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ON THE ERROR BOUNDS OF THE GAUSS-TYPE QUADRATURE FORMULAE ASSOCIATED WITH SPACES OF PARABOLIC AND CUBIC SPLINE FUNCTIONS WITH DOUBLE EQUIDISTANT KNOTS

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In two papers from 1995 P. Köhler and G. Nikolov showed that Gauss-type quadrature formulae associated with spaces of spline functions with equidistant knots are asymptotically optimal in certain Sobolev classes of functions. In particular, Gauss-type quadratures associated with the spaces of spline functions of degree $r-1$ with double equispaced knots are asymptotically optimal definite quadrature formulae of order r when r is even, and it is conjectured that the asymptotical optimality property persists also in the case of odd r. For $r = 3$, 4, these quadrature formulae have been constructed by G. Nikolov, who also proved estimates for their error constants. The aim of this note is to refine the estimates for the error constant in the case $r = 3$, and to point out to some error estimates in both cases $r = 3$ and $r = 4$, which are easier to evaluate and could be sharper than those which involve the uniform norm of the r-th derivative of the integrand.

Keywords: spline functions, monosplines, Peano representation of linear functionals, definite quadrature formulae, error estimation of quadratures, Bernoulli polynomials

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1. Introduction and statement of the results

A standard way to evaluate approximately the definite integral

$$I[f] := \int_0^1 f(x)\,dx$$

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is to use quadrature formulae, which are linear functionals of the form

$$Q[f]=\sum_{i=1}^{n}a_if(\tau_i),\quad 0\le\tau_1<\cdots<\tau_n\le 1. \tag{1.1}$$

We start with introducing some notation and definitions. Throughout this paper, π_m stands for the set of algebraic polynomials of degree not exceeding m. A quadrature formula Q is said to have algebraic degree of precision m (in short, $ADP(Q)=m$) if m is the largest non-negative integer such that its remainder functional

$$R[Q;f]:=I[f]-Q[f]$$

vanishes on π_m.

The Sobolev classes of functions $W_p^r[0,1]$, ($r\in\mathbb{N}$, $p\ge 1$), are defined by

$$W_p^r[0,1]:=\{f\in C^{r-1}[0,1]\colon f^{(r-1)}\text{ loc. abs. cont.}, \int_0^1|f^{(r)}(t)|^p\,dt<\infty\}$$

(note that $C^r[0,1]\subset W_p^r[0,1]$ for every $p\ge 1$). Henceforth, $\|\cdot\|$ designates the supremum norm in $[0,1]$, and the usual $L_p[0,1]$-norm is shortly denoted by $\|\cdot\|_p$,

$$\|f\|_p=\begin{cases}\left(\int_0^1|f(t)|^p\,dt\right)^{1/p}, & \text{if } 1\le p<\infty,\\ \operatorname{vrai\,sup}_{t\in[0,1]}|f(t)|, & \text{if } p=\infty.\end{cases}$$

If $ADP(Q)=m\ge r-1$ and $f\in W_1^r[0,1]$, then by Peano representation theorem for linear functionals (cf. [14]), the remainder $R[Q;f]$ can be written in the form

$$R[Q;f]=\int_0^1 K_r(Q;t)f^{(r)}(t)\,dt, \tag{1.2}$$

where $K_r(Q;t)$ is referred to as the r-th Peano kernel of Q and is given by

$$K_r(Q;t)=\frac{1}{(r-1)!}R\big[Q;(\cdot-t)_+^{r-1}\big], \tag{1.3}$$

where $(x)_+^{r-1}=\max\{x,0\}^{r-1}$ is the truncated power function. In literature, $K_r(Q;t)$ is also termed as monospline of degree r. For quadrature formula Q in (1.1) the explicit form of $K_r(Q;t)$, $t\in[0,1]$, is

$$K_r(Q;t)=\frac{(1-t)^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(\tau_i-t)_+^{r-1} \tag{1.4}$$

$$=(-1)^r\left\{\frac{t^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(t-\tau_i)_+^{r-1}\right\}. \tag{1.5}$$

If $f\in W_p^r[0,1]$, then application of Hölder's inequality to (1.2) implies the unimprovable error estimate

$$\big|R[Q;f]\big|\le c_{r,p}(Q)\|f^{(r)}\|_p,\quad\text{where}\quad c_{r,p}(Q)=\|K_r(Q;\cdot)\|_q,\quad \frac{1}{p}+\frac{1}{q}=1.$$

Usually, $c_{r,p}(Q)$ is called the error constant of Q in the Sobolev class $W_p^r[0,1]$. In what follows, the subscript "n" in Q_n is used to emphasize that Q_n is an n-point quadrature formula, i.e., a quadrature formula which has n nodes. Quadrature formulae Q_n with the smallest possible error constant $c_{r,p}(Q_n)$ are called optimal quadrature formulae in $W_p^r[0,1]$. Without going into details, let us mention that the existence and uniqueness of optimal quadrature formulae in Sobolev classes of functions have been established by Bojanov [1–3] and Zhensykbaev [16, 17].

In the present paper we study certain definite quadrature formulae. A quadrature formula Q_n is said to be definite of order r ($r \geq 1$), if there exists a constant $c_r(Q_n) \neq 0$ such that

$$R[Q_n; f] = c_r(Q_n) f^{(r)}(\xi)$$

for every $f \in C^r[0,1]$ with some $\xi \in [0,1]$ depending on the integrand f. More precisely, Q_n is called positive, resp. negative, definite quadrature formula of order r if $c_r(Q_n) > 0$, resp. $c_r(Q_n) < 0$. Since $c_r(Q_n) = c_{r,\infty}(Q_n)$ if Q_n is positive definite and $c_r(Q_n) = -c_{r,\infty}(Q_n)$ if Q_n is negative definite, $c_r(Q_n)$ will also be referred to as the error constant of Q_n. The importance of definite quadrature formulae of order r stems from the fact that they provide one-sided approximation to $I[f]$ when $f^{(r)}$ has a permanent sign in $(0,1)$. The midpoint and the trapezium quadrature formulae are best-known examples of positive, resp. negative, definite quadrature formulae of order two.

