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COHESIVE POWERS OF COMPUTABLE STRUCTURES

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We develop the notion of cohesive power $\mathcal{B}$ of a computable structure $\mathcal{A}$ over a cohesive set R. In the main theorem of this paper we prove certain connections between satisfaction of different formulas and sentences in the original model $\mathcal{A}$ and its cohesive power $\mathcal{B}$. We also prove various facts about cohesive powers, isomorphisms between them and consider an example in which the structure $\mathcal{A}$ is a computable field.

1. INTRODUCTION

In the study of the structure of the lattice $\mathcal{L}^*(V_\infty)$ we came upon a field with elements that are partial computable functions. We noticed that the construction of the field had certain similarities with the classical model theoretic ultrapower construction. We are now studying similar structures in a more general setting. We introduce the notion of cohesive power of a computable structure and prove an analogue of the fundamental theorem for ultraproducts [1]) for cohesive powers. The connection of cohesive powers of computable fields and the structure of $\mathcal{L}^*(V_\infty)$ is described in the concluding remarks.

A set R is cohesive if for every computably enumerable (c.e.) set W either $W \cap R$ or $\overline{W} \cap R$ is finite. There are continuum many cohesive subsets of ω. There are cohesive sets with computably enumerable complements. The c.e. complements of such cohesive sets are called maximal. For a fixed computable structure $\mathcal{A}$ and a cohesive set R we define the $R-$cohesive power $\mathcal{B}$ of $\mathcal{A}$. The satisfaction of sentences in $\mathcal{B}$ is connected to the existence of decision procedures for different segments of the complete diagram of $\mathcal{A}$. If $\mathcal{A}$ is a decidable structure, then $\mathcal{A}$ and $\mathcal{B}$ will be elementarily equivalent. If $\mathcal{A}$ is computable then $\mathcal{A}$ and $\mathcal{B}$ satisfy the same Π_2 and Σ_2 sentences.

We will use $\varphi_0, \varphi_1, \ldots$ to refer to arbitrary partial computable (p.c.) functions. Also, we assume a fixed enumeration $\phi_0, \phi_1, \ldots$ of the (unary) partial computable functions. We will write $\phi_{e,s}(x) = y$ if $e, x, y < s$ and y is the result of the $e-th$ computation on input n in less than s steps. In this case we will also write $\phi_{e,s}(x)\downarrow$. By $\phi_e(x)\downarrow$ we mean that $\exists s[\phi_{e,s}(x) = y]$. The enumeration of the e-th c.e. set $W_e = dom(\phi_e)$ is given as $W_{e,s} = dom(\phi_{e,s})$. We let use normal equality symbol $=$ (instead of $\simeq$) between partial computable functions. In definitions of p.c. functions we will assume that the function on the left side is defined when all of the elements on the right hand side are defined and the expression is acceptable for the particular values of the functions. For example, $\varphi = \frac{\psi_1}{\psi_2}$ means that

$$\varphi(x) = \begin{cases} \frac{\psi_1(x)}{\psi_2(x)} & \text{if } \psi_1(x)\downarrow,\ \psi_2(x)\downarrow, \text{ and } \psi_2(x) \neq 0 \\ undefined & \text{otherwise} \end{cases}.$$

2. MAIN RESULT

Let $\mathcal{A}$ be a computable structure over a fixed computable language L and let $R \subset \omega$ be a cohesive set. If Ψ is a formula in L, then we will use $\{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$ as a shorthand for

$$\{x : \exists s \exists t_1 \ldots \exists t_n (\bigwedge_{i=1}^{n} (\varphi_{i,s}(x) = t_i) \wedge \mathcal{A} \models \Psi(t_1, \ldots, t_n))\}.$$

Definition 2.1. *The cohesive power of $\mathcal{A}$ over R is a structure $\mathcal{B}$ (denoted $\prod_R \mathcal{A}$) in L such that:*

