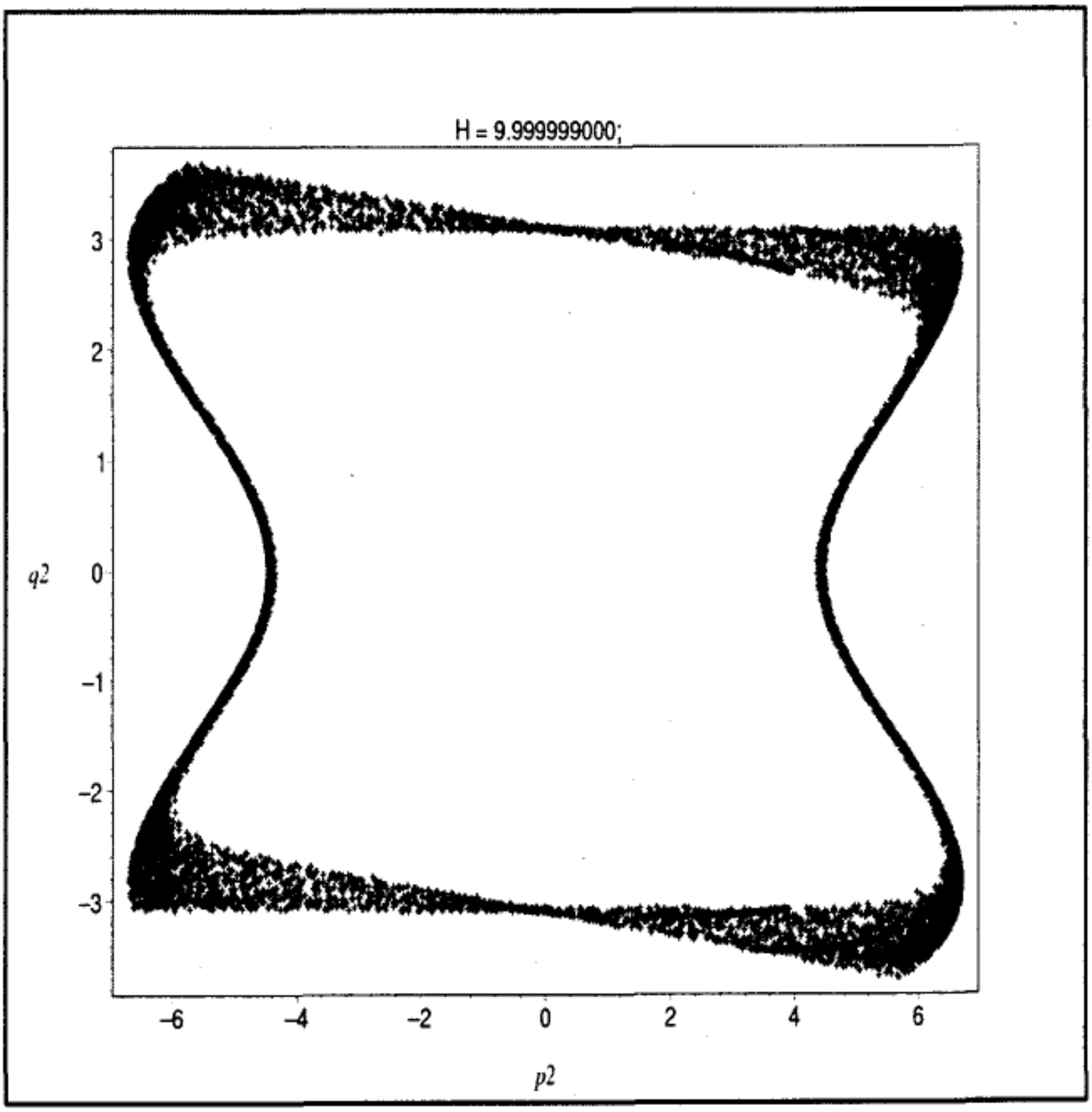

Fig. 1. $c_1 = -1.2, c_2 = 1.3, h_1 = 0, h_2 = -1.5, S = (p_1, q_1)$

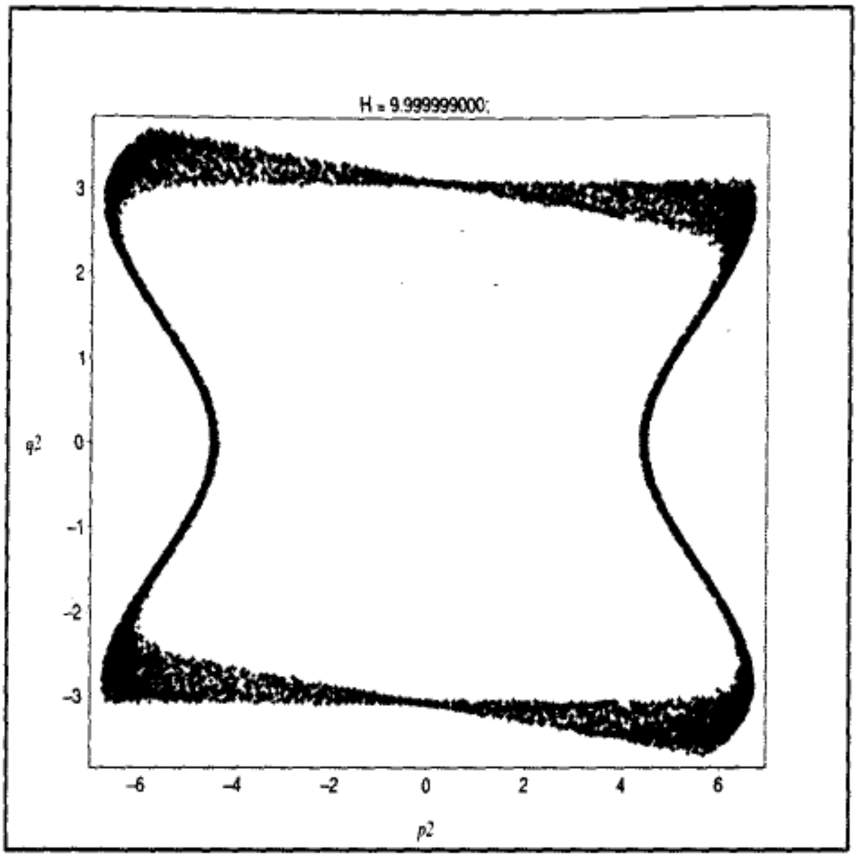

Fig. 2. $c_1 = -1.2, c_2 = 1.3, h_1 = 0, h_2 = -1.5, S = (p_2, q_2)$

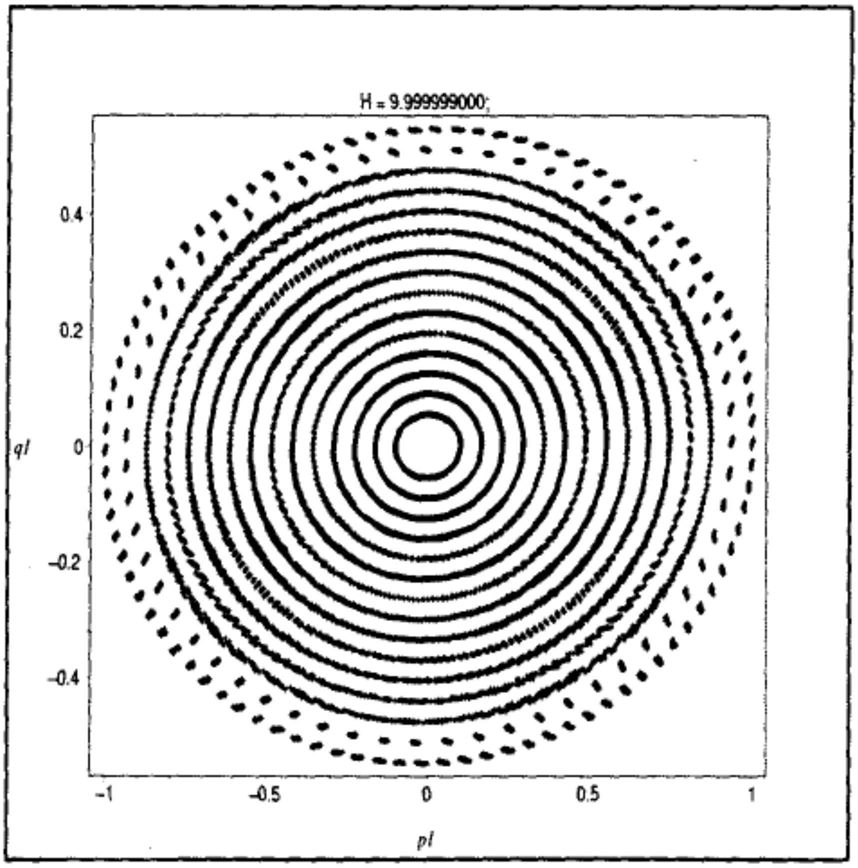

Fig. 3. $c_1 = 2.2, c_2 = -2.3, h_1 = 0, h_2 = 1.5, S = (p_1, q_1)$

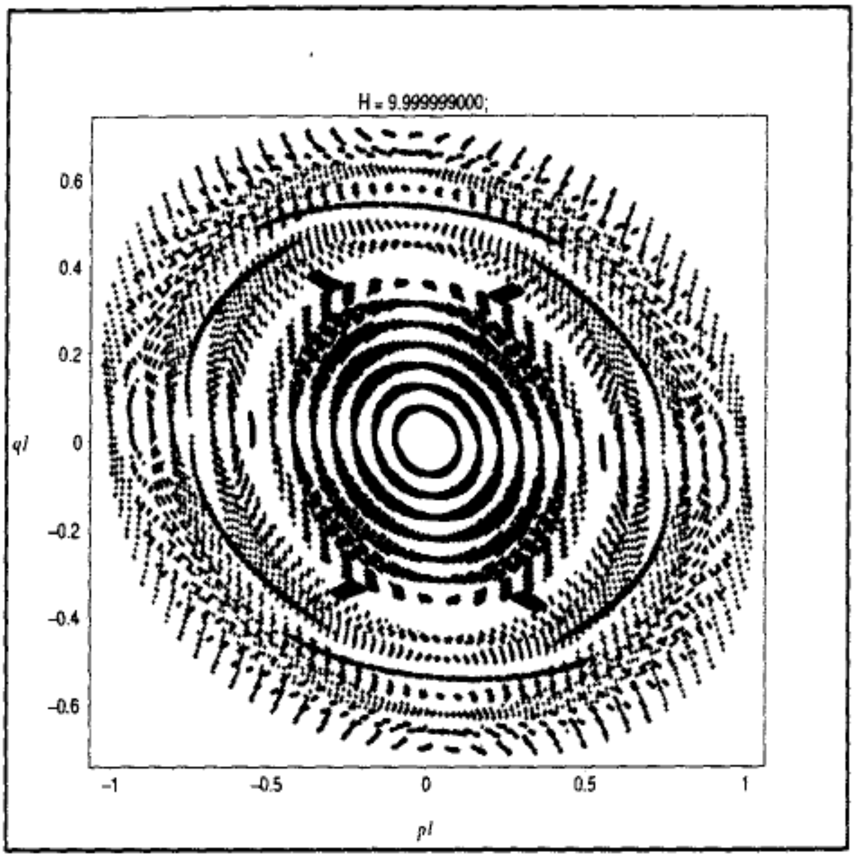

Fig. 4. $c_1 = -1.2, c_2 = 1.3, h_1 = 1, h_2 = -1.5, S = (p_1, q_1)$

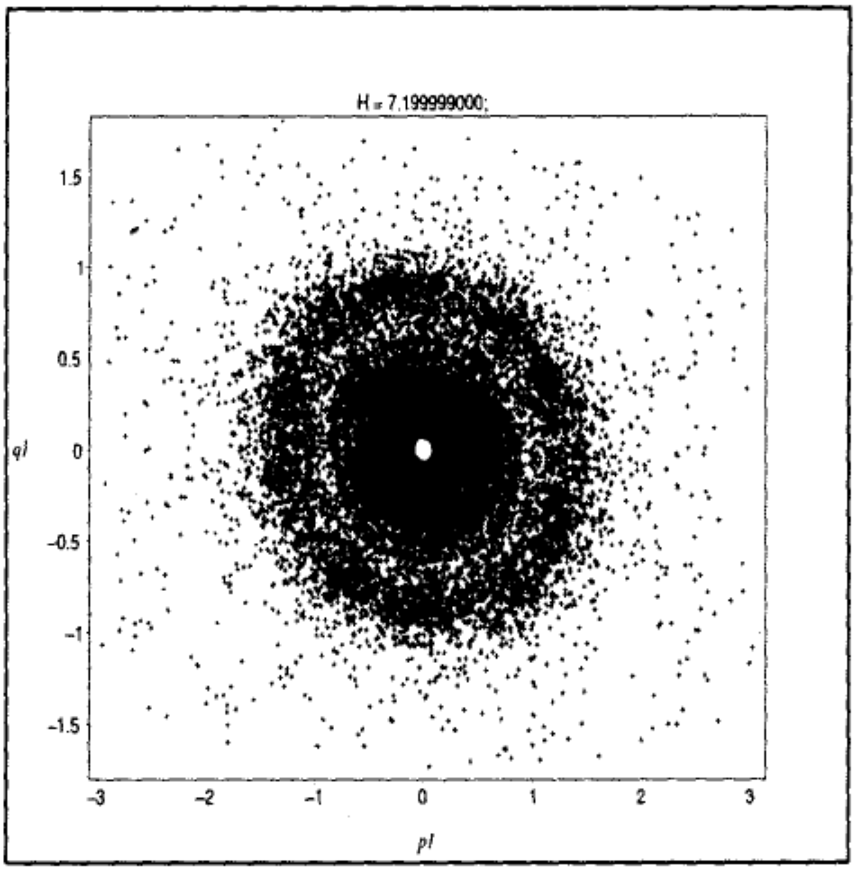

Fig. 5. $c_1 = -1.2, c_2 = 1.3, h_1 = 1.5, h_2 = -1.5, S = (p_1, q_1)$