Definite n-point quadrature formulae of order r with the smallest positive or the largest negative c_r are called optimal definite quadrature formulae. It is known that optimal definite quadrature formulae exist and are unique, cf. [6, 11, 15] and [5, Chapter VII.8]. We denote by $c_{n,r}^{+}$ and $c_{n,r}^{-}$ the error constants of the optimal n-point definite quadrature formulae of order r:

$$\begin{aligned} c_{n,r}^{+} &:= \inf\{c_r(Q_n)\colon Q_n \text{ is positive definite of order } r\}, \\ c_{n,r}^{-} &:= \sup\{c_r(Q_n)\colon Q_n \text{ is negative definite of order } r\}. \end{aligned}$$

In [9] estimates have been established for the error constants of the Gauss-type quadrature formulae associated with the spaces of spline functions with double equidistant knots. These estimates in turn provide bounds for the error constants of the optimal definite quadrature formulae. Below we restate the main result from [9], denoting by B_r the Bernoulli polynomial of order r with leading coefficient $1/r!$.

Theorem A ([9, Theorem 1.1])**.** (a) *For even r with $2 \leq r \leq 2n$, there holds*

$$c_{n,r}^{+} \leq -\frac{B_r(j/2)}{(n+1-r/2)^r}, \quad \textit{if} \quad r = 4m + 2j,\ j = 0, 1.$$

(b) *For even r with $2 \leq r \leq 2n-2$, there holds*

$$c_{n,r}^{-} \geq -\frac{B_r(j/2)}{(n-r/2)^r}, \quad \textit{if} \quad r = 4m + 2 - 2j,\ j = 0, 1.$$

(c) *For odd r with $1 \leq r \leq 2n-1$, there holds*

$$c_{n,r}^{+} \leq \frac{\|B_r\|}{(n-(r-1)/2)^r} \quad \textit{and} \quad c_{n,r}^{-} \geq -\frac{\|B_r\|}{(n-(r-1)/2)^r}.$$

Comparison with results of Lange [10] shows that Gauss-type quadrature formulae associated with the spaces of spline functions of degree $r-1$ with double equidistant knots are asymptotically optimal definite quadrature formulae of order r when r is even, and it is conjectured that the asymptotical optimality property persists also in the case of odd r. Two particular cases of Theorem A relevant to the object of this paper are

$$c^+_{n+1,3} \le \frac{\sqrt{3}}{216n^3}, \quad c^-_{n+1,3} \ge -\frac{\sqrt{3}}{216n^3}, \tag{1.6}$$

$$c^+_{n+1,4} \le \frac{1}{720n^4}. \tag{1.7}$$

The right-hand sides of the inequalities in (1.6) and (1.7) are in fact bounds for the error constants of Gauss-type quadrature formulae associated with the linear spaces of spline functions $S_{n,3}$ and $S_{n,4}$, respectively, where for $r \ge 3$ and $n \ge 2$,

$$\begin{aligned} &S_{n,r} = \{f\colon f \in C^{r-3}[0,1],\ f_{|(x_k,x_{k+1})} \in \pi_{r-1},\ k=0,\dots,n-1\},\\ &x_k = x_{k,n} := \frac{k}{n}, \quad k=0,\dots,n. \end{aligned} \tag{1.8}$$

The functions $\{1, x, x^2, (x-x_1)_+, (x-x_1)_+^2, \dots, (x-x_{n-1})_+, (x-x_{n-1})_+^2\}$ form a basis for $S_{n,3}$, therefore $\dim S_{n,3} = 2n+1$ and the Gauss-type quadratures associated with $S_{n,3}$ are left and right $(n+1)$-point Radau quadrature formulae. These quadrature formulae were found, among others, in [12].

Theorem B ([12, Theorem 2]). *The right Radau quadrature formula associated with the space of parabolic splines $S_{n,3}$ is*

$$Q^{R,r}_{n+1}[f] = \sum_{k=0}^{n-1} a_{k,n} f\Big(\frac{k+\theta_k}{n}\Big) + a_{n,n} f(1) \approx I[f],$$

where $\theta_0 = 1/3$,

$$\begin{aligned} \theta_k &= \frac{1-\theta_{k-1}}{5-6\theta_{k-1}}, \quad k=1,\dots,n-1,\\ a_{k,n} &= \frac{1}{6n\theta_k(1-\theta_k)}, \quad k=0,\dots,n-1, \end{aligned}$$

and

$$a_{n,n} = \frac{2-3\theta_{n-1}}{6n(1-\theta_{n-1})}.$$

$Q^{R,r}_{n+1}$ is negative definite quadrature formula of order three with error constant

$$c_3(Q^{R,r}_{n+1}) = -\frac{\sqrt{3}}{216n^3} + O(n^{-4}).$$

Remark 1.1. The left Radau quadrature formula $Q_{n+1}^{R,l}$ associated with $S_{n,3}$ is obtained from $Q_{n+1}^{R,r}$ by reflection, i.e., $Q_{n+1}^{R,l}[f(\cdot)] = Q_{n+1}^{R,r}[f(1-\cdot)]$. Clearly, $Q_{n+1}^{R,l}$ is positive definite of order three and $c_3\big(Q_{n+1}^{R,l}\big) = -c_3\big(Q_{n+1}^{R,r}\big)$.

Since $\dim S_{n,4} = 2n+2$, associated with $S_{n,4}$ are the $(n+1)$-point Gauss quadrature formula Q_{n+1}^{G} and the $(n+2)$-point Lobatto quadrature formula Q_{n+2}^{Lo}. These quadrature formulae were investigated in [13]. The following theorem gives the construction and summarizes some of the properties of Q_{n+1}^{G} (cf. [13, Section 2]).