1. $B = \{\varphi : \varphi$ *is a p.c. function,* $R \subseteq^* dom(\varphi)$, $rng(\varphi) \subseteq A\} / =_R$

   Here $\varphi_1 =_R \varphi_2$ *if* $R \subseteq^* \{x : \varphi_1(x)\downarrow = \varphi_2(x)\downarrow\}$. *The equivalence class of* φ *w.r.t.* $=_R$ *will be denoted by* $[\varphi]_R$ *or simply* $[\varphi]$ *when the set* R *is fixed.*

2. *If* $f \in L$ *is an n-ary functional symbol, then* $[f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ *is the equivalence class of a p.c. function such that*

   $$f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)).$$

3. *If* $P \in L$ *is an m-ary predicate symbol, then* $P^{\mathcal{B}}$ *is a relation such that*

   $$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \text{ iff } R \subseteq^* \{x : P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

4. *If* $c \in L$ *is a constant symbol, then the interpretation of* c *in* $\mathcal{B}$ *is the equivalence class of the total computable function with constant value* c^A.

The domains of the partial computable functions in the definition above contain the set R and form a filter in the lattice $\mathcal{E}$. The role that the cohesiveness of R plays in the theorem below is similar to the role the maximality of the ultrafilter plays in the ultraproduct construction.

Theorem 2.1. *(Fundamental theorem of cohesive powers)*

1. *If $\tau(y_1, \ldots, y_n)$ is a term in L and $[\varphi_1], \ldots, [\varphi_n] \in B$, then $[\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ is the equivalence class of a p.c. function such that $\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = \tau^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x))$.*

2. *If $\Phi(y_1, \ldots, y_n)$ is a formula in L that is a boolean combination of Σ_1 and Π_1 formulas and $[\varphi_1], \ldots, [\varphi_n] \in B$, then*

$$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n]) \text{ iff } R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$

3. *If Φ is a Π_3 sentence in L, then $\mathcal{B} \models \Phi$ implies $\mathcal{A} \models \Phi$.*

4. *If Φ is a Π_2 (or Σ_2) sentence in L, then $\mathcal{B} \models \Phi$ iff $\mathcal{A} \models \Phi$.*

Proof. (1) The proof is straightforward but we note that we essentially use the fact that the operations in $\mathcal{A}$ are computable.

(2) We proceed by induction:

(2.1) Let $\Phi(y_1, \ldots, y_n) = P(\tau_1(y_1, \ldots, y_n), \ldots, \tau_m(y_1, \ldots, y_n))$ be an atomic formula and suppose $[\psi_i] = \tau_i^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$
iff
$\mathcal{B} \models P([\psi_1], \ldots, [\psi_m])$
iff
$R \subseteq^* \{x : \mathcal{A} \models P(\psi_1(x), \ldots, \psi_m(x))\}$
iff
$R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_m(x))\}$

(2.2) Suppose $\Phi(y_1, \ldots, y_n) = \Phi_1(y_1, \ldots, y_n) \wedge \Phi_2(y_1, \ldots, y_n)$ and the claim is true for $\Phi_i(y_1, \ldots, y_n)$ $i = 1, 2$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$
iff
$\mathcal{B} \models \Phi_1([\varphi_1], \ldots, [\varphi_n])$ and $\mathcal{B} \models \Phi_2([\varphi_1], \ldots, [\varphi_n])$
iff
$R \subseteq^* \{x : \mathcal{A} \models \Phi_1(\varphi_1(x), \ldots, \varphi_n(x))\}$ and
$R \subseteq^* \{x : \mathcal{A} \models \Phi_2(\varphi_1(x), \ldots, \varphi_n(x))\}$

iff

$R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}$.

(2.3) Suppose $\Phi(y_1, \ldots, y_n) = \exists y \Psi(y, y_1, \ldots, y_n)$ and $\Psi(y, y_1, \ldots, y_n)$ is a quantifier free formula for which the claim is true.

