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THE BIQUARD CONNECTION ON RIEMANNIAN QUATERNIONIC CONTACT MANIFOLDS [1]

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The conformal infinity of a quaternionic-Kähler metric on a $4n$ dimensional manifold with boundary is a codimension 3-distribution on the boundary called quaternionic contact structure. In order to study such structures O.Biquard [1] has introduced a unique connection which preserves the structure and whose torsion tensor satisfies some conditions. This paper is devoted to obtaining an explicit formula for the torsion tensor and for the connection itself.

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1. INTRODUCTION

The quaternionic contact structures have been introduced by O.Biquard in [1] and [2]. Namely, a quaternionic contact structure on a (4n+3)-dimensional smooth manifold X is a codimension 3 distribution V such that at each point $x \in X$ the nilpotent Lie algebra $V_x \oplus T_xX/V_x$ is isomorphic to the quaternionic Heisenberg algebra $H^m \oplus Im\mathbf{H}$, where nilpotent Lie algebra structure is defined by

$$[a,b] = \begin{cases} \pi_{T_xX/V_x}[a,b] & \text{if } a,b \in V_x \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

and the Heisenberg algebra structure is given by the formula

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$$\left[\sum_{i=1}^{m} x_i e_i, \sum_{i=1}^{m} x_i e_i\right] = Im \sum_{i=1}^{m} \bar{x}_i y_i. \tag{1.2}$$

This is equivalent to the existence of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$such that $V = Ker\ \eta$ and the three 2-forms $d\eta_{i|V}$ are the fundamental 2-forms of a quaternionic structure on V. The 1-form η is given up to the action of SO_3 on $\mathbb{R}^3$ and up to a conformal factor.

If we pick up such a 1-form η (globally defined), we obtain the quaternionic structure on V defining the three endomorphisms $I_i = (d\eta_{k|V})^{-1} \circ (d\eta_{j|V}) : V \to V$, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$. Obviously, this quaternionic structure does not depend on the choice of η. We also define the metric g on V by $g(X, Y) = d\eta_s(X, I_s Y)$. This metric is given up to the conformal factor because it depends on the conformal factor of η. Further, Biquard has shown ([1]) that there exists a unique triple of vector fields $\{R_1, R_2, R_3\}$, which satisfy $\eta_s(R_k) = \delta_{sk}$, $(i_{R_s} d\eta_s)_{|V} = 0$ and $(i_{R_s} d\eta_k)_{|V} = -(i_{R_k} d\eta_s)_{|V}$. Using this triple we define the metric g on the whole $T_x X$, putting $sp\{R_1, R_2, R_3\} \perp V$ and $g(R_s, R_k) = \delta_{sk}$. This metric does not depend on the action of SO_3 but it depends on the conformal factor of η. In my exposition I will assume that this metric is fixed. Also, in order to capture the 3-Sasaki structures, I will assume that the fundamental 2-forms of a quaternionic structure on V are $\frac{1}{2} d\eta_{i|V}$ instead of $d\eta_{i|V}$. Obviously, these assumptions make no restriction to the general case.

I also assume throughout the paper that the dimension of the base manifold is $4m + 3 > 7$. The case dim X=7 needs a special approach ([3]).

The interest in quaternionic contact structures is motivated by the result of Biquard [1] on Einstein deformations of $\mathbf{H}H^m$, which asserts that if a quaternionic contact structure on S^{4m-1} is close enough to the standard one, then it is a conformal infinity of complete Einstein metric. This result of Biquard is a generalization of a Graham-Lee [4] theorem on Einstein deformations of real hyperbolic space.

In the Salamon's [5] construction of the twistor space of a quaternionic Kähler manifold one uses the Levi-Civita connection to define the horizontal space for the fibration. In the case of quaternionic contact structure, there is no canonical connection. So using the analogy with the Tanaka-Webster [6] connection in CR geometry, Biquard [1] has introduced a unique contact quaternionic connection which I will call the Biquard connection.

This paper is devoted to study the properties of the Biquard connection. Many of its properties have been proved by Biquard [1], but he did not prove an explicit formula for the torsion tensor. This I have done in Theorem 5.3 - (i), corollary 5.1 (together with Theorem 5.4) and corollaries 5.4, 5.5 and 5.6. The key point in calculating the torsion tensor is the formula for the tensor $\tilde{u}$ (see Corollary 5.1) which I have obtained redoing in completely different way the proof of the theorems 5.3 and 5.4.

2. BASIC DEFINITIONS

Let (M, g) be an orientable Riemannian manifold of dimension $4n+3 \geq 11$.

