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ALMOST ZERO AND HIGH ω-TURING DEGREES

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We show that the class of the high degrees is definable in the terms of the almost zero degrees in the local substructure of the ω-Turing degrees. Namely, we prove that a degree is high if and only if it bounds all almost zero degrees.

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1. Introduction

Soskov [15] introduces the ω-enumeration reducibility between sequences of sets of natural numbers. As a preorder, this reducibility induces a degree structure – the upper semi-lattice $\mathcal{D}_\omega$ of the ω-enumeration degrees. Soskov also defines a jump operation, whose properties are later explored in [16]. Its local theory (i.e., the theory of the degrees below the first jump of the least ω-enumeration degree) is studied in [15, 16].

The ω-Turing reducibility $\leq_{T,\omega}$ is introduced by Sariev and Ganchev in [13] as an analogue of the ω-enumeration reducibility, which is based on the Turing reducibility. In this computational framework the informational content of a sequence is uniquely determined by the set of the Turing degrees of the sets that *code* the sequence. We say that a set codes a sequence if and only if *uniformly* in k, it can compute the k-th element of the considered sequence in its k-th Turing jump:

$$X \subseteq \omega \text{ codes } \{A_k\}_{k<\omega} \iff A_k \leq_T X^{(k)} \text{ uniformly in } k.$$

Having this, we shall say that the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ if and only if each set that codes $\mathcal{B}$ also codes $\mathcal{A}$:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}].$$

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The relation $\leq_{T,\omega}$ is a preorder on the set of the sequences of sets of natural numbers and in the standard way it induces a degree structure – the upper semi-lattice $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

The least ω-Turing degree $\mathbf{0}$ is that one containing all sequences coded by $\varnothing$. The requirement for uniformity in the definition of coding allows us to define a new notion of "lowness" or "closeness" to $\mathbf{0}$, which has no equivalent in the Turing degrees. Indeed, note that if $\{A_k\}_{k<\omega}$ is a sequence such that for each $k < \omega$, $A_k \leq_T \varnothing^{(k)}$, then its degree is not necessarily $\mathbf{0}$, because of the lack of uniformity of the reduction. We shall refer to such kind of sequences and to the degrees containing them as *almost zero*. There are continuum many almost zero degrees so they are not bounded by any degree. Also, the substructure of all almost zero degrees is sufficiently rich to embed each countable partial order [12].

In [13] a jump operation on sequences is defined. Namely the jump $\mathcal{A}'$ of the sequence $\mathcal{A}$ is defined in such a way that for each $X \subseteq \omega$

$$X \text{ codes } \mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \ \& \ Y \text{ codes } \mathcal{A}].$$

Being a degree invariant, the jump on sequences induces in the standard way a jump operation in the degree structure.

The structure of the ω-Turing degrees extends in a natural way the structure of the Turing degrees. The mapping

$$\deg_T(A) \mapsto \deg_{T,\omega}(A, \varnothing, \dots, \varnothing, \dots)$$

defines an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$ which preserves the least element, the order, the l.u.b. and jump operations. Also, the both structures have an isomorphic groups of automorphisms [13].

The jump operator gives rise to the *local structure* $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees – namely this is the substructure consisting of the degrees below the first jump $\mathbf{0}'$ of the least ω-Turing degree. As usual a *high/low hierarchy* can be introduced. This hierarchy partitions the local substructure on layers, based on the informational content of the jumps of the degrees. Intuitively, a degree is high$_n$ or low$_n$ according as its n-th jump takes its greatest or least value. The degrees which are high$_n$ for some n we shall refer as *high*; similarly, a degree is *low* if it is low$_n$ for some n.

The structures of ω-Turing and ω-enumeration degrees are closely related. There is a natural embedding of $\mathcal{D}_{T,\omega}$ into $\mathcal{D}_\omega$ (it is based on the natural embedding of the Turing degrees into the enumeration degrees), which preserves the least element, the l.u.b. operation and the jump, [13]. Although the both structures share a lot of common properties, they are not elementary equivalent. For example, there are minimal ω-Turing degrees [13], while the ω-enumeration degrees are dense [15].

Since the underlying structures of the Turing and the enumeration degrees are quite different, in many cases the results from $\mathcal{D}_\omega$ can not be transferred directly to $\mathcal{D}_{T,\omega}$. For example, the so called Kalimullin pairs which are extensively used in a sequence of definability results concerning both enumeration and ω-enumeration degrees, do not even exist in the Turing and the ω-Turing degrees.

This paper explores definability issues concerning the high and almost zero degrees. In [13] Sariev and Ganchev show that each of the classes of the high$_n$ and the low$_n$ degrees are first order definable in the local structure. In [5] they prove that the classes of the high and the low degrees are definable in $\mathcal{G}_{T,\omega}$. As a consequence, the first order definability of the almost zero degrees in the local structure is shown. Here we continue the investigation by showing a tighter connection between the definability of the high and the almost zero degrees. Namely, we show that in the local structure a degree is high if and only if it bounds all almost zero degrees.

A similar result holds in the structure of the ω-enumeration degrees, [16]. Namely Soskov and Ganchev show that for each non-high degree there is an almost zero degree which is not below it. Just like us, they use a modification of the Sacks agreement method [9, 11]. In order to make their construction below the first jump of the least element, they use the so-called *good approximation* of a sequence of sets of natural numbers. Since such approximations are not usable in the case of the ω-Turing degrees, we consider Δ^0_2-approximations for sequences – a uniform version of the Δ^0_2-approximations for sets. First, we characterize the degrees in the local structure exactly as the degrees which contain a sequence with a Δ^0_2-approximation. This allows us to build degrees below $\mathbf{0}'$. Since the Δ^0_2-approximations have a different nature form the good approximations used in [16] and the Σ^0_1-approximations used in the original Sacks' construction, specific modifications in the construction are made. For example, in order to make the constructed approximation to converge, we consider *maximum length of agreement* function, instead of the standard length of agreement function used in both Sacks' and Soskov and Ganchev constructions.