(2.3a) Suppose $\mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n])$ and suppose that the p.c. function φ is such that $\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n])$. By the inductive hypothesis $R \subseteq^* \{x : \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\}$ and so

$$R \subseteq^* \{x : \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\}.$$

(2.3b) Suppose $R \subseteq^* \{x : \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\}$. Since the structure $\mathcal{A}$ is computable and $\Psi(y, y_1, \ldots, y_n)$ is quantifier free we can define a partial computable

$$\varphi(x) = \mu y \in A[\mathcal{A} \models \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))].$$

Then

$$\begin{aligned} \{x \;&:\; \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\} = \\ \{x \;&:\; \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\} \end{aligned}$$

and $R \subseteq^* \{x : \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\}$. By the inductive hypothesis $\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n])$ and so $\mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n])$.

(2.4) Suppose $\Phi(y_1, \ldots, y_n) = \neg \Psi(y_1, \ldots, y_n)$ and $\Psi(y_1, \ldots, y_n)$ is a Σ_1 formula for which the hypothesis is true.

(2.4a) Suppose $\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$ and let

$$D = \{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$

Since $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$, then $R \not\subseteq^* D$. Because $\Psi(y_1, \ldots, y_n)$ is a Σ_1 formula and φ_i for $i \leq n$ are p.c., then D is a c.e. set. Since R is cohesive we have $R \cap D =^* \emptyset$. Also, since $R \subseteq^* \bigcap_{i=1}^{n} dom(\varphi_i)$, then for almost all $x \in R$ we have $\mathcal{A} \not\models \Psi(\varphi_1(x), \ldots, \varphi_n(x))$.Therefore $R \subseteq^* \{x : \mathcal{A} \models \neg \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$.

(2.4b) Suppose $R \subseteq^* \{x : \mathcal{A} \models \neg \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$. Then

$$R \cap \{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\} =^* \emptyset$$

and by the inductive hypothesis $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$. Therefore

$$\mathcal{B} \models \neg \Psi([\varphi_1], \ldots, [\varphi_n]).$$

(3) Let $\Phi = \forall y \exists z \forall t \Psi(y, z, t)$ where $\Psi(y, z, t)$ is a quantifier free formula. Let $c \in A$ be arbitrary and let $\varphi_c(x) = c$ for every $x \in \omega$. Let $[\varphi] \in B$ be such that $\mathcal{B} \models \forall t \Psi([\varphi_c], [\varphi], t)$. By (2) above we have $R \subseteq^* \{x : \mathcal{A} \models \forall t \Psi(\varphi_c(x), \varphi(x), t)\}$.

Then $R \subseteq^* \{x : \mathcal{A} \models \exists z \forall t \Psi(c, z, t)\}$. The set is R is nonempty and x is not a free variable of $\exists z \forall t \Psi(c, z, t)$. Therefore $\mathcal{A} \models \exists z \forall t \Psi(c, z, t)$ and so $\mathcal{A} \models \Phi$.

(4) Let $\Phi = \forall y \exists z \Psi(y, z)$ where $\Psi(y, z)$ is a quantifier free formula.

(4a) The fact, that $\mathcal{A} \models \Phi$ whenever $\mathcal{B} \models \Phi$, follows from (3).

(4b) Suppose that $\mathcal{A} \models \Phi$ and let $[\varphi] \in B$ be arbitrary. We have that $R \subseteq^* dom(\varphi) = \{x : \mathcal{A} \models \exists z \Psi(\varphi(x), z)\}$. By (2), $\mathcal{B} \models \exists z \Psi([\varphi], z)$ and so $\mathcal{B} \models \Phi$. □

Note that if the structure $\mathcal{A}$ is decidable, then we can similarly prove the following:

Theorem 2.2. *If $\mathcal{A}$ is a decidable structure, then*

1. *If $\Phi(y_1, \dots, y_n)$ is a formula in L, and $[\varphi_1], \dots, [\varphi_n] \in B$, then*

$$\mathcal{B} \models \Phi([\varphi_1], \dots, [\varphi_n]) \text{ iff } R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \dots, \varphi_n(x))\}.$$

2. *If Φ is a sentence, then*

$$\mathcal{B} \models \Phi \text{ iff } \mathcal{A} \models \Phi.$$

Proof. (1) The proof is almost identical to the proof of part (2) of the main theorem. We note only that for any formula $\Psi(y_1, \dots, y_n)$ the set $\{(a_1, \dots, a_n) : \mathcal{A} \models \Psi(a_1, \dots, a_n)\}$ is computable. Then the set $\{x : \mathcal{A} \models \Psi(\varphi_1(x), \dots, \varphi_n(x))\}$ is c.e. and steps 2.3 and 2.4 of the proof above can be carried for any formula Ψ.

(2) Follows directly from (1). □

Definition 2.2. *For $c \in A$ let $[\varphi_c] \in B$ be the equivalence class of the total function φ_c such that $\varphi_c(x) = c$ for every $x \in \omega$. The map $d : A \to B$ such that $d(c) = [\varphi_c]$ is called the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$.*

Proposition 2.1. *The following hold:*

1. *If the structure $\mathcal{A}$ is finite, then $\mathcal{B} \cong \mathcal{A}$.*

2. *If the structure $\mathcal{A}$ is decidable, then the canonical map d is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$.*

3. *If $\Phi(y_1, \dots, y_n)$ is a Π_2 or a Σ_2 formula in L and $c_1, \dots, c_n \in A$, then*

$$\mathcal{A} \models \Phi(c_1, \dots, c_n) \text{ iff } \mathcal{B} \models \Phi(d(c_1), \dots, d(c_n)).$$

Proof. (1) Let $[\varphi] \in B$ be arbitrary. For any $c \in A$ let $X_c = \{x : \varphi(x) = c\}$ and notice that X_c is a c.e. set. Since $dom(\varphi) = \bigcup_{c \in A} X_c$ and A is finite, then for some $c_1 \in A$ the set $X_{c_1} \cap R$ is infinite. Since R is cohesive we have $R \subseteq^* X_{c_1}$ and therefore $[\varphi] = [\varphi_{c_1}]$. Therefore all equivalence classes in B correspond to the constants in A and the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$ is a 1-1 map. So $\mathcal{B} \cong \mathcal{A}$ follows directly from the definition of $\mathcal{B}$.

(2) Let $\Phi(y_1, \ldots, y_n)$ be a Σ_2 (or Π_2) formula and let $c_1, \ldots, c_n \in A$. If $\mathcal{A}$ is decidable, then

$\mathcal{B} \models \Phi([\varphi_{c_1}], \ldots, [\varphi_{c_n}])$ iff

$R \subseteq^* \{x : \mathcal{A} \models \Phi(c_1, \ldots, c_n)\}$ iff

$\mathcal{A} \models \Phi(c_1, \ldots, c_n)$.

(3) Let $c_1, \ldots, c_n \in A$ and let $L_C = L \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be the language L expanded by adding a constant symbol for each c_i. Let $\mathcal{A}_C$ be the structure $\mathcal{A}$ with the constant symbols $\mathbf{c}_1, \ldots, \mathbf{c}_n$ interpreted as $c_1, \ldots, c_n$ correspondingly. Let $\mathcal{B}_C$ be the $R-$cohesive power of $\mathcal{A}_C$. Then $\Phi(c_1, \ldots, c_n)$ will be a Σ_2 (or Π_2) sentence in L_C and by the Fundamental theorem part (4)

$$\mathcal{A}_C \models \Phi(\mathbf{c}_1, \ldots, \mathbf{c}_n) \text{ iff } \mathcal{B}_C \models \Phi(\mathbf{c}_1, \ldots, \mathbf{c}_n)$$

which is equivalent to

$$\mathcal{A} \models \Phi(c_1, \ldots, c_n) \text{ iff } \mathcal{B} \models \Phi(d(c_1), \ldots, d(c_n)). \square$$