Definition 2.1. A triple (V, Q, φ) will be called an Almost Contact Quaternionic Structure on M if

(i) V is codimension 3 distribution on M

(ii) Q is an almost quaternionic structure on V and

(iii) φ is a linear map from $V^{\perp}$ to Q which preserves orientation and which sends the unit sphere of $V^{\perp}$ into a set of complex structures of Q.

Let $J_1, J_2, J_3 \in Q, J_1^2 = J_2^2 = J_3^2 = -1, J_1 J_2 = -J_2 J_1 = J_3$ be the usual quaternionic basis of Q. Then the set of all complex structures in Q could be thought as a two dimensional sphere $\{ \sum_i a^i J_i \mid \sum_i (a^i)^2 = 1 \}$. It is easy to see that another triple $\hat{J}_1, \hat{J}_2, \hat{J}_3 \in Q,\ \hat{J}_i = \sum_k a_i^k J_k$ forms quaternionic basis, too, if and only if the matrix $(a_i^k)_{3x3}$ belongs to $SO(3)$.

We will denote with W the 3 dimensional distribution $V^{\perp}$ and with S^2 the unit sphere in W . Let $\xi_1, \xi_2, \xi_3 \in S^2$. Then by definition $(\varphi(\xi_i))^2 = -1$ and

Lemma 2.1. *The triple $\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3)$ forms a quaternionic basis of Q if and only if ξ_1, ξ_2, ξ_3 is orthogonal and oriented basis of W.*

We will say that the map φ originates from the exterior derivative if across any point in M one can find orthonormal local basis $\{\xi_1, \xi_2, \xi_3\}$ of W such that $g(\varphi(\xi_i)X, Y) = \frac{1}{2} d(\flat\xi_i)(X, Y),\ X, Y \in V,\ i = 1, 2, 3$, where $\flat\xi_i(X) = g(\xi, X),\ X \in TM$.

Definition 2.2. An almost contact quaternionic structure (V, Q, φ) on M is called contact quaternionic structure if φ originates from the exterior derivative.

3. THE STRUCTURE GROUP

We consider the space $\mathbb{R}^{4n+3} = \mathbb{R}^{4n} \times \mathbb{R}^3 = V_0 + W_0$ with standard quaternionic structure Q_0 on $V_0 = \mathbb{R}^{4n}$. Let I_0, J_0, K_0 be the standard quaternionic basis on Q_0 and $\{e_1, e_2, e_3\}$ the standard basis on $W_0 = \mathbb{R}^3$. We consider the map $\varphi_0 : W_0 \to Q_0,\ \varphi_0(e_1) = I_0,\ \varphi_0(e_2) = J_0,\ \varphi_0(e_3) = K_0$. So we obtain a constant contact quaternionic structure (V_0, Q_0, φ_0) in $\mathbb{R}^{4n+3}$.

Let G denote the group of all endomorphisms of $O(4n+3)$ which preserve the structure (V_0, Q_0, φ_0). Obviously G is a subgroup of $SO(4n+3)$.

Theorem 3.1. *The manifold M admits an almost contact quaternionic structure if and only if its structural group could be reduced to the subgroup of G.*

Let $A \in Sp(n)Sp(1)$ and

$$\begin{aligned} AI_0A^{-1} &= a_1^1 I_0 + a_1^2 J_0 + a_1^3 K_0 \\ AJ_0A^{-1} &= a_2^1 I_0 + a_2^2 J_0 + a_3^3 K_0 \\ AK_0A^{-1} &= a_3^1 I_0 + a_3^2 J_0 + a_3^3 K_0 \ . \end{aligned}$$

Then the matrix $(a_i^k)_{3x3}$ belongs to $SO(3)$ and we obtain a homomorphism $\tau : Sp(n)Sp(1) \to SO(3)$.

Lemma 3.1. *The group G can be represented by*

$$G = \{ \ (A, \tau(A)) \mid A \in Sp(n)Sp(1) \ \}$$

Corollary 3.1. *The group G is isomorphic to $Sp(n)Sp(1)$.*

We denote this isomorphism with $\lambda : Sp(n)Sp(1) \to G$, $\lambda(A) = (A, \tau(A))$, $A \in Sp(n)Sp(1)$.

Let g be the Lie algebra of G. We will identify $\mathbb{R}^3$ with $sp(1)$. For any matrix $A \in sp(n) \oplus sp(1)$ let $a = (a_1, a_2, a_3)$ be its projection in $sp(1)$.