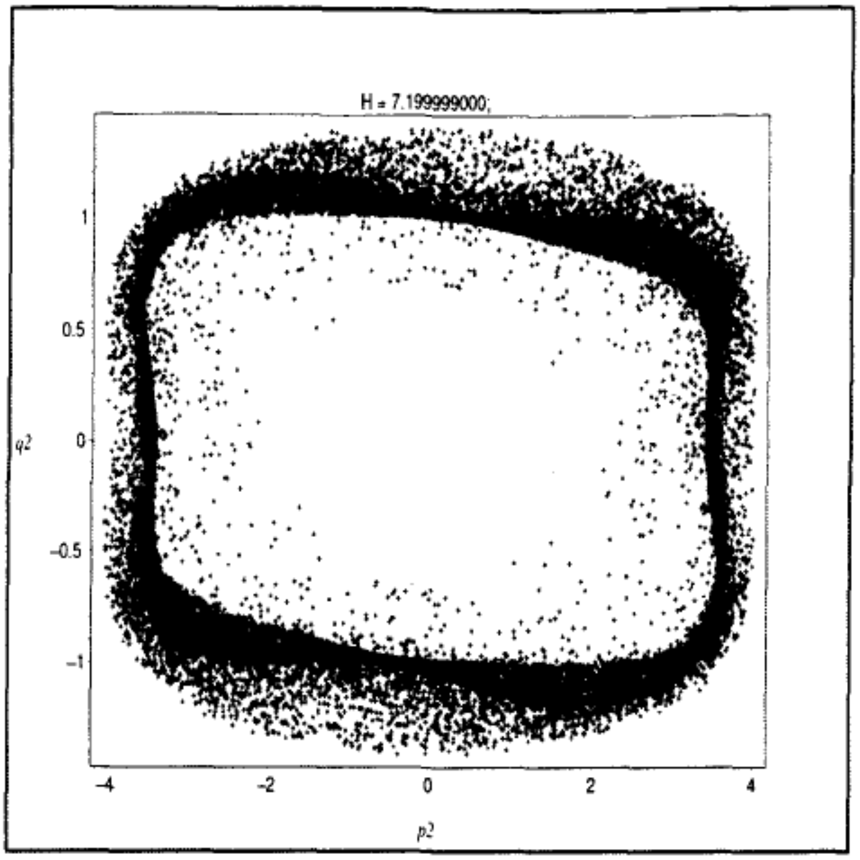

Fig. 6. $c_1 = -1.2, c_2 = 1.3, h_1 = 1.5, h_2 = -1.5, S = (p_2, q_2)$

Acknowledgements. The first author is partially supported by Grant23/2006 with FNI of Sofia University. The second author acknowledges the support from grant MM 1504/05 with NSF Bulgaria.

REFERENCES

1. Nishida, T. A note on the nonlinear vibrations of the elastic string, *Mem. Fac. Eng. Kyoto Univ.*, 1971, 329-341.
2. Dickey, R. Stability of periodic solutions of the nonlinear string, *Quaterly of Appl. Math.*, 1980, 253-259.