Definition 2.3. *Two sets A, B have the same 1-degree up to $=^*$ (denoted $A \equiv_1^* B$) if there are $A_1 =^* A$ and $B_1 =^* B$ such that $A_1 \equiv_1 B_1$.*

Proposition 2.2. *If $M_1 \equiv_1^* M_2$ are maximal sets, $\mathcal{B}_1 = \prod_{\overline{M_1}} \mathcal{A}$. and $\mathcal{B}_2 = \prod_{\overline{M_2}} \mathcal{A}$. then $\mathcal{B}_1 \cong \mathcal{B}_2$.*

Proof. Let $M_i' =^* M_i$ for $i = 1, 2$ be such that $M_1' \equiv_1 M_2'$. Let $\mathcal{B}_i' = \prod_{\overline{M_i'}} \mathcal{A}$ and notice that $\mathcal{B}_i' \cong \mathcal{B}_i$ for $i = 1, 2$. Using Myhill Isomorphism Theorem (see [6, p.24]) we let σ be a computable permutation of ω such that $\sigma(M_1') = M_2'$. Define a map $\Phi : B_2' \to B_1'$ as follows:

$\Phi([\psi]) = [\varphi]$ where $\varphi(x) = \psi(\sigma(x))$. We now prove that Φ is an isomorphism of $\mathcal{B}_2'$ and $\mathcal{B}_1'$:

(1) Notice that $\psi_1 =_{\overline{M_2'}} \psi_2$ iff $\overline{M_2'} \subseteq^* \{x : \psi_1(x) = \psi_2(x)\}$ iff $\overline{M_1'} \subseteq^* \{x : \psi_1(\sigma(x)) = \psi_2(\sigma(x))\}$ iff $\Phi(\psi_1) =_{\overline{M_1'}} \Phi(\psi_2)$. So Φ is correctly defined and injective. Finally, if $[\varphi] \in B_1'$ and $\psi(x) = \varphi(\sigma^{-1}(x))$, then $\Phi([\psi]) = [\varphi]$.

(2) Let $f \in L$ be an n-ary functional symbol.

Then $\Phi([f^{\mathcal{B}'_2}([\psi_1],\ldots,[\psi_n])])$ is the equivalence class of a p.c. function such that $\Phi([f^{\mathcal{B}'_2}([\psi_1],\ldots,[\psi_n])])(x) = f^{\mathcal{A}}(\psi_1(\sigma(x)),\ldots,\psi_n(\sigma(x)))$.

That means that $\Phi([f^{\mathcal{B}'_2}([\psi_1],\ldots,[\psi_n])]) = [f^{\mathcal{B}'_1}(\Phi([\psi_1]),\ldots,\Phi([\psi_n]))]$.

(3) If $P \in L$ is an m-ary predicate symbol, then
$P^{\mathcal{B}'_2}([\psi_1],\ldots,[\psi_n])$ iff
$\overline{M'_2} \subseteq^* \{x : P^{\mathcal{A}}(\psi_1(x),\ldots,\psi_m(x))\}$ iff
$\overline{M'_1} \subseteq^* \{x : P^{\mathcal{A}}(\psi_1(\sigma(x)),\ldots,\psi_m(\sigma(x)))\}$ iff
$P^{\mathcal{B}'_1}(\Phi([\psi_1]),\ldots,\Phi([\psi_n]))$.□

Proposition 2.3. *Every computable automorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{B}$.*

Proof. Let σ be a computable automorphism of $\mathcal{A}$. Define a map $\widetilde{\sigma}$ on B as follows:

$$\widetilde{\sigma}([\varphi]) = [\psi]$$

where $\psi(n) = \sigma(\varphi(n))$. The proof that $\widetilde{\sigma}$ is an automorphism of $\mathcal{B}$ is straightforward. Notice also that if $c \in A$ and $\varphi(n) = c$ for almost every $n \in R$, then $\widetilde{\sigma}([\varphi])(n)$ is the constant $\sigma(c)$ for almost every $n \in R$. □

