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ON INFINITESIMAL RIGIDITY OF HYPERSURFACES [1] IN EUCLIDEAN SPACE

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The infinitesimal rigidity of hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$, is considered. In section 1 we remind some definitions in the theory of the infinitesimal bendings (inf. b.). In section 2 we discuss the results in the papers [5 - 9]. In section 3 we consider our main result in the paper [10] and we give some geometric interpretations of the investigations in [10]. Finally we consider an example.

Keywords: hypersurfaces, Euclidian space, infinitesimal rigidity
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1. INTRODUCTION

The theory of the bendings of the surfaces is one of the most important sections of the classical differential geometry. The first definitions of the notion bending of the surfaces are in some 19th century works and concern only 2-dimensional surfaces in $\mathbb{R}^3$ (3-dimensional euclidean space). In these works the difference between bending and infinitesimal bending was not made. First Darboux in the end of the 19th century pointed out the difference between these two notions. The first results of the infinitesimal bendings (inf. b.) of the surfaces in $\mathbb{R}^3$ belong to Cauchy (1813) - for a closed convex polyhedron and to Liebmann (1901, 1919) - for an analytic convex surface. During the 20th century too many results on the inf. b. of the

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surfaces in $\mathbb{R}^{n+1}$ have been obtained (see [1 – 4]). In this paper we shall discuss the inf. b. of the hypersurfaces in $\mathbb{R}^{n+1}, n \geq 3$. First we shall give some definitions.

Let $S : r = r(u^1, ..., u^n)$ be a smooth hypersurface in $\mathbb{R}^{n+1}$ and let

$$S_t : r_t = r + 2tU + o(t), \quad t \longrightarrow 0\,,$$

be an infinitesimal deformation of S. Let c be an arbitrary smooth curve on S and let c_t be its corresponding curve on S_t. We denote the lengths of c and c_t with l and l_t correspondingly. The infinitesimal deformation S_t is called an inf. b. of S if the equality

$$l_t - l = o(t), \quad t \longrightarrow 0, \tag{1.1}$$

is true, i.e. the vector field U of inf. b. satisfies the equation

$$dr\, dU = 0. \tag{1.2}$$

The field U of inf. b. is called trivial if it has the form

$$U = \Omega r + \omega, \tag{1.3}$$

where Ω is a constant skew-symmetric matrix and ω is a constant vector. If the equation (1.2) has, under some conditions, only a trivial solution, i.e. the vector field U is of the form (1.3), then the hypersurface S is called infinitesimally rigid under these conditions.

2. THE RESULTS IN THE PAPERS [5]-[9]

There are 6 known papers ([5] - [10]) on inf. rigidity of hypersurfaces in $R^{n+1}, n \geq 3$, in the literature. The first results concerning inf. rigidity of hypersurfaces belong to Sen'kin [5]. He investigated inf.b. of general convex hypersurfaces, i.e. the convex hypersurfaces for which smoothness was not assumed. The theory of inf. b. of such surfaces in R^3 was developed by A. D. Alexandrov (1936). Sen'kin used this theory and the results of A. V. Pogorelov (1959) for inf. b. of general convex surfaces in $\mathbb{R}^3$ and proved ([5]) the following

Theorem 2.1 (Sen'kın, 1972). *A closed convex hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is infinitesimally rigid. If S contains n-dimensional domains it is inf. rigid outside of these domains.*

Theorem 2.2 (Sen'kin, 1975). *A closed convex hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is inf. rigid in neighborhood of each point, which does not lie in a flat $(n-1)$-dimensional and $(n-2)$-dimensional domain. If S contains flat n-dimensional domains, it is inf. rigid in neighborhood of the indicated points outside of the flat n-dimensional domains.*

In 1975 Goldstein and Ryan, using the theory of the conformal vector fields, proved ([6]) that the sphere S^n in $\mathbb{R}^{n+1}, n \geq 3$, is inf. rigid and in 1980 Nannicini proved ([7]) that a C^∞ smooth compact strictly convex hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, is inf. rigid. It is obvious that these two results are contained in Theorem 2.1 of Sen'kin.