3. Iliev, Hr. On the non-integrability of a Hamiltonian system resulting from a problem for elastic string, *Ann. de L'Univ. de Sofia, Fac. de Math. et Inform.*, **88**, 1994, 57 - 69.
4. Yagasaki, K. Nonintegrability of an Infinite-Degree-of-Freedom Model for Unforced and Undampted, Straight Beams, *Appl. Mechanics*, **70**, 2003, 732-738.
5. Perelomov, A. M. Integrable Systems of Classical Mechanics and Lie Algebras, **1**, Birkhäuser, 1990, 186-187.
6. Morales Ruiz, J. J. Differential Galois theory and non-integrability of Hamiltonian systems. Birkhäuser, Basel, 1999.
7. Singer, M. F. An outline of Differential Galois Theory. In: Computer Algebra and Differential Equations. E. Tournier, Ed., Academic Press, New York, 1989, 3-57.

8. Beukers, F. Differential Galois Theory. In: From Number Theory to Physics. M. Waldschmidt (Eds.), Springer-Verlag, New York, 1992, 413-439.

9. Arnold, V. I. Mathematical methods of classical mechanics. Springer, Berlin - Heidelberg - New York, 1978.

10. Poincaré, H. Methodes nouvelle de la méchanique céleste, v. 1 - 3, Gauthier - Villars, Paris, 1899.

11. Ziglin, S. Branching of solutions and non-existence of first integrals in Hamiltonian mechanics, *Func. Anal. Appl.*, I - **16**, 1982, 30 - 41; II - **17**, 1983, 8-23.

12. Coddington, E. A., N. Levinson. Theory of ordinary differential equations, McGraw-Hill, New York - Toronto - London, 1965.

13. Golubev, V. V. Lectures on Analytical Theory of Differential Equations, Gosud. Isdat. Teh. Teor. Lit., Moscow, 1950.

14. Wittaker, E., G. Watson. A Course of Modern Analysis, Cambridge University Press, 1927.

Received on November 15, 2006

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: petia_brainova@abv.bg
E-mail: christov@fmi.uni-sofia.bg