2. Preliminaries

2.1. Basic notions

We shall denote the set of natural numbers by ω. We also shall consider each natural number n as the set of natural numbers strictly less than n: $n = \{m \in \omega \mid m < n\}$. In this way when we write $D \subseteq n$, we mean that D is a finite set of natural numbers and all of its members are less than n.

If not stated otherwise, a, b, c, ... shall stand for natural numbers, A, B, C, ... for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

2.2. The Turing reducibility and degrees

We assume that the reader is familiar with the notion of Turing reducibility $\leq_T$, and with the structure of the Turing degrees $\mathcal{D}_T$ (for a survey of basic results on the Turing degree structure we refer the reader to [8–10, 14]).

Let $\{\widehat{P}_e\}_{e<\omega}$ be a fixed effective numbering of all Turing machine oracle programs. Note that $\widehat{P}_e$ is independent of the oracle A. We shall write $\Phi_e^A(x) = y$ if oracle program $\widehat{P}_e$ with A on its oracle tape and input x halts and yields output y. In this case we say that $\Phi_e^A(x)$ *converges* ($\Phi_e^A(x)\downarrow$); otherwise, $\Phi_e^A(x)$ diverges ($\Phi_e^A(x)\uparrow$). We refer to Φ_e^A as a e-th *Turing functional.* We shall write $\Phi_{e,s}^A(x) = y$ if $x, y, e < s$, and y is the output from $\Phi_e^A(x)$ in less that s steps of the oracle Turing program $\widehat{P}_e$. Note that always $\mathrm{dom}(\Phi_{e,s}^A) \subseteq s$.

We shall write W_e^A and $W_{e,s}^A$ for $\mathrm{dom}(\Phi_e^A)$ and $\mathrm{dom}(\Phi_{e,s}^A)$ respectively. By the Post's Theorem, the computably enumerable in A sets are exactly the sets $\{W_e^A\}_{e<\omega}$. Note that for each e,

$$W_{e,0}^A \subseteq W_{e,1}^A \subseteq \cdots \subseteq W_{e,s}^A \subseteq \cdots$$

and $W_e^A = \bigcup_s W_{e,s}^A$. Thus $x \in W_e^A$ if and only if $(\exists s)[x \in W_{e,s}^A]$ for all $x < \omega$. Also, each one of the sets $W_{e,s}^A$ is finite and computable in A. Later, we shall call such a sequence a Σ_1^A-approximation of W_e^A. By the Post's Theorem, a set is computably enumerable in A if and only if it has a Σ_1^A-approximation.

In order to simplify the constructions, we need the following corollary of the Recursion Theorem.

Corollary 2.1. *Let f be a computable function. Then there exists $e < \omega$, such that for all $A \subseteq \omega$,*

$$\Phi_{f(e)}^A = \Phi_e^A.$$

Proof. By the uniform version of the relativized Recursion Theorem [14], there is a computable function e such that for all sets $A \subseteq \omega$ and all $n \in \omega$, if Φ_n^A is a total function, then

$$\Phi_{\Phi_n^A(e(n))}^A = \Phi_{e(n)}^A.$$

Note that if f is computable function, then there is a natural number a such that for all A, $f = \Phi_a^A$. Indeed, just let $\widehat{P}_a$ be an oracle program which computes f that does not use the oracle tape. Now by the Recursion Theorem

$$\Phi_{f(e(a))}^A = \Phi_{\Phi_a^A(e(a))}^A = \Phi_{e(a)}^A.$$

Since the index a is universal for all sets A, we can take $e = e(a)$. □

We shall say that A is *Turing reducible* to (or *computable* in) B (written $A \leq_T B$) if there is a Turing functional Φ_e such that $A = \Phi_e^B$. The relation $\leq_T$ is a preorder on the powerset 2^ω of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Turing degrees. The Turing degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Turing reducibility between sets induces a partial order $\leq_T$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq_T \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $\langle \mathbf{D}_T, \leq_T \rangle$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\varnothing$. Also, the degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_T$ is an upper semi-lattice with least element.

The (Turing) jump A' of $A \subseteq \omega$ is defined as the halting problem for machines with an oracle A,

$$A' = \{e \mid \Phi_e^A(e)\downarrow\}.$$

The jump operation preserves the Turing reducibility, so we can define $\deg_T(A)' = \deg_T(A')$. Since $A <_T A'$, then we have $\mathbf{a} <_T \mathbf{a}'$ for every Turing degree $\mathbf{a}$. The jump operator is uniform, i.e., there exists a computable function j such that for every sets A and B, if $A \leq_T B$ via the Turing operator with index e, then $A' \leq_T B'$ via the operator with index $j(e)$.

The jump A' of A is computable enumerable in A uniformly in A. That is, there exists z such that for all sets A, $A' = W_z^A$. Thus $\{W_{z,s}^A\}_{s<\omega}$ is a Σ_1^A-approximation of A'.

2.3. The ω-Turing degrees

The ω-Turing reducibility and the corresponding degree structure $\mathcal{D}_{T,\omega}$ are introduced by Sariev and Ganchev in [13]. An equivalent, but more approachable definition in the terms of the uniform Turing reducibility is derived again in the same paper. Here we shall present only on the latter. It is based on the notion of *jump sequence* – it was introduced by Soskov [15] in order to describe the ω-enumeration reducibility.