Example. Let F be a computable field and let I be a maximal set. Then $\widetilde{F} = \prod_{\overline{I}} F$ is a field such that:

1. $\widetilde{F} \cong F$ if F is finite,

2. If $[\varphi] \in \widetilde{F}$ is algebraic over F, then φ is a constant function on $\overline{I}$.

3. Every computable automorphism σ of F can be extended naturally to an automorphism $\widetilde{\sigma}$ of $\widetilde{F}$.

Proof. We will prove only (2), (1) and (3) follow directly from the propositions above. Suppose $[\varphi] \in \widetilde{F}$ is root of a polynomial $g(x) \in F[x]$. Extend the language of F by adding new constants for each coefficient of the polynomial g. Let $\widetilde{F}_1$ be the cohesive power of F over $\overline{I}$ in the extended language. By the fundamental theorem of cohesive powers we have

$$\widetilde{F}_1 \models (g([\varphi]) = 0^{\widetilde{F}_1}) \text{ iff } \overline{I} \subseteq^* \{x : F \models (g(\varphi(x)) = 0^F)\}.$$

This means that $\varphi(x) \in F$ is a root of the polynomial $g(x)$ for almost every $x \in \overline{I}$. Since $g(x)$ can have finitely many roots, then $C = \{c : \exists x[(\varphi(x) = c) \wedge (g(c) = 0)]\}$ is finite. For each $c \in C$ let $X_c = \{x : \varphi(x) = c\}$. Notice that X_c is c.e.. Using the fact that $\overline{I}$ is cohesive we notice that

$$\forall c_1, c_2 \in C[c_1 \neq c_2 \rightarrow (|X_{c_1} \cap \overline{I}| < \infty \text{ or } |X_{c_2} \cap \overline{I}| < \infty)].$$

Since C is finite this implies that for some $c \in C$ we will have $\overline{I} \subseteq^* X_c$. This means that $[\varphi]$ is the equivalence class of a function that has value c on $\overline{I}$. □

3. CONCLUDING REMARKS

As we mentioned an example of cohesive powers appears naturally in the study of the structure of the lattice of subspaces of the fulliy effective vector space V_∞ over a computable field F. The lattice of computably enumerable subspaces of V_∞ modulo finite dimension is denoted $\mathcal{L}^*(V_\infty)$. The study of V_∞ was initiated by Metakides and Nerode in [5]. The lattice $\mathcal{L}^*(V_\infty)$ is an interesting modular analog of $\mathcal{E}^*$, the extensively studied (see [6]) lattice of c.e. sets modulo finite sets. Different cohesive powers of the field F appear (see [3]) in the characterization of principal filters of closures of quasimaximal sets.

Let $Q = \bigcap_{i=1}^{n} I_i$ where I_i $(i \le n)$ are maximal subsets of I_0–a fixed computable basis of V_∞. Suppose that I_i $(i \le n)$ are partitioned into k equivalence classes with respect to the relation $\equiv_1^*$ of having the same 1-degree up to $=^*$. Suppose that the $i-th$ equivalence class has n_i elements. In [3] we proved that the principal filter in $\mathcal{L}^*(V_\infty)$ of the linear span of Q is isomorphic to the product of the lattices $(\mathcal{L}(n_i, \widetilde{F}_i))_{i=1}^k$. Here $\mathcal{L}(n_i, \widetilde{F}_i)$ is the lattice of subspaces of an n_i-dimensional vector space over a field $\widetilde{F}_i$. The field $\widetilde{F}_i$ is the cohesive power of F w.r.t. a cohesive set R_i that is the complement of a maximal set from the i-th equivalence class described above.

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