ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 97

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 97

AN EXAMPLE OF ROTATIONAL HYPERSURFACE IN $\mathbb{R}^{n+1}$ WITH INDUCED IP METRIC FROM $\mathbb{R}^{n+1}$

YULIAN TSANKOV

We find a rotated hypersurface M^n whose induced metric from $\mathbb{R}^{n+1}$ is isometric to metric of IP manifolds and therefore the hypersurface is conformally flat. In the case of 4-dimensional hypersurface with IP metric we have presented explicitly a skew-symmetric curvature operator and have proved directly that its eigenvalues are pointwise. We find the mean curvature of the hypersurface.

Keywords: IP manifolds, curvature operator, rotated hypersurfaces

2000 MSC: 53A05, 53B20

Let ∇ be the Levi-Civita connection of a Riemannian manifold (M^m, g). Let x, y and z be tangent vector fields on M^m. Then the associated curvature tensor $R(x, y, z)$ is defined by

$$R(x,y,z) = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The value of $R(x, y, z)$ at a point p of M depends only of values of x, y and z at p.

The skew-symmetric curvature operator $K_{x,y}$ is defined by

$$K_{x,y}(u) = R(x,y,u)$$

for any orthonormal pair (x, y) of tangent vectors at any point p in M and $u \in T_pM$. It is easy to see that the curvature operator $K_{x,y}$ does not depend on the orientated orthonormal basis which is chosen for the orientated 2-plane $E^2 = span\{x, y\}$ [1]. A unit vector u is an eigenvector to $K_{x,y}$ with the corresponding eigenvalues c iff

$$K_{x,y}(u) = cu,$$

where, generally, c is a function of the point p and the plane E^2, $c = c(p; E^2)$. G. Stanilov first has stated a problem for the investigation of Riemannian manifolds of pointwise constant eigenvalues of $K_{x,y}$ [5].

In [4] Ivanov and Petrova have given a local classification of four dimension manifolds, where the skew-symmetric curvature operator $K_{x,y}$ has pointwise eigenvalues:

Theorem 1. *Let (M,g) be a four dimensional Riemannian manifold such that the eigenvalues of the skew-symmetric curvature operator are pointwise constants at any point p of the manifold M. Then (M,g) is locally (almost everywhere) isometric to one of the following spaces:*

a) *real space form;*

b) *a warped product $B \times_F N$, where B is an open interval on the real line, N is a 3-dimensional space form of the constant sectional curvature K, and F is a smooth function on B given by $F(u) = \sqrt{Ku^2 + Cu + D}$, K, C, D being constants such that $C^2 - 4KD \neq 0$.*

We say that (M,g) is IP if the eigenvalues of $K_{x,y}$ depend only on the point p in M and do not depend on the plane $E^2 = span\{x,y\}$.

This result is generalized for n-dimensional manifolds for $n \geq 4$ and $n \neq 7$ by Gilkey, Leahy and Sadofsky in [3], [2].

Later, we are going to give an example of a rotational hypersurface in $\mathbb{R}^{n+1}$, whose induced metric from $\mathbb{R}^{n+1}$ is isometric of IP n-dimensional manifolds.

Every rotated hypersurface M^n in $\mathbb{R}^{n+1}$ can be represented locally by

$$\begin{cases} x^1 & = & f(u^1)\sin(u^2)\sin(u^3)\dots\sin(u^n), \\ x^2 & = & f(u^1)\sin(u^2)\sin(u^3)\dots\cos(u^n), \\ \vdots & & \\ x^{n-1} & = & f(u^1)\sin(u^2)\cos(u^3), \\ x^n & = & f(u^1)\cos(u^2), \\ x^{n+1} & = & h(u^1), \end{cases} \tag{1}$$

$$u^i \in J_i, \ J_i \subset \mathbb{R}^1, \quad i = 1, \dots, n.$$

We write

$$x = (x^1, x^2, \dots, x^{n-1}, x^n, x^{n+1}),$$
$$x^i = x^i(u^1, u^2, \dots, u^n), \quad i = 1, \dots, n+1.$$

Suppose that

$$h(u^1) = \varepsilon \int_{u_o^1}^{u^1} \sqrt{1 - \left(\frac{df(v)}{dv}\right)^2} dv, \quad \varepsilon = \pm 1, \tag{2}$$

or

$$\left(\frac{df(u^1)}{du^1}\right)^2 + \left(\frac{dh(u^1)}{du^1}\right)^2 = 1.$$

This means that u^1 is a natural parameter of the curve

$$c \begin{cases} x^1 &= f(u^1), \\ x^2 &= 0, \\ \vdots & \\ x^n &= 0, \\ x^{n+1} &= h(u^1). \end{cases} \tag{3}$$

Let

$$f(u^1) = \sqrt{q(u^1)}, \quad q(u^1) > 0, \ u^1 \in J_1. \tag{4}$$

Then we can evaluate $h(u^1)$ from (2).