For each $\mathcal{X} = \{X_k\}_{k<\omega}$, its corresponding *jump sequence* $\mathcal{P}(\mathcal{X})$ is defined as the sequence $\{P_k(\mathcal{X})\}_{k<\omega}$ such that:

- $P_0(\mathcal{X}) = X_0$
- $P_{k+1}(\mathcal{X}) = (P_k(\mathcal{X}))' \oplus X_{k+1}$, for each $k < \omega$.

Now, the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ (written $\mathcal{A} \leq_{T,\omega} \mathcal{B}$), if and only if

$$A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n.$$

Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \text{ and } \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Turing degree of $\mathcal{A}$. The relation $\leq$ defined by

$$\mathbf{a} \leq \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_{T,\omega} \mathcal{B})$$

is a partial order on the set of all ω-Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $\langle \mathbf{D}_{T,\omega}, \leq \rangle$. The ω-Turing degree $\mathbf{0}$ of the sequence $\varnothing_\omega = \{\varnothing\}_{k<\omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

It is not difficult to notice that each sequence and its jump sequence belong to the same ω-Turing degree, i.e., for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2.1}$$

In this way, $\mathcal{P}(\mathcal{A})$ is equivalent to an $\mathcal{A}$ sequence, whose members are monotone with respect to $\leq_T$ and each of its member decides the halting problems of the previous members.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \varnothing, \varnothing, \dots, \varnothing, \dots)$. The definition of $\leq_{T,\omega}$ and the uniformity of the jump operation imply that for all sets of natural numbers A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{2.2}$$

The last equivalence means, that the mapping $\kappa \colon \mathbf{D}_T \to \mathbf{D}_{T,\omega}$, defined by

$$\kappa(\deg_T(X)) = \deg_{T,\omega}(X \uparrow \omega),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Further, the so defined embedding κ preserves the order, the least element and the binary least upper bound operation.

We shall refer to κ as *the natural embedding* of the Turing degrees into the ω-Turing degrees. The range of κ shall be denoted by $\mathbf{D}_1$ and shall be called *the natural copy* of the Turing degrees.

2.4. The jump operator and jump inversion

In Computability Theory, often in addition to the considered reducibility between objects (sets, sequences of sets or functions, countable structures, etc.) a *jump operation* $'$ is introduced – for example the Turing jump ([6]) and the enumeration jump ([7]) over subsets of ω, or the jump of a structure ([17]). Usually the jump operator is monotone ($\alpha \leq \beta$ implies $\alpha' \leq \beta'$) and strictly expansive ($\alpha < \alpha'$). Its monotonicity allows to transfer the jump on the degrees (usually the degrees are the equivalence classes with respect to the relation $\equiv$ defined as: $\alpha \equiv \beta \iff \alpha \leq \beta \;\&\; \beta \leq \alpha$).

The notion of ω-Turing jump of a sequence of sets of natural numbers is defined by Sariev and Ganchev [13]. All the properties of the ω-Turing jump mentioned in this section are proved in the same paper. Following their lines, the ω-Turing jump $\mathcal{A}'$ of $\mathcal{A} = \{A_k\}_{k<\omega}$ is defined as the sequence

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \dots, A_k, \dots). \tag{2.3}$$

Note that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}_{k<\omega}$. The jump operator is strictly expansive and monotone with respect to the ω-Turing reducibility. This allows us to define a jump operation on the ω-Turing degrees by setting

$$\deg_{T,\omega}(\mathcal{A})' = \deg_{T,\omega}(\mathcal{A}').$$

Clearly for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$, $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, i.e., we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \dots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{2.4}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the degree $\mathbf{a}$.

Having a jump operation, a natural question that arises is what is the range of the operation? The Friedberg Jump Inversion Theorem [4] gives us that the range of the Turing jump over the upper cone of the degree $\mathbf{b}$ is exactly the upper cone of $\mathbf{b}'$ and the range of the enumeration jump consists exactly of the total degrees above $\mathbf{b}'$. Similar inversion result holds for the operation jump of structure, [17]. So, as one may expect, the range of the ω-Turing jump over the upper cone of $\mathbf{b}$ is exactly the upper cone over $\mathbf{b}'$, [13]. Further, stronger jump inversion properties can be asked, by specifying additional characteristics for the preimage of the jump.

For example in the case of the Turing degrees we have the Cooper Jump Inversion Theorem [2], by which each Turing degree above $\mathbf{0}'_T$ is a jump of a minimal degree (a degree is minimal if and only if it bounds only the least element $\mathbf{0}_T$). A similar result [7] by McEvoy in the enumeration degrees reveals that each total degree above $\mathbf{0}'_e$ is a jump of a quasi-minimal degree (an enumeration degree is quasi-minimal if and only if the only total degree below it is $\mathbf{0}_e$). As it is shown in [13], a ω-Turing degree $\mathbf{m}$ is minimal (i.e., bounds only the least element) if and only if there is a natural number n and a set M such that the Turing degree of M is a low over $\mathbf{0}_T^{(n)}$ (i.e., $\varnothing^{(n)} <_T M <_T M' \equiv_T \varnothing^{(n+1)}$) minimal cover of $\mathbf{0}_T^{(n)}$ (i.e., there are no A such that $\varnothing^{(n)} <_T A <_T M$) and

$$(\underbrace{\varnothing, \dots, \varnothing}_{n}, M, \varnothing, \dots) \in \mathbf{m}.$$

Therefore all minimal ω-Turing degrees are low – the jump of each of them is equal to $\mathbf{0}'$; so such a jump inversion theorem does not hold in $\mathcal{D}_{T,\omega}$. There is a sequel of jump inversion theorems in the Computable Structure Theory as well ([1, 17, 18]).