Lemma 3.2. *An endomorphism $t \in gl(4n+3, \mathbb{R})$ belongs on g if and only if there exists a matrix $A \in sp(n) \oplus sp(1)$ such that*

$$t(x+y) = Ax - 2a \wedge y, \ x \in \mathbb{R}^n, \ y \in \mathbb{R}^3.$$

Proof: We compute $(\lambda_{*1} A)(x+y) = Ax + y_1[A, I_0] + y_2[A, J_0] + y_3[A, K_0] = Ax + 2a \wedge y$, where a is the $sp(1)$ component of A, considered as an element of $\mathbb{R}^3$. □

4. CONTACT QUATERNIONIC CONNECTIONS

Let on the Riemannian manifold M be fix an almost contact quaternionic structure (V, Q, φ).

Definition 4.1. A Riemannian connection is called contact quaternionic connection if its holonomy group is contained in the group G.

Theorem 4.1. *An arbitrary Riemannian connection ∇ is contact quaternionic if and only if it satisfies the conditions:*

(i) *for any vector fields $X \in TM$ and $v \in V$, $\nabla_X v \in V$ (i.e. ∇ preserves V and $W = V^{\perp}$),* .

(ii) *∇ preserves Q,*

(iii) *$\nabla\varphi = 0$.*

Note: In fact, the condition (ii) follows from the other two.

For the rest of the paper we will assume that the fixed structure (V, Q, φ) on M is contact quaternionic, i.e. the map φ originates from the exterior derivative. This means that in some neighborhood of any point on M we can find a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, such that $V = Ker\ \eta$, the three 2-forms ω_i, defined by

$$\begin{aligned} \omega_{i|V} &= \frac{1}{2} d\eta_{i|V} \\ i_X\omega_i &= 0,\ X \in W \end{aligned} \tag{5.1}$$

are the fundamental 2-forms of the quaternionic structure Q and the three vector fields $(\#\eta_1, \#\eta_2, \#\eta_3)$ form an orthonormal basis of W such that $g(\varphi(\#\eta_i)(X), Y) = \omega_i(X, Y)$. We fix this form η and denote $\xi_i = \#\eta_i,\ I_i = \#\omega_i$. In particular we have

$$I_{i|W} = 0,\ I_iI_j = I_k,\ and\ I^2_{i|V} = -id_V \tag{5.2}$$

where (i,j,k) is any cyclic permutation of (1,2,3)

Let D denote the Levi-Civita connection on M and let π be the orthogonal projection from TM to V. We define $\nabla_X Y = \pi(D_X Y)$ for any two vector fields $X, Y \in V$. We may regard ∇ as a part of Riemannian connection which preserves the distribution V. Our purpose is to extend ∇ to the connection on all TM which preserves our contact quaternionic structure.

Let T denote the torsion of ∇. It is easy to see (Biquard [1]) that

$$T(X, Y) = -[X, Y]_W, \qquad X, Y \in V. \tag{5.3}$$

Theorem 5.1 (Biquard [1]). *∇ preserves the quaternionic structure Q of V if and only if*

(i) $i_{\xi_\alpha} d\eta_{\alpha|V} = 0,\ \alpha = 1, 2, 3$ *and*

(ii) $i_{\xi_\alpha} d\eta_{\beta|V} = -i_{\xi_\beta} d\eta_{\alpha|V},\ \alpha \neq \beta.$

More precisely, if these two conditions hold, we have

$$\nabla_X \omega_\alpha = -d\eta_\alpha(\xi_\beta, X)\omega_\beta + d\eta_\gamma(\xi_\alpha, X)\omega_\gamma, \tag{5.4}$$

where $X \in V$ and (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

For any p-form ω we denote $\pi\omega(X_1, ..., X_p) = \omega(\pi X_1, ..., \pi X_p)$.

Lemma 5.1. *We have*

$$2\omega_i = \pi d\eta_i = d\eta_i - \sum_{s=1}^{3} \eta_s \wedge (i_{\xi_s} d\eta_i) + \sum_{1 \leq s < t \leq 3} d\eta_i(\xi_s, \xi_t)\eta_s \wedge \eta_t$$

Proof: We have $2\omega_i(X,Y) = d\eta_i(X - \eta(X), Y - \eta(Y)) =$
$= d\eta_i(X,Y) - d\eta_i(X,\eta(Y)) - d\eta_i(\eta(X),Y) + d\eta_i(\eta(X),\eta(Y))$, etc. □