Theorem C. *Let*

$$Q_{n+1}^{G}[f] = \sum_{i=1}^{n+1} a_{i,n+1}^{G} f(\tau_{i,n+1}^{G}), \quad 0 < \tau_{1,n+1}^{G} < \cdots < \tau_{n+1,n+1}^{G} < 1,$$

be the Gauss quadrature formula associated with $S_{n,4}$, i.e., determined uniquely by the property $I[f] = Q_{n+1}^{G}[f]$ for every $f \in S_{n,4}$. Then:

(a) *Q_{n+1}^{G} is symmetrical: $a_{k,n+1}^{G} = a_{n+2-k,n+1}^{G}$ and $\tau_{k,n+1}^{G} = 1 - \tau_{n+2-k,n+1}^{G}$ for $k = 1,\ldots,n+1$.*

(b) *Let $a_{i,n+1}^{G} = \delta_i/n$, $\tau_{i,n+1}^{G} = (i-\theta_i)/n$ for $i = 1,\ldots,[n/2]+1$. Then the sequences $\{\delta_i\}$ and $\{\theta_i\}$ are determined by $\delta_1 = 16/27$, $\theta_1 = 3/4$ and, for $i = 1,\ldots,[n/2]-1$, by the recurrence relations*

$$\theta_{i+1} = \frac{1-\delta_i(1-\theta_i)^2(5\theta_i+1)}{1-\delta_i(1-\theta_i)^2(4\theta_i+1)}, \quad \delta_{i+1} = \frac{1-\delta_i(1-\theta_i)^2(4\theta_i+1)}{\theta_{i+1}^2}.$$

If n is even ($n = 2m$), then $\theta_{m+1} = 1$ and $\delta_{m+1} = 1 - 2\delta_m(1-\theta_m)^2(2\theta_m+1)$; if n is odd ($n = 2m-1$), then $\delta_m = 1 - \delta_{m-1}(1-\theta_{m-1})^2(2\theta_{m-1}+1)$ and θ_m is the greater root of the equation

$$\theta_m(1-\theta_m) = \frac{\delta_{m-1}\theta_{m-1}(1-\theta_{m-1})^2}{1-\delta_{m-1}(1-\theta_{m-1})^2(2\theta_{m-1}+1)}.$$

(c) *Q_{n+1}^{G} is positive definite quadrature formula of order four and its error constant $c_4(Q_{n+1}^{G})$ obeys the representation*

$$c_4(Q_{n+1}^{G}) = \frac{1}{720n^4} - \frac{1}{12}\sum_{i=1}^{[(n+1)/2]} a_{i,n+1}^{G}(x_{i-1}-\tau_{i,n+1}^{G})^2(x_i-\tau_{i,n+1}^{G})^2.$$

For all $n \ge 4$ there holds

$$\frac{1}{720n^4} - \frac{1}{551.9775n^5} \le c_4(Q_{n+1}^{G}) \le \frac{1}{720n^4} - \frac{1}{552n^5}. \tag{1.9}$$

(d) *Let $f \in C^4[0,1]$. Then $R[Q_{n+1}^{G};f] = o(n^{-4})$ if and only if $f'''(0) = f'''(1)$. Moreover, if $\operatorname{sign}\{f'''(1)-f'''(0)\} = \epsilon \ne 0$, then there exists $n_0 \in \mathbb{N}$ such that $\epsilon R[Q_{n+1}^{G};f] \ge 0$ for all $n \ge n_0$.*

(e) *If $f \in W_1^4[0,1]$ and $f^{(4)} \ge 0$ a.e. in $[0,1]$, then for all $n \ge 2$,*

$$0 \le R\big[Q_{2n+1}^{G};f\big] \le R\big[Q_{n+1}^{G};f\big].$$

Remark 1.2. In [13] recurrence formulae have been proposed also for computation of the weights and nodes of the Lobatto quadrature formula Q_{n+2}^{Lo} associated with $S_{n,4}$, which is negative definite of order four. However, unlike the case with Q_{n+1}^{G}, this procedure is of numerical nature, as it requires determination of an initial parameter, cf. [13, Theorem 2.5].

Our first goal in this paper is to prove properties of the Radau quadrature formulae associated with $S_{n,3}$, which are the analogues of those of Q_{n+1}^{G}, presented in parts (c), (d) and (e) of Theorem C.

Theorem 1.1. *Let $Q_{n+1}^{R,l}$ and $Q_{n+1}^{R,r}$ be the $(n+1)$-point left and right Radau quadrature formulae associated with $S_{n,3}$, i.e., determined uniquely by the property $R[Q_{n+1}^{R,l};f]=R[Q_{n+1}^{R,r};f]=0$ for every $f\in S_{n,3}$. Then:*

(a) *The error constants of $Q_{n+1}^{R,l}$ and $Q_{n+1}^{R,r}$ are given by*

$$c_3\big(Q_{n+1}^{R,l}\big)=-c_3\big(Q_{n+1}^{R,r}\big)=\frac{\sqrt{3}}{216n^3}-\frac{\sqrt{3}}{108n^4}\sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1}. \tag{1.10}$$

With $*$ standing for both r and l, the following inequalities hold true for all $n\geq 4$:

$$\frac{\sqrt{3}}{216n^3}-\frac{1}{269.13n^4}<\big|c_3\big(Q_{n+1}^{R,*}\big)\big|<\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}. \tag{1.11}$$

(b) *Let $f\in C^3[0,1]$. With $*$ standing for both r and l, $R[Q_{n+1}^{R,*};f]=o(n^{-3})$ as $n\to\infty$ if and only if $f''(0)=f''(1)$. Moreover, if $\operatorname{sign}\{f''(1)-f''(0)\}=\epsilon\neq 0$, then there exists $n_0\in\mathbb{N}$ such that $\epsilon\,R[Q_{n+1}^{R,r};f]\leq 0$ and $\epsilon\,R[Q_{n+1}^{R,l};f]\geq 0$ for all $n\geq n_0$.*

(c) *If $f\in W_1^3[0,1]$ and $f'''\geq 0$ a.e. in $[0,1]$, then for all $n\geq 2$,*

$$0\geq R\big[Q_{2n+1}^{R,r};f\big]\geq R\big[Q_{n+1}^{R,r};f\big]\quad\text{and}\quad 0\leq R\big[Q_{2n+1}^{R,l};f\big]\leq R\big[Q_{n+1}^{R,l};f\big].$$

Remark 1.3. Theorem C and Theorem 1.1 provide the following improvement of the estimates (1.6) and (1.7) for the error constants of the optimal positive definite quadrature formulae of orders two and three with $n+1$ nodes, $n\geq 4$:

$$c_{n+1,3}^{+}\leq c_3\big(Q_{n+1}^{R,l}\big)<\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4},$$
$$c_{n+1,4}^{+}\leq c_4\big(Q_{n+1}^{G}\big)<\frac{1}{720n^4}-\frac{1}{552n^5}.$$

As a consequence we have the following

Corollary 1.1. (a) *If $f\in C^3[0,1]$ and $f'''\geq 0$ in $[0,1]$, then for all $n\geq 4$,*

$$\begin{aligned}0&\leq R\big[Q_{n+1}^{R,l};f\big]\leq\Big(\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}\Big)\|f'''\|,\\ 0&\geq R\big[Q_{n+1}^{R,r};f\big]\geq-\Big(\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}\Big)\|f'''\|.\end{aligned} \tag{1.12}$$

(b) *If* $f \in C^4[0,1]$ *and* $f^{(4)} \geq 0$ *in* $[0,1]$, *then for all* $n \geq 4$

$$0 \leq R\big[Q_{n+1}^{G}; f\big] \leq \Big(\frac{1}{720n^4} - \frac{1}{552n^5}\Big)\|f^{(4)}\|. \tag{1.13}$$

Alternative error estimates are provided by the following theorem.