Below, we will consider a rotational surface generated from the rotation of a curve that satisfies conditions (2) and (4).

Using that $g_{ij} = \dfrac{\partial x}{\partial u^i}\dfrac{\partial x}{\partial u^j}$, we can evaluate directly the components of the metric tensor g of surface M, induced from the inner product of $\mathbb{R}^{n+1}$. The matrix of g is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q(u^1) & 0 & \cdots & 0 \\ 0 & 0 & q(u^1)\sin^2(u^2) & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & q(u^1)\sin^2(u^2)\ldots\sin^2(u^{n-1})\sin^2(u^n) \end{pmatrix}. \tag{5}$$

Therefore, the metric of the rotational surface given by (1) coincides with the metric of a warped product $B^1 \times_q S^{n-1}$, where B is an open interval in $\mathbb{R}$ and S^{n-1} is the $(n-1)$-dimensional sphere. The radius of the sphere S^{n-1} is 1.

If we set $q(u^1) = (u^1)^2 + C(u^1) + D$, $C^2 - 4D < 0$, the metric of (1) will coincide with the metric of IP manifolds which are not with constant sectional curvature. *This kind of rotational surfaces we will call rotational IP hypersurfaces.*

We can check directly that for rotational IP hypersurfaces in $\mathbb{R}^5$, generated by the rotation of the curve (3) when $q(u^1) = (u^1)^2 + Cu^1 + D$, it holds that the skew-symmetric curvature operator has pointwise eigenvalues. For this purpose we are going to use a local parametrization of 4-dimensional hypersurface. Explicitly, the parametrization of this surface is

$$\begin{cases} x^1 &= ((u^1)^2 + Cu^1 + D)\sin(u^2)\sin(u^3)\sin(u^4), \\ x^2 &= ((u^1)^2 + Cu^1 + D)\sin(u^2)\sin(u^3)\cos(u^4), \\ x^3 &= ((u^1)^2 + Cu^1 + D)\sin(u^2)\cos(u^3), \\ x^4 &= ((u^1)^2 + Cu^1 + D)\cos(u^2), \\ x^5 &= \dfrac{1}{2}\sqrt{4D - C^2}\ln(C + 2(u^1 + \sqrt{(u^1)^2 + Cu^1 + D})). \end{cases} \tag{6}$$

Its metric tensor g is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q(u^1) & 0 & 0 \\ 0 & 0 & q(u^1)\sin^2(u^2) & 0 \\ 0 & 0 & 0 & q(u^1)\sin^2(u^2)\sin^2(u^3) \end{pmatrix},$$

where $q(u^1) = D + Cu^1 + (u^1)^2$. The inverse matrix g^{-1} of g is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{D+Cu^1+(u^1)^2} & 0 & 0 \\ 0 & 0 & \dfrac{csc^2(u^2)}{D+Cu^1+(u^1)^2} & 0 \\ 0 & 0 & 0 & \dfrac{csc^2(u^2)csc^2(u^3)}{D+Cu^1+(u^1)^2} \end{pmatrix}.$$

When we have the metric tensor of a given manifold, we can calculate the Christofel symbols Γ_{ij}^k and the components of the curvature tensor R_{ijk}^l on the following way:

$$\Gamma_{ij}^k = \frac{1}{2} g^{hk} \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right),$$

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial u^i} - \frac{\partial \Gamma_{ik}^l}{\partial u^j} + \Gamma_{jk}^s \Gamma_{si}^l - \Gamma_{ki}^s \Gamma_{sj}^l.$$

After some algebra we find that the sectional curvature $k_{i,j}$, $i,j = 1,\ldots,4$, of two-dimensional planes given from the base vectors $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^3}, \frac{\partial}{\partial u^4}$ is

$$\begin{aligned} k_{1,2} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\ k_{1,3} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\ k_{1,4} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\ k_{2,3} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\ k_{2,4} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\ k_{3,4} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}. \end{aligned}$$

Using the components of the curvature tensor, we can find the matrix of the skew-symmetric curvature operator K_{e_i,e_j}, $i \neq j$, $i,j = 1,\ldots,4$, where $e_i = \dfrac{\frac{\partial}{\partial u^i}}{\sqrt{g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^i})}}$, $i = 1,\ldots,4$, relative to orthonormal base e_i. For example, the matrix of K_{e_1,e_2} is

$$K_{e_1,e_2} = \begin{pmatrix} 0 & -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 \\ \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The eigenvalues λ_i, $i = 1, \ldots, 4$, of this operator are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}.$$