Using the Cooper Jump Inversion Theorem, one can show that there is no least jump inversion – there are countably many minimal degrees with jump equal to $\mathbf{a} >_T \mathbf{0}'_T$ and they are mutually incomparable; so there is no least degree with jump equal to $\mathbf{a}$. In the enumeration degrees the jump cannot be inverted to least degree, too. In a difference to $\mathcal{D}_T$ and $\mathcal{D}_e$, the structure of the ω-Turing degrees possesses a least jump inversion. More precisely, for each natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then the set

$$\{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x} \ \&\ \mathbf{x}^{(n)} = \mathbf{b}\}$$

has a least element. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_k, \dots), \tag{2.5}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

In the case when $\mathbf{a} = \mathbf{0}$, for the sake of simplicity, we shall use the notation $\mathbf{I}^n$ instead of $\mathbf{I}^n_{\mathbf{0}}$. The operation $\mathbf{I}^n$ is monotone:

$$\mathbf{0}^{(n)} \leq \mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{I}^n(\mathbf{x}) \leq \mathbf{I}^n(\mathbf{y}).$$

3. The local structure and the a.z. degrees

The degree substructure lying beneath the first jump of the least element is usually referred to as the local structure of the degree structure. In the case of the ω-Turing degrees, we shall denote this structure by $\mathcal{G}_{T,\omega}$,

$$\mathcal{G}_{T,\omega} = \langle \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid \mathbf{x} \leq \mathbf{0}'\}, \leq \rangle,$$

where $\leq$ is the inherited from $\mathcal{D}_{T,\omega}$ order. Proofs of all stated properties in this section can be found in [12, 13].

As usual, a degree in the local structure is said to be *high*$_n$ if and only if its n-th jump is as high as possible. A degree is *high* if and only if it is high$_n$ for some n. We shall denote the set of all high degrees in $\mathcal{G}_{T,\omega}$ by

$$\mathbf{H} = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \ \&\ (\exists n)[\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}]\}.$$

We need the following sufficient condition when a degree is in $\mathbf{H}$.

Proposition 3.1. *Let $\mathbf{a} \leq \mathbf{0}'$ and $\mathcal{A} \in \mathbf{a}$. Then $\mathbf{a} \in \mathbf{H}$, if there is a k such that $\varnothing^{(k+1)} \equiv_T P_k(\mathcal{A})$.*

Proof. Let $\mathcal{A} \leq_{T,\omega} \varnothing'_\omega$ and k be such that $\varnothing^{(k+1)} \equiv_T P_k(\mathcal{A})$. Then for each $r \geq 0$, $\varnothing^{(k+r+1)} \leq_T P_{k+r}(\mathcal{A})$ uniformly in r. Hence $\varnothing^{(k+1)}_\omega \leq_{T,\omega} \{P_{k+r}(\mathcal{A})\}_{r<\omega} \equiv_{T,\omega} \mathcal{A}^{(k)}$. Since the ω-Turing jump is monotone, then $\mathbf{a}$ is high$_k$ and so in $\mathbf{H}$. $\square$

Similarly a degree in the local structure is said to be *low*$_n$ if and only if its n-th jump is as low as possible. A degree is *low* if and only if it is low$_n$ for some n. We shall denote the set of all low degrees in $\mathcal{G}_{T,\omega}$ by

$$\mathbf{L} = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \ \&\ (\exists n)[\mathbf{a}^{(n)} = \mathbf{0}^{(n)}]\}.$$

The degrees that are neither high nor low, shall be referred to as *intermediate* degrees. The corresponding degree class is denoted by $\mathbf{I}$.

Further, for each $n < \omega$ define $\mathbf{o}_n$ as the least degree, satisfying the equation $\mathbf{x}^{(n)} = \mathbf{0}^{(n+1)}$:

$$\mathbf{o}_n = \mathbf{I}^n(\mathbf{0}^{n+1}).$$

In other words, $\mathbf{o}_n$ is the least high$_n$ degree. Thus, a degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$ is high$_n$ if and only if $\mathbf{o}_n \leq \mathbf{a}$. Note also that $\mathbf{o}_0 = \mathbf{0}'$. Since each high$_n$ degree is also high$_{n+1}$ degree but not all high$_{n+1}$ degrees are high$_n$ we have that

$$\mathbf{0}' = \mathbf{o}_0 > \mathbf{o}_1 > \cdots > \mathbf{o}_n > \cdots$$

By (2.3) and (2.5), for each $n < \omega$ the degree $\mathbf{o}_n$ contains the sequence $\mathcal{O}_n = \{O_n^k\}_{k<\omega}$, where $O_n^k = \varnothing$ for $k < n$, and $O_n^k = \varnothing^{(k+1)}$ for $k \geq n$.

Finally, we introduce the notion of *almost zero* (a.z.) degrees.

Definition 3.2. We shall say that the ω-Turing degree $\mathbf{a}$ is almost zero if and only if there is a sequence $\mathcal{A} \in \mathbf{a}$ such that

$$(\forall k)[P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}].$$

Note that the class of a.z. degrees is downward closed; also it is closed under the operation $\vee$ for l.u.b. There are continuum many a.z. degrees, hence not all of them are in $\mathcal{G}_{T,\omega}$. In fact, the almost zero degrees less than $\mathbf{0}'$ are fully characterized by the $\mathbf{o}_n$ degrees.