Theorem 1.2. (a) *If* $f \in C^3[0,1]$ *and* $f''' \geq 0$ *in* $[0,1]$, *then for all* $n \geq 2$,

$$\begin{aligned} 0 &\leq R\big[Q_{n+1}^{R,l}; f\big] \leq \frac{\sqrt{3}}{108n^3}\Big(f''(1) - f''(0)\Big), \\ 0 &\geq R\big[Q_{n+1}^{R,r}; f\big] \geq \frac{\sqrt{3}}{108n^3}\Big(f''(0) - f''(1)\Big). \end{aligned} \tag{1.14}$$

(b) *If* $f \in C^4[0,1]$ *and* $f^{(4)} \geq 0$ *in* $[0,1]$, *then for all* $n \geq 2$,

$$0 \leq R\big[Q_{n+1}^{G}; f\big] \leq \frac{1}{384n^4}\Big(f'''(1) - f'''(0)\Big). \tag{1.15}$$

Since the supremum norms of f''' and $f^{(4)}$ may be not accessible or difficult to evaluate, evidently the error bounds in Theorem 1.2 are easier to apply than those in Corollary 1.1. Even in the cases when $\|f'''\|$ or $\|f^{(4)}\|$ is known, it can still happen that the estimates in Theorem 1.2 are superior to those from Corollary 1.1.

Before concluding this section, we find appropriate to briefly mention a few more facts about Peano kernel representation of the remainders of quadrature formulae, for more details the reader is referred to [5].

It follows from (1.3) that the requirement $K_r(Q; u) = 0$ for some $u \in (0,1)$ is equivalent to $I[f_u] = Q[f_u]$, where $f_u(x) = (x-u)_+^{r-1}$. Hence, in order that Q evaluates to the exact value definite integrals of functions from a linear space of splines of degree $r-1$ with maximal dimension, it is necessary that the monospline $K_r(Q; \cdot)$ has the maximal possible number of zeros in $(0,1)$. The problem of the existence and uniqueness of monosplines satisfying boundary conditions and having maximal number of prescribed zeros in $(0,1)$ (the fundamental theorem of algebra for monosplines) has been resolved by Karlin and Micchelli [7]. Quadrature formulae corresponding to monosplines of the form (1.4)–(1.5) with maximal number of pre-assigned zeros in $(0,1)$ are called Gauss-type quadratures associated with the space of spline functions of degree $r-1$ having knots at these zeros. The results in [7] assert that Gauss-type quadratures for spaces of splines exist and are unique, and as in the case of classical Gauss-type quadratures associated with spaces of algebraic polynomials, all their weights are positive.

We finally point out that, in view of (1.2), a quadrature formula Q is definite of order r if and only if $ADP(Q) = r-1$ and $K_r(Q; t)$ does not change its sign in $(0,1)$. Therefore, all zeros of $K_r(Q; \cdot)$ in $(0,1)$ must have even multiplicities.

Theorem 1.1 is proved in the next section, and in Section 3 we present the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In view of Remark 1.1, it suffices to prove only the claims of Theorem 1.1 concerning $Q_{n+1}^{R,r}$. We denote the right Radau quadrature formula associated with $S_{n,3}$ by

$$Q_{n+1}^{R,r}[f] = \sum_{k=0}^{n-1} a_k\, f(\tau_k) + a_n\, f(1), \quad 0 < \tau_0 < \cdots < \tau_{n-1} < 1,$$

where, for the sake of simplicity, we skip the second indices in the weights and nodes (we also write $x_k = k/n$, $k = 0, \ldots, n$, see (1.8)).

According to Theorem B, we have

$$\tau_k = \frac{k+\theta_k}{n}, \quad k = 0, \ldots, n-1, \tag{2.1}$$

with

$$\theta_0 = \frac{1}{3}, \quad \theta_k = \frac{1-\theta_{k-1}}{5-6\theta_{k-1}}, \quad k = 1, \ldots, n-1, \tag{2.2}$$

$$a_k = \frac{1}{6n\theta_k(1-\theta_k)}, \quad 0 \le k \le n-1, \tag{2.3}$$

and

$$a_n = \frac{2-3\theta_{n-1}}{6n(1-\theta_{n-1})}. \tag{2.4}$$

Lemma 2.1. *The sequence $\{\theta_k\}$ in (2.2) has the explicit representation*

$$\theta_k = \frac{s_k}{s_k + s_{k+1}}, \quad s_k = (2+\sqrt{3})^k + (2-\sqrt{3})^k, \quad k \in \mathbb{N}.$$

Proof. We apply induction with respect to k. The statement is true for $k = 0$, since $s_0 = 2$ and $s_1 = 4$. Assuming $\theta_{k-1} = \dfrac{s_{k-1}}{s_{k-1}+s_k}$ for some $k \in \mathbb{N}$, then

$$\theta_k = \frac{1-\theta_{k-1}}{5-6\theta_{k-1}} = \frac{s_k}{5s_k - s_{k-1}} \overset{?}{=} \frac{s_k}{s_k + s_{k+1}}.$$

The last equality follows from the identity $s_{k-1} + s_{k+1} = 4s_k$, which is verified using $(2 \pm \sqrt{3})^2 + 1 = 4(2 \pm \sqrt{3})$. This accomplishes the induction step and thereby the proof of Lemma 2.1. The proposed method of proof does not give a clue about the way the explicit form of the solution of this recurrence equation was deduced. Equations like (2.2) are called Riccati difference equations, see e.g. [4] for a general approach to their solutions. □