From now on we assume that the conditions of the Theorem 5.1 hold. Using the equation $\nabla\varphi = 0$, we are able to determine (Biquard [1]) the covariant derivative $\nabla_X\xi$, $X \in V, \xi \in W$. We have

$$\nabla_X\xi = [X,\xi]_W \tag{5.5}$$

Let H be a subgroup of $Gl(4n,\mathbb{R})$ and h be the corresponding Lie algebra. Suppose that on V is given an H-structure and an extension of our connection in form $\nabla_\xi X$ wich preserves the H-structure. Then, for the torsion T, we obtain

$$T(\xi,X) = \nabla_\xi X - \nabla_X\xi - [\xi,X] = \nabla_\xi X - [\xi,X]_V \ \in V \tag{5.6}$$

In particular, we may regard $T(\xi,.)$ as an endomorphism T_ξ of V.

Lemma 5.2 (Biquard [1]). *For any H-structure on V there exists a unique extension of our connection in form $\nabla_\xi X$ which preserves this structure and such that*

$$T_\xi \in h^\perp,\ \xi \in W.$$

Proof: Let $\hat\nabla$ be an arbitrary extension of the covariant derivative which preserves the H-structure. Then for any other extension ∇ which preserves the H-structure we have $\nabla_\xi X = \hat\nabla_\xi X + a_\xi(X)$, where a_ξ is an endomorphism of V and $a_\xi \in h$. We obtain

$$T(\xi,Y) = \hat T(\xi,X) + a_\xi(X),\ \xi \in W, X \in V.$$

Obviously the tenzor $a_\xi(X)$ might be chosen in a unique way. □

It follows the main theorem.

Theorem 5.2 (Biquard [1]). *If the conditions*
(i) $i_{\xi_\alpha} d\eta_{\alpha|V} = 0,\ \alpha = 1,2,3$
(ii) $i_{\xi_\alpha} d\eta_{\beta|V} = -i_{\xi_\beta} d\eta_{\alpha|V},\ \alpha \neq \beta$
are satisfied, there exists a unique contact quaternionic connection ∇ with torsion T such that
(i) $T(X,Y) = -[X,Y]_W,\ X,Y \in V$
(ii) $T_\xi \in (sp(n) \oplus sp(1))^\perp,\ \xi \in W$

We call this connection the Biquard connection.

One may decompose the tensor T_ξ(we regard it as an endomorphism of V which by the definition belongs to $(sp(n) \oplus sp(1))^\perp$) in two components: T^0_ξ - the symmetric one and a_ξ - the anti-symmetric. We have

$$T_\xi = T^0_\xi + a_\xi \tag{5.7}$$

Note: Through the Lemma 5.2 one can construct connection ∇^0 using the group $H = SO(4n)$ instead of $H = Sp(n)Sp(1)$. Then, according to the Lemma, $T'_\xi \in so(4n)^\perp$, where T' is the torsion of ∇^0. We have

$$T_\xi(X) = T'_\xi(X) + b_\xi(X), \ \xi \in W, X \in V \tag{5.8}$$

and since ∇ and ∇' both preserve the metric, $b_\xi \in so(V)$. So we obtain again the decomposition (5.7) and in particular $T^0_\xi = T'_\xi$.

My next aim is to calculate the torsion tensor T of the Biquard connection. Theorem 5.3 (ii) and (iii) and Theorem 5.4 were originally proved by Biquard [1], but in order to obtain an explicit formula for T I will remake there proofs in completely different way.

We will use the following well known lemma:

Lemma 5.3 (Biquard [1]). *Any endomorphism u of V might be decomposed uniquely:*

$$u = u^{+++} + u^{+--} + u^{-+-} + u^{--+},$$

where u^{+++} commutes with all three I_i, u^{+--} commutes with I_1 and anti-commutes with the others two and etc. In fact we have

$$4u^{+++} = u - I_1uI_1 - I_2uI_2 - I_3uI_3.$$

$$4u^{+--} = u - I_1uI_1 + I_2uI_2 + I_3uI_3.$$

$$4u^{-+-} = u + I_1uI_1 - I_2uI_2 + I_3uI_3.$$

$$4u^{--+} = u + I_1uI_1 + I_2uI_2 - I_3uI_3.$$

We define $L'_X(Y) = \pi L_X(Y)$, $X, Y \in TM$, where L denotes the Lee differentiation. If we regard the distribution V as a vector bundle over M, then we may regard L'_X and ∇_X as two differentiations of the tensor algebra of this vector bundle. In fact, for any differentiation of V we have the following useful lemma.