Proposition 3.3. *Let $\mathbf{a} \leq \mathbf{0}'$. Then $\mathbf{a}$ is a.z. if and only if for each n, $\mathbf{a} < \mathbf{o}_n$.*

Proof. Let $\mathbf{a}$ be an a.z. degree below $\mathbf{0}'$. Clearly $\mathbf{a} < \mathbf{o}_0 = \mathbf{0}'$. Now, fix a natural number $n \geq 1$. Fix a sequence $\mathcal{A}$ in it such that for each k, $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}$. Since $\mathbf{a} \leq \mathbf{0}'$, then uniformly in $k \geq n$:

$$P_k(\mathcal{A}) \leq_T \varnothing^{(k+1)} \equiv_T P_k(\mathcal{O}_n).$$

But for $k < n$, $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)} \equiv_T P_k(\mathcal{O}_n)$. Hence $P_k(\mathcal{A}) \leq_T P_k(\mathcal{O}_n)$ uniformly in k and thus $\mathcal{A} \leq_{T,\omega} \mathcal{O}_n$.

For the converse direction, let $\mathbf{a} \leq \mathbf{0}'$ be such that $\mathbf{a} < \mathbf{o}_n$ for each n. Let $\mathcal{A}$ be a sequence in $\mathbf{a}$. Then its jump sequence $\mathcal{P}(\mathcal{A})$ has the same degree. Since for each n, $\mathcal{P}(\mathcal{A}) \leq_{T,\omega} \mathcal{O}_{n+1}$, then for each n

$$P_n(\mathcal{A}) \leq_T P_n(\mathcal{O}_{n+1}) \equiv_T \varnothing^{(n)}.$$

Therefore $\mathbf{a}$ is an a.z. degree. □

There are no nonzero almost zero degrees neither in the high degrees, nor in the low degrees.

Proposition 3.4. *Let $\mathbf{a}$ be a nonzero a.z. degree. Then $\mathbf{a} \in \mathbf{I}$.*

Proof. Let $\mathbf{a}$ be a nonzero a.z. degree. Then for each n, $\mathbf{a} < \mathbf{o}_n$ and hence $\mathbf{a}$ is a non-high$_n$. So $\mathbf{a} \notin \mathbf{H}$.

Let $\mathcal{A}$ be a sequence in $\mathbf{a}$ such that $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}$ for each k. Then $\mathcal{A}^{(n)} \equiv_{T,\omega} (P_n(\mathcal{A}), P_{n+1}(\mathcal{A}), \dots)$. Thus, $I^n_{\varnothing_\omega}(\mathcal{A}^{(n)}) = (\varnothing, \dots, \varnothing^{(n-1)}, P_n(\mathcal{A}), P_{n+1}(\mathcal{A}), \dots) \equiv_{T,\omega} \mathcal{P}(\mathcal{A}) \equiv_{T,\omega} \mathcal{A}$. Hence $\mathbf{I}^n(\mathbf{a}^{(n)}) = \mathbf{a}$.

Now, suppose that $\mathbf{a}$ is low$_n$ for some n. Then $\mathbf{a} = \mathbf{I}^n(\mathbf{a}^{(n)}) = \mathbf{I}^n(\mathbf{0}^{(n)}) = \mathbf{0}$. A contradiction. Therefore $\mathbf{a}$ is a non-low. Thus $\mathbf{a}$ is intermediate. $\square$

4. Δ^0_2-APPROXIMATIONS

The aim of this section is to provide a machinery for constructing degrees below $\mathbf{0}'$. The idea is to use uniform relativization of the Δ^0_2-approximations used in the Turing degrees. We start with some basic notions and properties.

Definition 4.1. We say a computable sequence $\{A^s\}_{s<\omega}$ of finite sets is a Δ^0_2-approximation for A if for all x, $\lim_{s\to\infty} A^s(x)$ exists and it is equal to $A(x)$.

If A has Δ^0_2-approximation $\{A^s\}_{s<\omega}$, then we shall call a natural number $e < \omega$ an Δ^0_2-*index* for A corresponding to the approximation $\{A^s\}_{s<\omega}$ if and only if for all $s, x < \omega$, $A^s(x) = \Phi^{\varnothing}_e(s, x)$.

The sets possessing Δ^0_2-approximations are exactly those which are Turing reducible to the Halting problem (see, for example [2]).

Proposition 4.2 (The Limit Lemma)**.** *$A \leq_T \varnothing'$ if and only if A has a Δ^0_2-approximation.*

Moreover, we can pass effectively between a Δ^0_2-index for A and index for the reduction of A to $\varnothing'$ (i.e., such e that $A = \Phi^{\varnothing'}_e$), [14, Ch. 4.2].

The definition of Δ^0_2-approximation can be easily relativized to oracle B (see [14, Ch. 4.4]).

Definition 4.3. We say a computable in B sequence $\{A^s\}_{s<\omega}$ of finite sets is a Δ^B_2-approximation for A if for all x, $\lim_{s\to\infty} A^s(x)$ exists and it is equal to $A(x)$.

If $\{A^s\}_{s<\omega}$ is a Δ^B_2-approximation for A and $e < \omega$ is such that for all s, x, $A^s(x) = \Phi^B_e(s, x)$, then we shall call e an Δ^B_2-index for A, corresponding to the approximation $\{A^s\}_{s<\omega}$.

We have the relativized version of the Limit Lemma.

Proposition 4.4 (The Relativized Limit Lemma)**.** *$A \leq_T B'$ if and only if A has a Δ^B_2-approximation.*

Again, we can pass effectively between Δ^B_2-indices for A and indices for the reduction of A to B'.

Using a Δ^B_2-approximation of A we can build a Δ^B_2-approximation of each computable in A set.