For $f \in C^3[0,1]$ the remainder of $Q_{n+1}^{R,r}$ admits the representation

$$R[Q_{n+1}^{R,r}; f] = \int_0^1 K_3(Q_{n+1}^{R,r}; t) f'''(t)\, dt,$$

where, according to (1.5),

$$K_3(Q_{n+1}^{R,r};t) = -\frac{t^3}{6} + \frac{1}{2}\sum_{i=0}^{n-1} a_i(t-\tau_i)_+^2 \le 0, \quad t \in (0,1).$$

The zeros of $K_3(Q_{n+1}^{R,r};\cdot)$ in $(0,1)$ are $\{x_k\}_{k=1}^{n-1}$, and each of them is double. The error constant of $Q_{n+1}^{R,r}$ is given by

$$c_3\big(Q_{n+1}^{R,r}\big) = \int_0^1 K_3(Q_{n+1}^{R,r};t)\,dt = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} K_3(Q_{n+1}^{R,r};t)\,dt =: \sum_{k=0}^{n-1} I_k. \tag{2.5}$$

Clearly,

$$I_k = \frac{1}{24}\big(x_k^4 - x_{k+1}^4\big) + \frac{1}{6}\,J_k, \tag{2.6}$$

where

$$J_k = \sum_{i=0}^{k} a_i(x_{k+1}-\tau_i)^3 - \sum_{i=0}^{k-1} a_i(x_k-\tau_i)^3.$$

By using $x_{k+1}-x_k = 1/n$, we obtain

$$\begin{aligned}
J_k &= \frac{1}{n}\sum_{i=0}^{k} a_i\Big[(x_{k+1}-\tau_i)^2 + (x_{k+1}-\tau_i)(x_k-\tau_i) + (x_k-\tau_i)^2\Big] - a_k(\tau_k-x_k)^3\\
&= \frac{1}{n}\sum_{i=0}^{k} a_i\Big[2(x_{k+1}-\tau_i)^2 - \frac{1}{n}(x_{k+1}-\tau_i) + (x_k-\tau_i)^2\Big] - a_k(\tau_k-x_k)^3\\
&= \frac{2}{n}Q_{n+1}^{R,r}\big[(x_{k+1}-\cdot)_+^2\big] - \frac{1}{n^2}Q_{n+1}^{R,r}\big[(x_{k+1}-\cdot)_+\big] + \frac{1}{n}Q_{n+1}^{R,r}\big[(x_k-\cdot)_+^2\big]\\
&\quad + a_k(\tau_k-x_k)^2(x_{k+1}-\tau_k).
\end{aligned}$$

Since $Q_{n+1}^{R,r}[f] = I[f]$ for every $f \in S_{n,3}$, we have

$$\begin{aligned}
J_k &= \frac{2}{n}I\big[(x_{k+1}-\cdot)_+^2\big] - \frac{1}{n^2}I\big[(x_{k+1}-\cdot)_+\big] + \frac{1}{n}I\big[(x_k-\cdot)_+^2\big]\\
&\quad + a_k(\tau_k-x_k)^2(x_{k+1}-\tau_k)\\
&= \frac{2}{3n}x_{k+1}^3 - \frac{1}{2n^2}x_{k+1}^2 + \frac{1}{3n}x_k^3 + a_k(\tau_k-x_k)^2(x_{k+1}-\tau_k).
\end{aligned}$$

Substituting this expression for J_k in (2.6) and replacing x_k, x_{k+1}, a_k and τ_k using (1.8), (2.1) and (2.3), we obtain

$$I_k = \frac{2\theta_k - 1}{72n^4}. \tag{2.7}$$

By using Lemma 2.1, we find

$$\begin{aligned} 1-2\theta_k &= \frac{s_{k+1}-s_k}{s_{k+1}+s_k} = \frac{(\sqrt{3}+1)(2+\sqrt{3})^k-(\sqrt{3}-1)(2-\sqrt{3})^k}{(3+\sqrt{3})(2+\sqrt{3})^k+(3-\sqrt{3})(2-\sqrt{3})^k} \\ &= \frac{\sqrt{3}}{3}\,\frac{(2+\sqrt{3})^k-\frac{\sqrt{3}-1}{\sqrt{3}+1}(2-\sqrt{3})^k}{(2+\sqrt{3})^k+\frac{\sqrt{3}-1}{\sqrt{3}+1}(2-\sqrt{3})^k} = \frac{\sqrt{3}}{3}\,\frac{(2+\sqrt{3})^k-(2-\sqrt{3})^{k+1}}{(2+\sqrt{3})^k+(2-\sqrt{3})^{k+1}} \\ &= \frac{\sqrt{3}}{3}\Big(1-\frac{2(2-\sqrt{3})^{k+1}}{(2+\sqrt{3})^k+(2-\sqrt{3})^{k+1}}\Big) \\ &= \frac{\sqrt{3}}{3}\Big(1-\frac{2}{(2+\sqrt{3})^{2k+1}+1}\Big). \end{aligned}$$

By plugging this expression in (2.7), we arrive at

$$I_k = -\frac{\sqrt{3}}{216n^4}\Big(1-\frac{2}{(2+\sqrt{3})^{2k+1}+1}\Big), \qquad k=0,\dots,n-1. \tag{2.8}$$

The representation (1.10) of $c_3(Q_{n+1}^{R,r})$ in Theorem 1.1(a) now follows from (2.5) and (2.8). As was already mentioned, $c_3(Q_{n+1}^{R,l}) = -c_3(Q_{n+1}^{R,r})$. The two-sided estimates (1.11) are derived using the inequalities

$$\sum_{k=0}^{3}\frac{1}{(2+\sqrt{3})^{2k+1}+1} \le \sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1} \le \sum_{k=0}^{3}\frac{1}{(2+\sqrt{3})^{2k+1}+1} + \sum_{k=4}^{\infty}\frac{1}{(2+\sqrt{3})^{2k+1}}.$$