Lemma 5.4. *Let D be any differentiation of the tensor algebra of V. Then we have*

(i) $D(I_i)I_i = -I_iD(I_i), \ i = 1, 2, 3$

(ii) $I_1D(I_1)^{-+-} = I_2D(I_2)^{+--}$ (*The other two identities could be obtained through cyclic permutation of* $(1, 2, 3)$).

Proof: We calculate

$$0 = D(-Id_V) = D(I_iI_i) = D(I_i)I_i + I_iD(I_i)$$

and we obtain (i). To get (ii), we calculate

$$0 = D(I_1I_2 + I_2I_1) = I_1D(I_2) + D(I_1)I_2 + D(I_2)I_1 + I_2D(I_1) =$$

$$= I_2(D(I_1) - I_2D(I_1)I_2) + I_1(D(I_2) - I_1D(I_2)I_1) =$$

$$= I_2D(I_1)^{-+-} + I_1D(I_2)^{+--}. \qquad \square$$

Theorem 5.3. *For any $X, Y \in V$ the symmetric component T^0 of the torsion T satisfies:*

(i) $g(T^0_{\xi_i}(X), Y) = \frac{1}{2} L_{\xi_i} g(X, Y),\ i = 1, 2, 3;$

(ii) (Biquard [1]) $T^0_{\xi_i}(I_i X) = -I_i T^0_{\xi_i}(X),\ i = 1, 2, 3;$

(iii) (Biquard [1]) $I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}$ (*The other two identities could be obtained through cyclic permutation of* (1,2,3)).

Lemma 5.5.

$$L'_{\xi_1} I_2 = -2T^{0\ --+}_{\xi_1} I_2 - 2I_3\tilde{u} + d\eta_1(\xi_2, \xi_1) I_1 + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) I_3 \tag{5.9}$$

$$L'_{\xi_2} I_1 = -2T^{0\ --+}_{\xi_2} I_1 + 2I_3\tilde{u} + d\eta_2(\xi_1, \xi_2) I_2 - \frac{1}{2}(-d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) I_3 \tag{5.10}$$

$$L'_{\xi_1} I_1 = -2T^0_{\xi_1} I_1 + d\eta_1(\xi_1, \xi_2) I_2 + d\eta_1(\xi_1, \xi_3) I_3 \tag{5.11}$$

and six more identities which may be obtained through cyclic permutation of $(1, 2, 3)$. *Here $\tilde{u}$ is symmetric endomorphism of V which commutes with I_1, I_2 and I_3.*

Proof: (Theorem 5.3 and Lemma 5.5) Let $X, Y \in V$. Using (5.6), we calculate

$$g(T^0_{\xi_i} X, Y) = \frac{1}{2}(g(T_{\xi_i} X, Y) + g(T_{\xi_i} Y, X)) =$$

$$= \frac{1}{2}(g(\nabla_{\xi_i} X - [\xi_i X]_V, Y) + g(\nabla_{\xi_i} Y - [\xi_i Y]_V, X)) =$$

$$= \frac{1}{2}(\xi_i g(X, Y) - g([\xi_i X]_V, Y) - g([\xi_i Y]_V, X)) = \frac{1}{2} L_{\xi_i} g(X, Y).$$

We also have

$$L_{\xi_i} \omega_j(X, Y) = \xi_i g(I_j X, Y) - g(I_j[\xi_i, X], Y) - g(I_j X, [\xi_i, Y]) =$$

$$= L_{\xi_i} g(I_j X, Y) + g((L_{\xi_i} I_j) X, Y)$$

which leads to

$$\# L'_{\xi_i} \omega_j = 2T^0_{\xi_i} I_j + L'_{\xi_i} I_j \tag{5.12}$$

On the other hand, using Lemma 5.1 and the well known identity $L_{\xi_i} \omega_i = i_{\xi_i} d\omega_i + d i_{\xi_i} \omega_i$, we compute

$$L_{\xi_i} \omega_{i|V} = d\eta_i(\xi_i, \xi_j) \omega_j + d\eta_i(\xi_i, \xi_k) \omega_k, \tag{5.13}$$

where (i, j, k) is cyclic permutation of $(1, 2, 3)$. So we obtain

$$L'_{\xi_i} I_i = -2T^0_{\xi_i} I_i + d\eta_i(\xi_i, \xi_j) I_j + d\eta_i(\xi_i, \xi_k) I_k \qquad (5.14)$$

Now we apply Lemma 5.4 (i) for $D = L'$ and this completes the proof of Theorem 5.3, (i) and (ii).