In a similar way, the matrix of the skew-symmetric curvature operator K_{e_1,e_3} is

$$K_{e_1,e_3} = \begin{pmatrix} 0 & 0 & -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

The eigenvalues of this operator are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}.$$

In general, let us consider two orthonormal vectors $a, b \in T_pM$, i. e.

$$a = a^i e_i, \quad b = b^i e_i,$$

$$g(a,a) = 1, \quad g(b,b) = 1, \quad g(a,b) = 0. \tag{9}$$

Then the matrix of the skew-symmetric curvature operator $K_{u,v}$ with respect to the orthonormal base e_i is

$$K_{u,v} = k \begin{pmatrix} 0 & a^2b^1 - a^1b^2 & a^3b^1 - a^1b^3 & a^4b^1 - a^1b^4 \\ -a^2b^1 + a^1b^2 & 0 & -a^3b^2 + a^2b^3 & -a^4b^2 + a^2b^4 \\ -a^3b^1 + a^1b^3 & a^3b^2 - a^2b^3 & 0 & -a^4b^3 + a^3b^4 \\ -a^4b^1 + a^1b^4 & a^4b^2 - a^2b^4 & a^4b^3 - a^3b^4 & 0 \end{pmatrix}, \tag{10}$$

where $k = \dfrac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}$.

The eigenvalues of the curvature operator $K_{u,v}$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0,$$
$$\lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}\sqrt{A}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}\sqrt{A},$$

where

$$\begin{aligned} A = \; & (a^4)^2(b^1)^2 + (a^1)^2(b^2)^2 + (a^4)^2(b^2)^2 + (a^1)^2(b^3)^2 \\ & +(a^4)^2(b^3)^2 - 2a^1a^4b^1b^4 + (a^1)^2(b^4)^2 - a^3b^3(a^1b^1 + a^4b^4) \\ & -2a^2b^2(a^1b^1 + a^3b^3 + a^4b^4) + (a^3)^2((b^1)^2 + (b^2)^2 + (b^4)^2) \\ & +(a^2)^2((b^1)^2 + (b^3)^2 + (b^4)^2). \end{aligned}$$

Using (9), we obtain that $A = 1$. Therefore, the eigenvalues of the operator $K_{u,v}$ do not depend on the two-dimensional plane determined by the vectors u, v.

We are going to generalize the derived results in the following

Theorem 2. *The rotational 4-dimensional hypersurface given by* (6) *has pointwise eigenvalues.*

We can prove directly also the similar results in dimensions 3, 5, 6, 7.

Let us point out that the two-dimensional IP hypersurface in $\mathbb{R}^3$

$$\begin{cases} x^1 &= ((u^1)^2 + Cu^1 + D)\sin(u^2), \\ x^2 &= ((u^1)^2 + Cu^1 + D)\cos(u^2), \\ x^3 &= \dfrac{1}{2}\sqrt{4D - C^2}\ln(C + 2(u^1 + \sqrt{(u^1)^2 + Cu^1 + D})) \end{cases}$$

has a vanishing mean curvature, i. e. this is a minimal surface in $\mathbb{R}^3$.

But we prove directly that when we have k-dimensional IP rotational surfaces for $n = 3, 4, 5, 6, 7$, they are not minimal. More exactly, the mean curvature H of n-dimensional IP rotational surfaces is:

$$H = (n-2)\frac{\sqrt{-C^2 + 4D}}{2((u^1)^2 + Cu^1 + D)}.$$

We remark that the IP rotational hypersurfaces are conformally flat as rotational hypersurfaces. We can also see this from (5) or, using the local parametrization of the IP rotational surfaces, we can directly evaluate the components of the Weyl tensor W_{ijkl} and find that $W_{ijkl} = 0$.

We have used computer for some of the evaluations above.

REFERENCES

1. Belger, M., G. Stanilov. About the Riemannian geometry of some curvature operators. *Ann. Sofia Univ., Fac. Math and Inf.*, **84**, 1995 (in press).

2. Gilkey, P. Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues II. In: *Differential Geometry and Applications*, eds. Kolar, Kowalski, Krupka and Solvak, Massaryk University, Brno, Czech Republic, 1999, 73-87.

3. Gilkey, P., J. V. Leahy and H. Sadofsky. Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues. *Indiana Univ. Math. J.*, **48**, 1999, 615-634.

4. Ivanov, S. and I. Petrova. Riemannian manifold in which the skew-symmetric curvature operator has constant eigenvalues. *Geometriae Dedicata*, **70**, 1998, 269-282.

5. Ivanova, R. and G. Stanilov. A skew-symmetric curvature operator in Riemannian geometry. In: *Symposia Gaussiana, Conf. A*, eds. Behara, Fritsch, Lintz, Berlin, New York, 1995, 391-395.

Received February 28, 2003

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: ucankov@fmi.uni-sofia.bg