With this Theorem 1.1(a) is proved, and we proceed with the proof of part (b). If $f\in C^3[0,1]$, then by the mean value theorem,

$$R[Q_{n+1}^{R,r};f] = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} K_3(Q_{n+1}^{R,r};t)f'''(t)\,dt = \sum_{k=0}^{n-1} I_k\, f'''(\xi_k)$$

with $\xi_k\in(x_k,x_{k+1})$, $k=0,\dots,n-1$. We split the last sum into two parts:

$$R[Q_{n+1}^{R,r};f] = -\frac{\sqrt{3}}{216n^3}\sum_{k=0}^{n-1}\frac{1}{n}\,f'''(\xi_k) + \sum_{k=0}^{n-1}\Big[I_k+\frac{\sqrt{3}}{216n^4}\Big]\,f'''(\xi_k) =: A+B.$$

The sum in A is a Riemann sum for the continuous (hence integrable) function f''' on $[0,1]$, therefore

$$A = -\frac{\sqrt{3}}{216n^3}\big(f''(1)-f''(0)\big)+o(n^{-3}).$$

For B we have, in view of (2.8),

$$B = \frac{\sqrt{3}}{108n^4}\sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1}f'''(\xi_k) = O(n^{-4}).$$

Hence,

$$R[Q_{n+1}^{R,r};f] = A + B = -\frac{\sqrt{3}}{216n^3}\,\big(f''(1) - f''(0)\big) + o(n^{-3}),$$

which proves Theorem 1.1(b) for the remainders of $Q_{n+1}^{R,r}$.

For the proof of Theorem 1.1(c) we need the estimate for the number of zeros of a spline function in a given interval (a,b), provided by the Budan-Fourier theorem for splines. For a real-valued function f defined on the finite interval $[a,b]$, $Z_f(a,b)$ stands for the total number of the zeros of f in (a,b) counted with their multiplicities. By $S^-(a_1,a_2,\dots,a_m)$ and $S^+(a_1,a_2,\dots,a_m)$ we denote the number of strong and weak sign changes, respectively, in the finite sequence of real numbers $a_1,a_2,\dots,a_m$.

Lemma 2.2 ([8], Theorem 2.1). *If f is a polynomial spline function of exact degree r on (a,b) (i.e., of degree r with $f^{(r)}(t)\neq 0$ for some $t\in(a,b)$ with finitely many (active) knots in (a,b), all simple), then*

$$\begin{aligned} Z_f(a,b) \le Z_{f^{(r)}}(a,b) &+ S^-\big(f(a),f'(a),\dots,f^{(r-1)}(a),f^{(r)}(\sigma+)\big)\\ &- S^+\big(f(b),f'(b),\dots,f^{(r-1)}(b),f^{(r)}(\tau-)\big),\end{aligned}$$

where $[\sigma,\tau]\subset[a,b]$ is the largest interval such that $f^{(r)}(\sigma+)\neq 0$ and $f^{(r)}(\tau-)\neq 0$.

The difference $s(t) = K_3(Q_{n+1}^{R,r};t) - K_3(Q_{2n+1}^{R,r};t)$ of the third Peano kernels of the right Radau quadrature formulae associated with $S_{n,3}$ and $S_{2n,3}$ is a spline function of degree two with $3n$ knots in $(0,1)$, which has double zeros at the points $x_k = k/n$, $k=1,\dots,n-1$. In view of (1.4) and (1.5), s can be represented in two alternative ways,

$$s(t) = \frac{1}{2}\Big(\sum_{k=0}^{n-1} a_{k,n}(t-\tau_{k,n})_+^2 - \sum_{k=0}^{2n-1} a_{k,2n}(t-\tau_{k,2n})_+^2\Big), \tag{2.9}$$

$$s(t) = \frac{1}{2}\Big(\sum_{k=0}^{2n} a_{k,2n}(\tau_{k,2n}-t)_+^2 - \sum_{k=0}^{n} a_{k,n}(\tau_{k,n}-t)_+^2\Big). \tag{2.10}$$

Recall that all weights $a_{k,n}$ and $a_{k,2n}$ of Radau quadrature formulae are positive, therefore $s(t)$ is a spline function of exact degree two. Indeed,

$$s''(t) = \sum_{k=0}^{n-1} a_{k,n}(t-\tau_{k,n})_+^0 - \sum_{k=0}^{2n-1} a_{k,2n}(t-\tau_{k,2n})_+^0$$

is a piecewise constant function whose n positive jumps cannot be canceled out by the $2n$ negative jumps. This observation implies also that the number of sign changes of s'' in $(0,1)$ does not exceed $2n$, i.e.,

$$Z_{s''}(0,1) \le 2n. \tag{2.11}$$

By Theorem B, $\tau_{0,2n} = \dfrac{1}{6n} < \tau_{0,n} = \dfrac{1}{3n}$, therefore $s''(\tau_{0,2n}+) = -a_{0,2n} < 0$ while $s(t) \equiv 0$ for $t \in [0, \tau_{0,2n})$. From $\tau_{n,n} = \tau_{2n,2n} = 1$ and (2.10) we obtain $s''(1-) = a_{2n,2n} - a_{n,n}$. We shall show that $a_{2n,2n} - a_{n,n} \neq 0$, in fact,

$$a_{2n,2n} - a_{n,n} < 0. \tag{2.12}$$

From (2.4) and Lemma 2.1 we find

$$a_{n,n} = \frac{2s_n - s_{n-1}}{6n\, s_n},$$

hence (2.12) is equivalent to inequality

$$2\,\frac{s_{n-1}}{s_n} - \frac{s_{2n-1}}{s_{2n}} < 2,$$

which obviously is true since $0 < s_{k-1} < s_k$, $k \in \mathbb{N}$.