We use the well known formula $L_{\xi_1}\omega_2 = i_{\xi_1} d\omega_2 + d i_{\xi_1}\omega_2$ and Lemma 5.1 to compute

$$(L_{\xi_1}\omega_2)_{|V} = \frac{1}{2}(d(i_{\xi_1} d\eta_2) - i_{\xi_1} d\eta_3 \wedge i_{\xi_3} d\eta_2)_{|V}. \qquad (5.15)$$

Next we apply the condition (ii) of Theorem 5.1 to obtain

$$(L_{\xi_1}\omega_2 + L_{\xi_2}\omega_1)_{|V} = \frac{1}{2}(d(i_{\xi_1} d\eta_2) + d(i_{\xi_2} d\eta_1))_{|V} =$$

$$= d\eta_1(\xi_2, \xi_1)\omega_1 + d\eta_2(\xi_1, \xi_2)\omega_2 + (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3))\omega_3 \qquad (5.16)$$

On the other hand, (5.12) leads to

$$2T^0_{\xi_1} I_2 + L'_{\xi_1} I_2 + 2T^0_{\xi_2} I_1 + L'_{\xi_2} I_1 =$$

$$= d\eta_1(\xi_2, \xi_1) I_1 + d\eta_2(\xi_1, \xi_2) I_2 + (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3)) I_3. \qquad (5.17)$$

Now we decompose (5.17) according to Lemma 5.3 to get

$$L'_{\xi_1} I_2{}^{+--} = -2T^0_{\xi_1}{}^{--+} I_2 + d\eta_1(\xi_2, \xi_1) I_1 \qquad (5.18)$$

$$L'_{\xi_2} I_1{}^{-+-} = -2T^0_{\xi_2}{}^{--+} I_1 + d\eta_2(\xi_1, \xi_2) I_2 \qquad (5.19)$$

$$(L'_{\xi_1} I_2 + L'_{\xi_2} I_1)^{--+} = (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_1, \xi_3)) I_3. \qquad (5.20)$$

$$T^0_{\xi_1}{}^{-+-} I_2 + T^0_{\xi_2}{}^{+--} I_1 = 0 \qquad (5.21)$$

Obviously, (5.21) completes the proof of Theorem 5.3. Using (5.20), we define

$$2\tilde{u} = I_3 L'_{\xi_1} I_2{}^{--+} + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) Id_V = \qquad (5.22)$$

$$= -I_3 L'_{\xi_2} I_1{}^{--+} + \frac{1}{2}(-d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) Id_V.$$

Applying Lemma 5.4 for $D = L'$, we obtain the formulas in Lemma 5.5.

Now we shall show that $\tilde{u}$ is symmetric. For any $X, Y \in V$ according to (5.12) we have

$$L_{\xi_1}\omega_2(X, Y) = 2g(T^0_{\xi_1} I_2 X, Y) + g(L_{\xi_1} I_2 X, Y).$$

But $L_{\xi_1}\omega_2(X,Y)$ is skew-symmetric and applying (5.9) we get

$$0 = symm(2T^0_{\xi_1} I_2 + L'_{\xi_1} I_2) = -I_3 antisymm(2\tilde{u}) \qquad \square$$

Let (i,j,k) be any cyclic permutation of $(1,2,3)$. We define three 2-forms

$$A_i = \frac{1}{2}\pi\{d(\pi(i_{\xi_j}d\eta_k)) + (i_{\xi_i}d\eta_j)\wedge(i_{\xi_i}d\eta_k)\} = \tag{5.23}$$

$$= \frac{1}{2}\pi\{d(i_{\xi_j}d\eta_k) + (i_{\xi_i}d\eta_j)\wedge(i_{\xi_i}d\eta_k)\} - d\eta_k(\xi_j,\xi_k)\omega_k + d\eta_k(\xi_i,\xi_j)\omega_i$$

We put this into (5.15) to get

$$(L_{\xi_1}\omega_2)_{|V} = A_3 + d\eta_2(\xi_1,\xi_2)\omega_2 - d\eta_2(\xi_3,\xi_1)\omega_3$$

On the other hand, using (5.9) and (5.12) we calculate

$$\#L'_{\xi_1}\omega_2 = 2T^{0\ -+-}_{\xi_1} I_2 - 2I_3\tilde{u} + d\eta_1(\xi_2,\xi_1)I_1 + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3$$

We decompose the last two identities to obtain

Lemma 5.6.

$$\begin{aligned}
\#A_3^{+++} &= 2T^{0\ -+-}_{\xi_1} I_2\\
\#A_3^{+--} &= d\eta_1(\xi_2,\xi_1)I_1\\
\#A_3^{-+-} &= -d\eta_2(\xi_1,\xi_2)I_2\\
\#A_3^{--+} &= -2I_3\tilde{u} + \tfrac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3
\end{aligned}$$

Analogous formulas for A_1 and A_2 may be obtained through cyclic permutation of $(1,2,3)$.