Proposition 4.5. *Let $\{A^s\}_{s<\omega}$ be a Δ_2^B-approximation of A and e be such that Φ_e^A is a total characteristic function. Then $\{\Phi_{e,s}^{A^s}\}_{s<\omega}$ is a Δ_2^B-approximation of Φ_e^A.*

Proof. Let $x<\omega$. Then $\Phi_e^A(x)\downarrow$. Fix a $n<\omega$ greater than the maximal use of the oracle A in the computation $\Phi_e^A(x)$. Then

$$\Phi_e^{A\restriction n}(x)\downarrow=\Phi_e^A(x).$$

Let s_0 and s_1 be such that

$$(\forall t\geq s_0)[A^t\restriction n=A\restriction n]$$

and

$$(\forall t\geq s_1)[\Phi_{e,t}^{A\restriction n}(x)\downarrow=\Phi_e^{A\restriction n}(x)].$$

Hence, for each $t\geq\max\{s_0,s_1\}$ we have that

$$\Phi_{e,t}^{A^t}(x)\downarrow=\Phi_{e,t}^{A^t\restriction n}(x)=\Phi_{e,t}^{A\restriction n}(x)=\Phi_e^{A\restriction n}(x)=\Phi_e^A(x).$$

Thus for each $x<\omega$, $\lim_s\Phi_{e,s}^{A^s}(x)=\Phi_e^A(x)$. Since $\mathrm{dom}(\Phi_{e,s}^{A^s})\subseteq s$, then $\Phi_{e,s}^{A^s}$ is finite for each s. Finally, it is not difficult to notice that the sequence $\{\Phi_{e,s}^{A^s}\}_{s<\omega}$ is computable in B uniformly in s. $\square$

Finally, we define the notion of Δ_2^0-approximation for a sequence.

Definition 4.6. We say a sequence $\{A_n^s\}_{n,s<\omega}$ is a Δ_2^0-approximation for the sequence $\{A_n\}_{n<\omega}$ if $\{A_n^s\}_{s<\omega}$ is $\Delta_2^{\varnothing^{(n)}}$-approximation for A_n uniformly in n.

Again, the sequences with Δ_2^0-approximations are exactly those below $\varnothing'_\omega$.

Proposition 4.7. *$\mathcal{A}\leq_{T,\omega}\varnothing'_\omega$ if and only if $\mathcal{A}$ has a Δ_2^0-approximation.*

Proof. First, let $\mathcal{A}\leq_{T,\omega}\varnothing'_\omega$. Let f be a total computable function such that for each n, $A_n=\Phi_{f(n)}^{\varnothing^{(n+1)}}$. Let z be such that for all n, $\varnothing^{(n+1)}=W_z^{\varnothing^{(n)}}$. Denote $\varnothing_s^{(n+1)}=W_{z,s}^{\varnothing^{(n)}}$ for each $s,n<\omega$. Thus uniformly in n, $\{\varnothing_s^{(n+1)}\}_{s<\omega}$ is a $\Sigma_1^{\varnothing^{(n)}}$-approximation for $\varnothing^{(n+1)}$. Let for each n and s, $A_n^s=\Phi_{f(n),s}^{\varnothing_s^{(n+1)}}$.

Then for each n and s, A_n^s is finite, because $A_n^s\subseteq s$. Using the computability of f, it is easy to notice that uniformly in n the sequence $\{A_n^s\}_{s<\omega}$ is computable in $\varnothing^{(n)}$. Let $x<\omega$ and u be greater than the maximum use of the oracle $\varnothing^{(n+1)}$ in the computation of $A_n(x)$ as $\Phi_{f(n)}^{\varnothing^{(n+1)}}$. Let s_0 and s_1 be such that

$$(\forall t\geq s_0)[\varnothing^{(n+1)}\restriction u=\varnothing_t^{(n+1)}\restriction u]$$

and

$$(\forall t\geq s_1)[\Phi_{f(n),t}^{\varnothing^{(n+1)}}(x)\downarrow=\Phi_{f(n)}^{\varnothing^{(n+1)}}(x)].$$

Now, for each $t \geq \max\{s_0, s_1\}$, we have that

$$A_n^s(x) = \Phi_{f(n),t}^{\varnothing_t^{(n+1)}}(x) = \Phi_{f(n),t}^{\varnothing^{(n+1)}\upharpoonright u}(x) = \Phi_{f(n),t}^{\varnothing^{(n+1)}}(x) = \Phi_{f(n)}^{\varnothing^{(n+1)}}(x) = A_n(x).$$

Hence, $\{A_n^s\}_{n,s<\omega}$ is a Δ_2^0-approximation for $\mathcal{A}$.

Conversely, let $\{A_n^s\}_{n,s<\omega}$ be a Δ_2^0-approximation for the sequence $\mathcal{A}$. Since for each n, $\{A_n^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(n)}}$-approximation for A_n, then $A_n \leq_T \varnothing^{(n+1)}$. It remains to show that this reduction is uniform in n. We shall use the standard relativizations of the Enumeration Theorem and the s-m-n Theorem, [14]. Fix a natural number z and a computable function q such that for all sets $A \subseteq \omega$ and all $x, y, z_1, z_2 < \omega$: $\Phi_z^A(x, y) = \Phi_x^A(y)$ and $\Phi_{q(x,y)}^A(z_1, z_2) = \Phi_x^A(y, z_1, z_2)$.