Lemma 2.2 applied with $r = 2$, $f = s$, $[a, b] = [0, 1]$ and $[\sigma, \tau] = [\tau_{0,2n}, 1]$ yields

$$\begin{aligned} Z_s(0,1) &\le Z_{s''}(0,1) + S^-\big(s(0), s'(0), s''(\tau_{0,2n}+)\big) - S^+\big(s(1), s'(1), s''(1-)\big) \\ &\le 2n + S^-\big(0, 0, -a_{0,2n}\big) - S^+\big(0, 0, a_{2n,2n} - a_{n,n}\big) \\ &\le 2n - 2. \end{aligned}$$

Recalling that s has double zeros at the points k/n, $k = 1, \ldots, n-1$, we conclude that s has no other zeros in $(0, 1)$. Since $s(1) = s'(1) = 0$ and $s''(1-) < 0$, it follows that $s(t) \le 0$, $t \in (0, 1)$, i.e.,

$$K_3(Q_{n+1}^{R,r}; t) \le K_3(Q_{2n+1}^{R,r}; t) \le 0, \quad t \in (0, 1).$$

If $f \in W_1^3[0, 1]$ and $f'''(t) \ge 0$ a.e. in $[0, 1]$, then

$$R\big[Q_{n+1}^{R,r}; f\big] - R\big[Q_{2n+1}^{R,r}; f\big] = \int_0^1 s(t) f'''(t)\, dt \le 0,$$

and consequently

$$R\big[Q_{n+1}^{R,r}; f\big] \le R\big[Q_{2n+1}^{R,r}; f\big] \le 0.$$

With this Theorem 1.1(c) is proved. □

Figure 1 illustrates the situation when $n = 4$. Its left part depicts the graphs of the third Peano kernels of the 5-point and 9-point right Radau quadrature formulae. We observe that the difference of the two Peano kernels, depicted on the right, vanishes on the interval $[0, \tau_{0,9}]$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2(a). In view of Remark 1.1, it suffices to prove the estimates (1.14) only for $R\big[Q_{n+1}^{R,r}; f\big]$. We apply the argument from [9] to the proof of (1.6), comparing $K_3(Q_{n+1}^{R,r}; t)$ with the adjusted one-periodic Bernoulli monospline

$$g(t) = \frac{1}{n^3}\Big(B_3(\theta) - B_3\big(\{nt + \theta\}\big)\Big), \quad \theta = \frac{3 + \sqrt{3}}{6}. \tag{3.1}$$

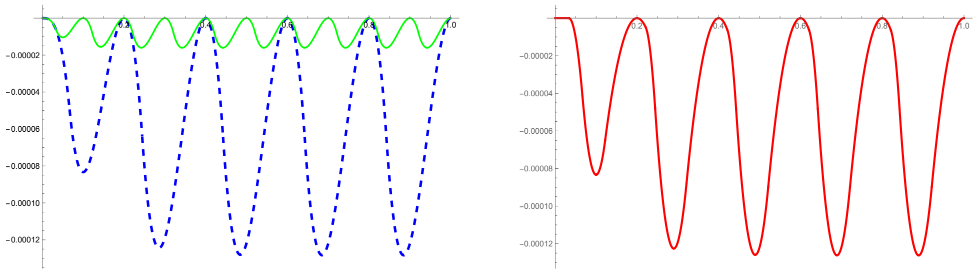

Figure 1. Left: graphs of $K_3(Q_5^{R,r};t)$ (dashed) and $K_3(Q_9^{R,r};t)$ (solid);
Right: graph of $K_3(Q_5^{R,r};t) - K_3(Q_9^{R,r};t)$.

Here, $\{\cdot\}$ is the fractional part function, B_3 is the third Bernoulli polynomial with leading coefficient $1/6$,

$$B_3(t) = \frac{1}{6}\Big(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t\Big),$$

and

$$B_3(\theta) = -\|B_3\| = -\frac{\sqrt{3}}{216}.$$

We need the following properties of $g(t)$, defined in (3.1):

(i) The zeros of $g(t)$ in $(0,1)$ are $x_k = \dfrac{k}{n}$, $k = 1,\dots,n-1$, each of them double;

(ii) $g(t)$ satisfies the inequalities

$$-\frac{\sqrt{3}}{108n^3} \le g(t) \le 0, \quad t \in [0,1];$$

(iii) $g(t)$ has n simple knots in $(0,1)$ located at the points $\dfrac{k-\theta}{n}$, $k = 1,\dots,n$;

(iv) $g(0) = g'(0) = g(1) = g'(1) = 0$ and

$$g''(0+) = g''(1-) = \frac{1-2\theta}{2n} < 0.$$

The set of zeros of $K_3(Q_{n+1}^{R,r};t)$ in $(0,1)$ coincides with that of the zeros of $g(t)$, namely, the double zeros at x_k, $k = 1,\dots,n-1$. Furthermore,

$$\begin{aligned} &K_3(Q_{n+1}^{R,r};0) = K_3'(Q_{n+1}^{R,r};0) = K_3(Q_{n+1}^{R,r};1) = K_3'(Q_{n+1}^{R,r};1) = 0, \\ &K_3''(Q_{n+1}^{R,r};0+) = 0, \quad K_3''(Q_{n+1}^{R,r};1-) = a_{n,n} > 0. \end{aligned} \tag{3.2}$$

Then $s(t) = g(t) - K_3(Q_{n+1}^{R,r};t)$ is a spline function of degree two with $2n$ knots in $(0,1)$. We apply Lemma 2.2 to s and obtain

$$\begin{aligned} 2n-2 \le Z_s(0,1) &\le Z_{s''}(0,1) + S^-\big(s(0),s'(0),s''(0+)\big) - S^+\big(s(1),s'(1),s''(1-)\big) \\ &\le 2n + S^-\Big(0,0,\frac{1-2\theta}{2n}\Big) - S^+\Big(0,0,\frac{1-2\theta}{2n} - a_{n,n}\Big) \\ &\le 2n-2. \end{aligned}$$

Hence, $s(t)$ has no other zeros in $(0,1)$ except the double ones at x_k, $k=1,\dots,n-1$, therefore $s(t)$ does not change its sign in $(0,1)$. From (iv) and (3.2) it follows that $s(t)\le 0$ on $[0,1]$, which together with (ii) implies

$$-\frac{\sqrt{3}}{108n^3}\le g(t)\le K_3(Q_{n+1}^{R,r};t)\le 0,\quad t\in[0,1]. \tag{3.3}$$

If $f\in C^3[0,1]$ and $f'''(t)\ge 0$, $t\in[0,1]$, then (3.3) implies

$$\begin{aligned}0\ge R[Q_{n+1}^{R,r};f]&=\int_0^1 K_3(Q_{n+1}^{R,r};t)f'''(t)\,dt\ge\min_{t\in[0,1]}K_3(Q_{n+1}^{R,r};t)\int_0^1 f'''(t)\,dt\\&\ge\min_{t\in[0,1]}g(t)\int_0^1 f'''(t)\,dt=-\frac{\sqrt{3}}{108n^3}\big(f''(1)-f''(0)\big).\end{aligned}$$

The proof of Theorem 1.2(a) is complete. □

Figure 2 shows how close to each other are the graphs of the third Peano kernel of a right Radau quadrature formula and the associated adjusted Bernoulli monospline $g(t)$ defined in (3.1) in the case $n=4$. For larger n, except for a small neighborhood of the left end-point of the interval, the two graphs are practically undistinguishable.