In [7] Nannicini proved the following

Theorem 2.3 (Nannicini, 1980). *Let $\widetilde{S}$ be a $(n-1)$-dimensional C^∞ smooth compact strictly convex surface which lie on a hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}, n \geq 3$. Let $\mu = \mathbb{R}^{n-1}$ be a subspace of $\mathbb{R}^n$ and $\widetilde{S} \cap \mu = \varnothing$. The rotation hypersurface $S = \widetilde{S} \times S^1$ in $\mathbb{R}^{n+1}$ obtained by rotation of $\widetilde{S}$ around μ is inf. rigid.*

In [8] Markov proved the following

Theorem 2.4 (Markov, 1980). *Let S be a C^3 smooth hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with type number $\tau \geq 3$, i.e. S has at least 3 nonzero principal curvatures at each point. Then each neighborhood on S of every point of S is inf. rigid.*

This result of Markov is infinitesimal analog of the well known classical result of Beez (1876) and Killing (1885) for isometric rigidity of hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$. The surface in theorem 4 can be compact or noncompact.

In [9] Dajczer and Rodrigues prove the following

Theorem 2.5 (Daiczer, Rodrigues, 1990). *Every smooth compact hypersurphace in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is infinitesimally rigid.*

This result is an infinitesimal version of a very beautiful result of Sacksteder (1960) for isometric rigidity.

3. THE MAIN RESULT IN THE PAPER [10] AND SOME GEOMETRIC INTERPRETATIONS

In the paper [10] we obtain sufficient conditions for inf. rigidity of a class of hypersurfaces with boundary in $\mathbb{R}^{n+1}, n \geq 3$, which are projected one-to-one orthogonally on a region G in a hyperplane. Such a hypersurface S is represented by:

$$S : x^{n+1} = f(x), x = (x^1, ..., x^n) \in G \tag{3.1}$$

We assume that G is a bounded finitely connected region with piecewise smooth boundary $\partial G = \Gamma$ and the function $f(x)$ and the field $U(x)$ $(\xi^1(x), \ldots, \xi^n(x), \zeta(x))$ of the inf. b. of S belong to the class $C^3(\overline{G})$. We assume that the inequalities

$$f_{\beta\beta\beta} > 0, f_{\beta\beta\beta} f_{\alpha\alpha\beta} - f^2_{\alpha\beta\beta} > 0 \text{ (respectively } f_{\beta\beta\beta} < 0, f_{\beta\beta\beta} f_{\alpha\alpha\beta} - f^2_{\alpha\beta\beta} > 0) \tag{3.2}$$

are fulfilled on a set $\widetilde{G}_{\alpha\beta}$ everywhere dense in $\overline{G} = G \cup \partial G$, $\beta := \alpha + 1$, $\alpha = 1, 3, ..., n-3, n-1$ for n even and $\alpha = 1, 3, ..., n-2, n-1$ for n odd. Here we denote with $f_\beta, f_{\beta\alpha}, f_{\beta\beta\beta}, ...$ the partial derivatives $f_{x^\beta}, f_{x^\beta x^\alpha}, f_{x^\beta x^\beta x^\beta}, ...$. Further our presentation will be for n even - when n is odd the things are analogous.

We shall give a geometric interpretation of the inequalities (3.2). Let $P(a^1, a^2, ..., a^n)$ be an arbitrary point of $\overline{G}$. We consider, for fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$, the 2-dimensional surface $S^{\alpha\beta} = S \cap \mathbb{R}^3_{\alpha\beta}$, where $\mathbb{R}^3_{\alpha\beta}$ is the 3-dimensional plane which contains P and is parallel to the coordinate 3-dimentional plane $O_{e_\alpha e_\beta e_{n+1}}$. The surface $S^{\alpha\beta}$ has the representation

$$S^{\alpha\beta} : x^{n+1} = f(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n), \ (x^\alpha, x^\beta) \in \overline{G}^P_{\alpha\beta} = \overline{G} \cap \mathbb{R}^3_{\alpha\beta} \quad (3.3)$$

with respect to the coordinate system $O'_{e_\alpha e_\beta e_{n+1}}$, $O'(a^1, ..., a^{\alpha-1}, 0, 0, a^{\alpha+2}, ..., a^n, 0)$. The following statement is valid