Corollary 5.1. *For the symmetric tensor $\tilde{u}$ we have*

$$2\tilde{u} = I_1\#A_1^{+--} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))Id_V =$$

$$= I_2\#A_2^{-+-} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))Id_V =$$

$$= I_3\#A_3^{--+} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))Id_V$$

and also

$$tr(\tilde{u}) = \frac{1}{2}tr(I_1\#A_1) + n(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)) =$$

$$= \frac{1}{2}tr(I_2\#A_2) + n(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)) =$$

$$= \frac{1}{2}tr(I_3\#A_3) + n(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)).$$

Theorem 5.4 (Biquard [1]). *For any $i = 1, 2, 3$ we have*

$$T_{\xi_i} = T^0_{\xi_i} + I_i u.$$

Here $u = \tilde{u} - \frac{tr(\tilde{u})}{4n} Id_V$ *and $\tilde{u}$ is given in Corollary 5.1.*

Proof: First we denote with Σ^2 the space of symmetric endomorphisms of V and with "*ant*" the projection

$$ant : End(V) = \#\Sigma^2(V) \oplus \#\Lambda^2(V) \to \#\Lambda^2(V).$$

We have

$$4[T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} = 3ant(T_{\xi_i}) + I_1 ant(T_{\xi_i}) I_1 + I_2 ant(T_{\xi_i}) I_2 + I_3 ant(T_{\xi_i}) I_3 = \tag{5.24}$$

$$= \sum_{s=1}^{3} (ant(T_{\xi_i}) + I_s ant(T_{\xi_i}) I_s).$$

We apply (5.6) and for any $X, Y \in V$, we obtain

$$g(4[T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} X, Y) = -\sum_{s=1}^{3} g((\nabla_{\xi_i} I_s)X, I_s Y) + \tag{5.25}$$

$$+\frac{1}{2} \sum_{s=1}^{3} \{g((L_{\xi_i} I_s)X, I_s Y) - g((L_{\xi_i} I_s)Y, I_s X)\}.$$

We have also

$$T_{\xi_i} = [T_{\xi_i}]_{(sp(n) \oplus sp(1))^\perp} = T^0_{\xi_i} + [T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} - [T_{\xi_i}]_{sp(1)}.$$

Now we apply Lemma 5.5 and the theorem follows. □

Corollary 5.2. $I_2(T_{\xi_2})^{+--} = I_1(T_{\xi_1})^{-+-}$ (*The other two identities could be obtained through cyclic permutation of* $(1, 2, 3)$).

Corollary 5.3. *For any $X, Y \in V$*

$$g(\nabla_{\xi_i} X, Y) = \frac{1}{2} L_{\xi_i} g(X, Y) + g([\xi_i, X], Y) + g(I_i u X, Y).$$

Corollary 5.4.

$$\nabla_{\xi_1} I_1 = -d\eta_1(\xi_3, \xi_1) I_3 + d\eta_1(\xi_1, \xi_2) I_2$$
$$\nabla_{\xi_1} I_2 = -d\eta_1(\xi_1, \xi_2) I_1 + (-\tfrac{tr(\tilde{u})}{2n} + \tfrac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))) I_3$$
$$\nabla_{\xi_1} I_3 = -(-\tfrac{tr(\tilde{u})}{2n} + \tfrac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))) I_2 + d\eta_1(\xi_3, \xi_1) I_1$$

$\nabla_{\xi_2} I_1 = -(-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)))I_3 + d\eta_2(\xi_1,\xi_2)I_2$
$\nabla_{\xi_2} I_2 = -d\eta_2(\xi_1,\xi_2)I_1 + d\eta_2(\xi_2,\xi_3)I_3$
$\nabla_{\xi_2} I_3 = -d\eta_2(\xi_2,\xi_3)I_2 + (-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)))I_1$

$\nabla_{\xi_3} I_1 = -d\eta_3(\xi_3,\xi_1)I_3 + (-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)))I_2$
$\nabla_{\xi_3} I_2 = -(-\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)))I_1 + d\eta_3(\xi_2,\xi_3)I_3$
$\nabla_{\xi_3} I_3 = -d\eta_3(\xi_2,\xi_3)I_2 + d\eta_3(\xi_3,\xi_1)I_1$

Of course we may write all this formulas briefly as follows

$$\nabla_{\xi_s} I_i = -\alpha_j(\xi_s)I_k + \alpha_k(\xi_s)I_j, \tag{5.26}$$

where $\alpha_i(\xi_s) = -\delta_{is}(\frac{tr(\tilde{u})}{2n} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2))) + d\eta_s(\xi_j,\xi_k)$, $s = 1,2,3$ *and* (i,j,k) *is any cyclic permutation of* $(1,2,3)$.