Proof of Theorem 1.2(b). The argument is similar to that in the proof of part (a). The fourth Peano kernel of the $(n+1)$-point Gaussian quadrature formula

$$Q_{n+1}^G[f]=\sum_{i=1}^{n+1}a_{i,n+1}^G f(\tau_{i,n+1}^G),\quad 0<\tau_{1,n+1}<\cdots<\tau_{n+1,n+1}<1,$$

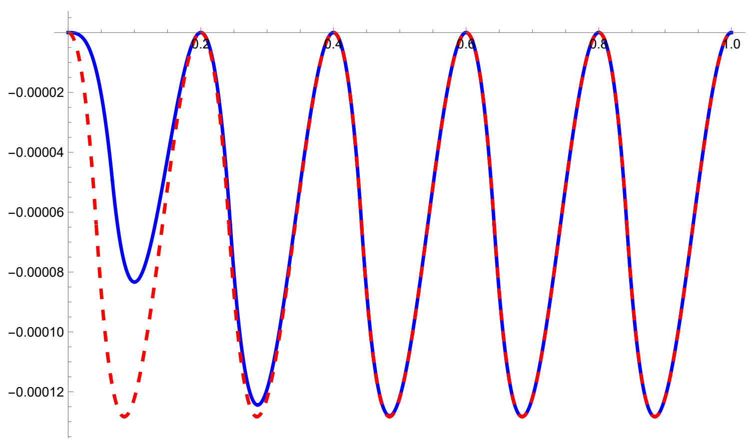

Figure 2. Graphs of Peano kernel $K_3(Q_{n+1}^{R,r};t)$ (solid) and of the associated adjusted Bernoulli monospline $g(t)$ defined in (3.4) (dashed), $n=4$.

associated with $S_{n,4}$, is compared with the adjusted Bernoulli monospline

$$g(t) = \frac{1}{n^4}\Big(B_4(\{nt\}) - B_4(0)\Big) = \frac{1}{n^4}\Big(B_4(\{nt\}) + \frac{1}{720}\Big), \tag{3.4}$$

where $B_4(t)$ is the fourth Bernoulli polynomial

$$B_4(t) = \frac{1}{24}\Big(t^4 - 2t^3 + t^2 - \frac{1}{30}\Big).$$

Now $g(t)$ is a monospline of degree four which has $n-1$ simple knots in $(0,1)$ located at the points $x_k = \dfrac{k}{n}$, $k = 1, \dots, n-1$. It follows from

$$-B_4(0) = -B_4(1) = \frac{1}{720} = \|B_4\|$$

that $g(t) \ge 0$, $t \in (0,1)$, and g has double zeros in $(0,1)$ at x_k, $k = 1, \dots, n-1$. Moreover,

$$\|g\| = \frac{1}{n^4}\Big(\max_{t\in[0,1]} B_4(t) + \frac{1}{720}\Big) = \frac{1}{384n^4}. \tag{3.5}$$

The difference $s(t) = g(t) - K_4(Q^G_{n+1}; t)$ is a spline function of degree three with $2n$ simple knots in $(0,1)$, namely, $\{x_k\}_{k=1}^{n-1} \cup \{\tau^G_{k,n+1}\}_{k=1}^{n+1}$. We have

$$\begin{aligned}
&s(0) = s'(0) = s(1) = s'(1) = 0,\\
&s''(0) = s''(1) = \frac{1}{12n^2},\\
&s'''(0+) = -\frac{1}{2n}, \quad s'''(1-) = \frac{1}{2n}.
\end{aligned}$$

The explicit form of $s'''(t)$ for $t \in (0,1)$ is

$$s'''(t) = -\frac{1}{2n} - \frac{1}{n}\sum_{k=0}^{n-1}(t - x_k)^0_+ + \sum_{k=1}^{n+1} a^G_{k,n+1}\big(t - \tau^G_{k,n+1}\big)^0_+.$$

Taking into account that all Gaussian weights $a^G_{k,n+1}$ are positive, we conclude that

$$Z_{s'''}(0,1) \le 2n - 1.$$

By applying Lemma 2.2 we obtain

$$\begin{aligned}
2n - 2 \le Z_s(0,1) &\le Z_{s'''}(0,1) + S^-\big(s(0), s'(0), s''(0), s'''(0+)\big)\\
&\qquad - S^+\big(s(1), s'(1), s''(1), s'''(1-)\big)\\
&\le 2n - 1 + S^-\Big(0, 0, \frac{1}{12n^2}, -\frac{1}{2n}\Big) - S^+\Big(0, 0, \frac{1}{12n^2}, \frac{1}{2n}\Big)\\
&= 2n - 2.
\end{aligned}$$

Hence, the only zeros of $s(t)$ in $(0,1)$ are the double zeros at x_k, $k = 1,\dots,n-1$, and $s(t)$ does not change its sign in $(0,1)$. Since $s(0) = s'(0) = 0$ and $s''(0) > 0$, it follows that $s(t) \ge 0$ on $[0,1]$, hence

$$g(t) \ge K_4(Q^G_{n+1};t) \ge 0, \quad t \in [0,1]. \tag{3.6}$$

If $f \in C^4[0,1]$ and $f^{(4)}(t) \ge 0$ on $[0,1]$, then (3.4) and (3.6) imply

$$\begin{aligned} 0 \le R[Q^G_{n+1};f] &= \int_0^1 K_4(Q^G_{n+1};t) f^{(4)}(t)\,dt \le \max_{t\in[0,1]} g(t) \int_0^1 f^{(4)}(t)\,dt \\ &= \frac{1}{384n^4}\Big(f'''(1) - f'''(0)\Big). \end{aligned}$$

The proof of Theorem 1.2(b) is complete. □

Remark 3.1. Using Lemma 2.2, error estimates analogous to those in Theorem 1.2 can be proved for all Gauss-type quadrature formulae associated with the spaces $S_{n,r}$, $r > 4$, defined in (1.8). However, since the Gauss-type quadrature formulae are not known for $r > 4$, these estimates are not of practical importance.

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