Proposition 3.1. *Let S be of the class* (3.1), (3.2). *Then the surfaces*

$$S^{\alpha\beta}_\beta : x^{n+1} = f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n), \alpha \in \{1, 3, ..., n-3, n-1\}, \beta = \alpha+1, \quad (3.4)$$

have Gaussian curvature $K > 0$ on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta}) = \pi^{-1}(\widetilde{G} \cap \mathbb{R}^3_{\alpha\beta})$ [2] *and they are convex (correspondingly locally convex) if $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ is convex (correspondingly nonconvex).*

Proof. From (3.2) for the sign of the Gaussian curvature K of $S^{\alpha\beta}_\beta$ on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta})$ we have

$$\operatorname{sgn} K = \operatorname{sgn}(f_{\beta\beta\beta} f_{\beta\alpha\alpha} - f^2_{\beta\alpha\beta}) > 0. \quad (3.5)$$

Let $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ be convex. For fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$ we consider the quadratic form

$$C(\xi_\alpha, \xi_\beta) = f_{\beta\alpha\alpha}\xi^2_\alpha + 2f_{\beta\alpha\beta}\xi_\beta\xi_\alpha + f_{\beta\beta\beta}\xi^2_\beta \quad (3.6)$$

of the function $f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n)$. From the inequalities (3.2) it follows that quadratic form (3.6) is nonnegative (nonpositive) in $\overline{G}^P_{\alpha\beta}$. Hence the function $f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n)$ is convex. Therefore the surface $S^{\alpha\beta}_\beta$ is convex.

Let $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ be nonconvex, where $\alpha \in \{1, 3, ..., n-3, n-1\}$ is fixed, $\beta = \alpha + 1$. For every point of $G^P_{\alpha\beta}$ we take a convex neighborhood and repeating the above reasonings we obtain that the surface $S^{\alpha\beta}_\beta$ is locally convex.

Let us consider the curve $c^{\alpha\beta}_\beta = S^{\alpha\beta}_\beta \cap \mathbb{R}^2_\beta$, where $\mathbb{R}^2_\beta$ is 2-dimensional plane across an arbitrary point of $\widetilde{G}^P_{\alpha\beta}$ and is parallel to $O_{e_\beta e_{n+1}}$. Let $\nu_{c^{\alpha\beta}_\beta}$ be the normal

[2] We denote with π the orthogonal projection $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n = O_{e_1, ..., e_n}$.

curvature of $c_\beta^{\alpha\beta}$ relative to the unit normal vector $\bar{l}$ to $S_\beta^{\alpha\beta}$ for which $(\bar{l}, e_{n+1})_e < \frac{\pi}{2}$. We have

$$\operatorname{sgn}(f_{\beta\beta\beta}) = \operatorname{sgn}(\nu_{c_\beta^{\alpha\beta}}) \tag{3.7}$$

on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta} \bigcap \mathbb{R}^2_\beta)$. Then from (3.2), (3.5) and (3.7)we obtain

Corollary 3.1. *If S is of the class* (3.1), (3.2$_1$)*(correspondingly* (3.1), (3.2$_2$)*. then the surfaces $S_\beta^{\alpha\beta}, \alpha = 1, 2, \ldots, n-3, n-1$, $\beta = \alpha + 1$ are convex below (correspondingly above).*

Let $\widetilde{n} = (\widetilde{n}^1, \ldots, \widetilde{n}^n)$ be the unit vector of the exterior normal to $\partial G = \Gamma$. Then $\widetilde{n}^\gamma = \cos\theta^\gamma$, where $\theta^\gamma = (e_\gamma, \widetilde{n})_e$ and e_γ is the unit vector of the axis $Ox^\gamma, \gamma = 1, ..., n$. We decompose ([10]) the smooth parts of Γ (for fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$) in nonintersecting subsets $\Gamma_i^{\alpha\beta}, i = 1, 2, 3, 4$, as follows:

1) on $\Gamma_1^{\alpha\beta} : H^{\alpha\beta}\widetilde{n}^\beta \geq 0, \;\; f_{\beta\beta}\widetilde{n}^\beta > 0$ (respectively $H^{\alpha\beta}\widetilde{n}^\beta \leq 0, f_{\beta\beta}\widetilde{n}^\beta < 0$);

2) on $\Gamma_2^{\alpha\beta}$: $\begin{cases} a)\, H^{\alpha\beta}\widetilde{n}^\beta < 0, \;\; f_{\beta\beta}\widetilde{n}^\beta \leq 0 \\ \text{(respectively } H^{\alpha\beta}\widetilde{n}^\beta > 0, \;\; f_{\beta\beta}\widetilde{n}^\beta \geq 0) \text{ or} \\ b)\, \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} \neq 0 \;\; \text{or} \;\; \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha > 0 \\ \text{(respectively } \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} \neq 0 \,\text{or}\, \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha < 0); \end{cases}$

3) on $\Gamma_3^{\alpha\beta} : H^{\alpha\beta}\widetilde{n}^\beta < 0, \;\; f_{\beta\beta}\widetilde{n}^\beta > 0$ (respectively $H^{\alpha\beta}\widetilde{n}^\beta > 0, f_{\beta\beta}\widetilde{n}^\beta < 0$);

4) on $\Gamma_4^{\alpha\beta}$: $\begin{cases} a)\, \widetilde{n}^\beta \neq 0, H^{\alpha\beta}\widetilde{n}^\beta \geq 0 \;\; f_{\beta\beta}\widetilde{n}^\beta \leq 0 \\ \text{(respectively } \widetilde{n}^\beta \neq 0, H^{\alpha\beta}\widetilde{n}^\beta \leq 0 \;\; f_{\beta\beta}\widetilde{n}^\beta \geq 0) \;\; \text{or} \\ b)\, \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha \leq 0 \\ \text{(respectively } \widetilde{n}^\beta = 0, \;\; f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha \geq 0); \end{cases}$

where $H^{\alpha\beta} = f_{\beta\beta}(\widetilde{n}^\alpha)^2 - 2f_{\alpha\beta}\widetilde{n}^\alpha\widetilde{n}^\beta + f_{\alpha\alpha}(\widetilde{n}^\beta)^2$.

The decomposition of $\Gamma = \partial G$ induces corresponding decomposition of the boundary $\partial S = \pi^{-1}(\partial G)$ in nonintersecting parts ${}^S\Gamma_i^{\alpha\beta} = \pi^{-1}(\Gamma_i^{\alpha\beta}), i = 1, 2, 3, 4$, which depends only on the geometric properties of ∂S. Indeed, a) $\widetilde{n}^\beta = \cos\theta^\beta, \theta^\beta = (e_\beta, \widetilde{n})_e, \widetilde{n}^\alpha = \cos\theta^\alpha, \theta^\alpha = (e_\alpha, \widetilde{n})_e$; b) $\operatorname{sgn} H^{\alpha\beta} = \operatorname{sgn} \nu_{\widehat{l}}^{\alpha\beta}$, where $\nu_{\widehat{l}}^{\alpha\beta}$ is the normal curvature to the curve $c^{\alpha\beta} = \partial S \cap \mathbb{R}^3_{\alpha\beta}$ relative to unit normal vector $\widehat{l}$ to the surface S for which $(\widehat{l}, e_{n+1})_e < \frac{\pi}{2}$ ($c^{\alpha\beta}$ has equations $x^i = c^i = const, \;\; i = 1, ..., n+1, \;\; i \neq \alpha, \beta, \;\; x^\alpha = x^\alpha(s), \;\; x^\beta = x^\beta(s), \;\; \dot{c}^{\alpha\beta}(0, ..., 0, \dot{x}^\alpha(s), \dot{x}^\beta(s), , 0, ..., 0)$ and $\dot{x}^\alpha = \sigma\widetilde{n}^\beta, \;\; \dot{x}^\beta = -\sigma\widetilde{n}^\alpha, \;\; \sigma \neq 0$, since $\widetilde{n} \perp \underline{c}^{\alpha\beta}$, where $\underline{c}^{\alpha\beta} = \pi(c^{\alpha\beta})$); c)$f_{\beta\beta}$ is the normal curvature of the x^β-line on S relative to $\widehat{l}$ and $f_{\alpha\beta}$ is the polar form of the second fundamental form relative to $\widehat{l}$ for x^α-line and x^β-line of S in the points of ∂S.