Proof: According to (5.6) we have

$$\nabla_{\xi_s} I_i = [T_{\xi_s}, I_i] + L'_{\xi_s} I_i = [T^0_{\xi_s}, I_i] + u[I_s, I_i] + L'_{\xi_s} I_i.$$

We apply Lemma 5.5 to get the corollary . □

Corollary 5.5.

$$\nabla_{\xi_s}\xi_i = -\alpha_j(\xi_s)\xi_k + \alpha_k(\xi_s)\xi_j,$$

Here $\alpha_i(\xi_s)$ *is the same as in* (5.26), (i,j,k) *is any cyclic permutation of* $(1,2,3)$ *and* $s = 1,2,3$.

Corollary 5.6.

$$T(\xi_i,\xi_j) = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_V.$$

Here i,j,k *is any cyclic permutation of* $(1,2,3)$.

Proof: Using Corollary 5.5 we compute

$$T(\xi_i,\xi_j) = \nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i - [\xi_i,\xi_j] = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_V. \qquad \square$$

6. THE 3-SASAKIAN CASE

Let on the Riemannian manifold (M,g) be given a 3-Sasakian structure. This means there are given three Killing vector fields $\{\xi_1,\xi_2,\xi_3\}$, which satisfy

(i) $g(\xi_i,\xi_j) = \delta_{ij},\ i,j = 1,2,3$

(ii) $[\xi_i,\xi_j] = -2\xi_k$, for any cyclic permuatation (i,j,k) of $(1,2,3)$

(iii) $(D_X\tilde{I}_i)Y = g(\xi_i,Y)X - g(X,Y)\xi_i,\ i = 1,2,3,\ X,Y \in TM$. Where $\tilde{I}_i(X) = D_X\xi_i$ and D denotes the Levi-Civita connection.

We denote $V = \{\xi_1,\xi_2,\xi_3\}^\perp$. We shall use without proof the next well known

Lemma 6.1. *Let (i,j,k) be any cyclic permutation of $(1,2,3)$. We have*

$$\tilde{I}_i(\xi_j) = \xi_k; \tag{6.1}$$

$$\tilde{I}_i \circ \tilde{I}_j(X) = \tilde{I}_k X, \ X \in V; \tag{6.2}$$

$$\tilde{I}_i \circ \tilde{I}_i(X) = -X, \ X \in V; \tag{6.3}$$

$$d\eta_i(X,Y) = 2g(\tilde{I}_i X, Y), \ X, Y \in V. \tag{6.4}$$

If we define $W = space\{\xi_1, \xi_2, \xi_3\}$, $I_{i|V} = \tilde{I}_{i|V}$, $I_{i|W} = 0$ and $\varphi(\xi_i) = I_i$ we clearly obtain a contact quaternionic structure $(V, Q = \{I_1, I_2, I_3\}, \varphi)$ on M. In this case it is easy to calculate

Lemma 6.2.

$$i_{\xi_i} d\eta_{j|V} = 0 \quad for\ all\ i,j = 1,2,3; \tag{6.5}$$

$$d\eta_1(\xi_2, \xi_3) = 2, \ d\eta_1(\xi_1, \xi_3) = d\eta_1(\xi_1, \xi_2) = 0; \tag{6.6}$$

$$A_1 = A_2 = A_3 = 0; \tag{6.7}$$

$$\tilde{u} = \frac{1}{2} Id_V. \tag{6.8}$$

Theorem 6.1. *The contact quaternionic structure (V, Q, φ) satisfies the conditions of the Theorem 5.2 and therefore it admits the Biquard connection ∇. We have*

(i) $\nabla_X I_i = 0, X \in V$.
(ii) $\nabla_{\xi_i} I_i = 0$.
(iii) $\nabla_{\xi_i} I_j = -2I_k$, $\nabla_{\xi_j} I_i = 2I_k$, *here (i,j,k) is cyclic permutation of $(1,2,3)$*.
(iv) $T(\xi_i, \xi_j) = -2\xi_k$.
(v) $T(\xi_i, X) = 0, \ X \in V$.

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