Since $\{A_n^s\}_{s<\omega}$ is computable in $\varnothing^{(n)}$ uniformly in n, then there is a computable function f such that for all $n, s < \omega$

$$A_n^s = \Phi_{f(n,s)}^{\varnothing^{(n)}}.$$

Then for all $x, n, s < \omega$, $\Phi_{f(n,s)}^{\varnothing^{(n)}}(x) = \Phi_z^{\varnothing^{(n)}}(f(n, s), x)$. Since f is computable, then there is z' (which is uniformly computable by z and an index for f as a partial computable function) such that $\Phi_z^{\varnothing^{(n)}}(f(n, s), x) = \Phi_{z'}^{\varnothing^{(n)}}(n, s, x)$. But $\Phi_{z'}^{\varnothing^{(n)}}(n, s, x) = \Phi_{q(z',n)}^{\varnothing^{(n)}}(s, x)$ and hence there is a computable function $r(n) = q(z', n)$ such that for all $x, s, n < \omega$, $A_n^s(x) = \Phi_{r(n)}^{\varnothing^{(n)}}(s, x)$.

In this way we have a computable function which gives a $\Delta_2^{\varnothing^{(n)}}$-index for A_n corresponding to the approximation $\{A_n^s\}_{s<\omega}$ and hence there is a computable function which gives an index for the reduction of A_n to $\varnothing^{(n+1)}$. □

5. Defining the high degrees

In this section we shall prove our main result – namely that in the local substructure of the ω-Turing degrees, the set of all high degrees is definable in the terms of the a.z. degrees and the structure order $\leq$. First, we need the following result, revealing that there is no intermediate degree which bounds all a.z. degrees. The proof uses Δ_2^0-approximations of sequences and the construction is based on that one used in the Sacks Splitting Theorem [3, 11, 14].

Lemma 5.1. *Let $\mathbf{a} \in \mathbf{I}$. Then there exists an a.z. degree $\mathbf{d}$ such that $\mathbf{d} \leq \mathbf{0}'$ and $\mathbf{d} \not\leq \mathbf{a}$.*

Proof. Let $\mathbf{a} \in \mathbf{I}$ and $\mathcal{A} \in \mathbf{a}$. Fix Δ_2^0-approximations $\{P_n^s\}_{n,s<\omega}$ and $\{P_n^s(\mathcal{A})\}_{n,s<\omega}$ for $\mathcal{P}(\varnothing_\omega{}')$ and $\mathcal{P}(\mathcal{A})$ respectively. We shall construct a sequence $\{D_k^s\}_{k,s<\omega}$ such that:

1. for each k and each x, $\lim_{s\to\infty} D_k^s(x)$ exists and $D_k = \lim_{s\to\infty} D_k^s$ is finite;
2. uniformly in k, $\{D_k^s\}_{s<\omega}$ is computable in $\varnothing^{(k)}$;

3. for each k, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$.

The satisfaction of 1. and 2. immediately gives us that $\{D_k\}_{k<\omega}$ has an a.z. degree. As we will see, 3. implies that $\mathcal{D} \not\leq_{T,\omega} \mathcal{A}$.

We shall need the following notations. For each k and s, denote by $\Psi_k^s = \Phi_{k,s}^{P_k^s(\mathcal{A})}$. By Proposition 4.5, $\{\Psi_k^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(k)}}$-approximation of $\Phi_k^{P_k(\mathcal{A})}$. Given two sets X and Y of natural numbers and $s < \omega$, denote by $l^s(X,Y)$ the *length of agreement function* up to s:

$$l^s(X,Y) = \max\{n \leq s \mid X \restriction (n+1) = Y \restriction (n+1)\}.$$

Finally, for each $k, s < \omega$ let

$$l_k^s = l^s(D_k^s, \Psi_k^s)$$

and

$$m_k^s = \max\{l_k^t \mid t \leq s\}$$

stay respectively for the length of agreement and the maximum length of agreement between D_k^s and Ψ_k^s.

Construction. For each $k < \omega$, set $D_k^0 = \varnothing$. Suppose that for each $k < \omega$, D_k^s is constructed. Then for each k set

$$D_k^{s+1} = \{x \mid x \leq m_k^s \ \&\ x \in P_k^s\}.$$

End of the construction.

Note that $\{D_k^s\}_{s<\omega}$ is computable in $\varnothing^{(k)}$ uniformly in k. The next claim shows that $\{D_k^s\}_{k,s<\omega}$ is a Δ_2^0-approximation.

Claim 1. *For each k and x, $\lim_{s\to\infty} D_k^s(x)$ exists.*

Proof. Consider arbitrary k and x. Let s be such that the approximations $\{P_k^t\}_{t<\omega}$ and $\{\Psi_k^t\}_{t<\omega}$ are stabilized up to x after stage s, i.e., for all $t \geq s$,

$$P_k^s \restriction (x+1) = P_k(\varnothing_\omega{}') \restriction (x+1),$$

and

$$\Psi_k^t \restriction (x+1) = \Phi_k^{P_k(\mathcal{A})} \restriction (x+1).$$

If $x \notin P_k(\varnothing_\omega{}')$, then for each $t \geq s$, $x \notin P_k^t$ and so $D_k^t(x) = 0$. Now let $x \in P_k(\varnothing_\omega{}')$. Then for each $t \geq s$, $P_k^t(x) = 1$. On stage $s+1$ of the construction we have two cases:

- $m_k^s < x$. Then $l_k^s < x$ and hence $D_k^{s+1}(x) = 0 \neq 1 = \Psi_k^{s+1}(x)$. Therefore $l_k^{s+1} < x$ and thus $m_k^{s+1} < x$. Now using an induction on $t \geq s$, one can easily show that for all $t \geq s$, $D_k^t(x) = 0$.
- $x \leq m_k^s$. Since for each $t \geq s$, then $m_k^t \geq m_k^s \geq x$. But for all $t \geq s$, $x \in P_k^t$ and thus $D_k^t(x) = 1$.

This proves the claim. $\square$

Denote by D_k for each $k < \omega$, the limit of $\{D_k^s\}_{s<\omega}$. Thus $\{D_k^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(k)}}$-approximation of the set D_k. So $\{D_k\}_{k<\omega} \leq_{T,\omega} \varnothing_\omega{}'$. The next claim is needed in order to show that the degree of $\{D_k\}_{k<\omega}$ is an a.z. and not below $\mathbf{a}$.

Claim 2. *For each $k < \omega$, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$.*

Proof. Suppose that k is such that $D_k = \Phi_k^{P_k(\mathcal{A})}$. Then $D_k = P_k(\varnothing_\omega{}')$.

Indeed, let $x \in D_k$. Let s be such that for all $t \geq s$, $D_k^t(x) = 1$. By the construction, for each $t \geq s$, $x \leq m_k^t$ and $x \in P_k^t$. Hence $x \in P_k(\varnothing_\omega{}')$.

Now, let $x \in P_k(\varnothing_\omega{}')$. Let s_0 be such that after stage s_0 the approximation $\{P_k^s\}_{s<\omega}$ is stabilized on x:

$$(\forall t \geq s_0)[P_k^t(x) = P_k(\mathcal{A})(x) = 1].$$

Let s_1 be such that after stage s_1, the approximations $\{D_k^s\}_{s<\omega}$ and $\{\Psi_k^s\}_{s<\omega}$ are stabilized on each $y \leq x$:

$$(\forall t \geq s_1)(\forall y \leq x)[D_k^t(y) = D_k(y) \;\&\; \Psi_k^t(y) = \Phi_k^{P_k(\mathcal{A})}(y)].$$

Thus for each $t \geq s_1$ and for all $y \leq x$, $D_k^t(y) = D_k(y) = \Phi_k^{P_k(\mathcal{A})} = \Psi_k^t(y)$. Thus for each $t \geq \max\{s_0, s_1\}$, $x \leq l_k^t \leq m_k^t$ and since $x \in P_k^t$, then $x \in D_k^{t+1}$. Therefore $x \in D_k$.

Having that $D_k = P_k(\varnothing_\omega{}')$ and $D_k = \Phi_k^{P_k(\mathcal{A})}$, then $\varnothing^{(k+1)} \equiv_T P_k(\varnothing_\omega{}') \leq_T P_k(\mathcal{A})$. By Proposition 3.1, $\mathbf{a}$ must be a high degree, which contradicts with the fact that $\mathbf{a}$ is an intermediate. $\square$

The following claim shows that the sequence $\{D_k\}_{k<\omega}$ has an a.z. degree.

Claim 3. *For each $k < \omega$, the set D_k is finite.*

Proof. Fix k. Then, by the previous claim, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$. Let x be the least natural number such that $D_k(x) \neq \Phi_k^{P_k(\mathcal{A})}(x)$. Fix s such that after it the approximations $\{D_k^t\}_{t<\omega}$ and $\{\Psi_k^t\}_{t<\omega}$ are stabilized on each $y \leq x$. Then for all $t \geq s$:

- for all $y < x$, $D_k^t(y) = \Psi_k^t(y)$,
- $D_k^t(x) = D_k^s(x) \neq \Psi_k^s(x) = \Psi_k^t(y)$.

Thus for each $t \geq s$, $l_k^t = x - 1$ and hence $m_k^t = \max\{m_k^s, x-1\}$. Therefore, $D_k^t \subseteq m_k^{s+1} + 1$, which implies that $D_k \subseteq m_k^{s+1} + 1$; so D_k is finite. $\square$

Finally, we use the Corollary 2.1 of the Recursion Theorem in order to prove that $\{D_k\}_{k<\omega}$ is not below $\mathcal{A}$.

Claim 4. *$\{D_k\}_{k<\omega} \not\leq_{T,\omega} \mathcal{A}$.*

Proof. Suppose that $\{D_k\}_{k<\omega} \leq_{T,\omega} \mathcal{A}$ and let f be a computable function such that for each k, $D_k = \Phi_{f(k)}^{P_k(\mathcal{A})}$. By Corollary 2.1, let $e < \omega$ be such that for all $A \subseteq \omega$, $\Phi_e^A = \Phi_{f(e)}^A$. Then

$$D_e = \Phi_{f(e)}^{P_e(\mathcal{A})} = \Phi_e^{P_e(\mathcal{A})},$$

which contradicts Claim 2. □

This proves the lemma. □

Finally in our main result, we characterize the high degrees exactly as the degrees bounding all almost zero degrees.

Theorem 5.2. *Let* $\mathbf{a} \leq \mathbf{0}'$. *Then* $\mathbf{a} \in \mathbf{H}$ *if and only if for each a.z. degree* $\mathbf{d} \leq \mathbf{0}'$, $\mathbf{d} \leq \mathbf{a}$.

Proof. Let $\mathbf{a}$ be a high degree and $\mathbf{d} \leq \mathbf{0}'$ be an a.z. degree. Let n be such that $\mathbf{a}$ is a high$_n$. Hence $\mathbf{d} \leq \mathbf{o}_n \leq \mathbf{a}$.

Now let $\mathbf{a} \leq \mathbf{0}'$ be a degree that bounds all a.z. degrees below $\mathbf{0}'$. By the previous lemma, $\mathbf{a} \notin \mathbf{I}$. Let $\mathbf{d} \leq \mathbf{0}'$ be an a.z. degree. Since $\mathbf{d} \notin \mathbf{L}$, then for each n, $\mathbf{0}^{(n)} < \mathbf{d}^{(n)} \leq \mathbf{a}^{(n)}$. Hence $\mathbf{a}$ is not low$_n$ for any n. So $\mathbf{a} \notin \mathbf{L}$ and thus $\mathbf{a} \in \mathbf{H}$. □

The similar problem of the definability of the low degrees in the terms of the almost zero degrees still remains an open question.

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