Let $L \subset S$ be a surface of dimension $k, \;\; 1 \leq k \leq n-1$. Every inf. b. S_t of S with a field U which satisfies the condition

$$Ue_{n+1}|_L = 0 \quad \text{(respectively} \;\; Ue_{n+1}|_L = const) \tag{3.8}$$

is called inf. b. with sliding (respectively generalized sliding) along L with respect to $\gamma = Ox^1...x^n$. Let L_1 be the orthogonal projection of L on γ i.e. $L_1 = \pi(L)$.

We shall call x^β - inf. b. along L with respect to γ every inf. b. S_t of S for which the field U of inf. b. satisfy the condition

$$(Ue_{n+1})_\beta|_{L_1} = 0. \tag{3.9}$$

We denote

$$^S\Gamma_{13} = \bigcup_{\substack{\alpha=1,3,\dots,n-3,n-1 \\ \beta=\alpha+1}} ({}^S\Gamma_1^{\alpha\beta} \cup {}^S\Gamma_3^{\alpha\beta}),$$

$$^S\Gamma_4 = \bigcap_{\substack{\alpha=1,3,\dots,n-3,n-1, \\ \beta=\alpha+1}} {}^S\Gamma_4^{\alpha\beta},$$

$$^S\Gamma_1 = \bigcup_{\substack{\alpha=1,3,\dots,n-3,n-1, \\ \beta=\alpha+1}} {}^S\Gamma_1^{\alpha\beta}.$$

We proved in [10] the following

Theorem 3.1. *The hypersurface* (3.1), (3.2) *is rigid under inf. b. with sliding (or generalized sliding) along* $^S\Gamma_{13}$ *with respect to the hyperplane* γ *and* x^β *- inf. b. along* $^S\Gamma_2^{\alpha\beta} \cup {}^S\Gamma_3^{\alpha\beta}, \alpha = 1, 3, ..., n-3, n-1,\ \beta = \alpha + 1,$ *with respect to* γ.

Remark. There are ([10]) $m \leq \frac{n}{2}$ conditions on the field U of the inf. b. at every point $P \in \partial S, P \notin {}^S\Gamma_4$, since there are $\frac{n}{2}$ decompositions of the boundary ∂S. Certainly we assume that these conditions are consistent.

We denote with $^S\Gamma_{21}^{\alpha\beta}$ for $\alpha \in \{1, 3, \dots, n-3, n-1\}$ and $\beta = \alpha + 1$ this part of $^S\Gamma_2^{\alpha\beta}$, whose orthogonal project on the hyperplane $\gamma = 0e_1 \dots e_n$ is composed of $(n-1)$-dimensional planes parallel to the coordinate vectors e_β, $\beta = 2, 4, \dots, n$ or it is composed of $(n-1)$-dimensional ruled surfaces, whose generatrixes are parallel to l_β, $\beta = 2, 4, \dots, n$. Let

$$^S\Gamma_{21} = \bigcup_{\substack{\alpha=1,3,\dots,n-3,n-1, \\ \beta=\alpha+1}} {}^S\Gamma_{21}^{\alpha\beta}.$$

Then we have

Corollary 3.2. *The hypersurface* (3.1), (3.2), *which has a boundary* $\partial S = {}^S\Gamma_1 \cup {}^S\Gamma_{21} \cup {}^S\Gamma_4$ *is rigid under inf.b. with sliding (or generalized sliding) along* $\partial S \setminus {}^S\Gamma_4$ *with respect to the hyperplane* γ.

Corollary 3.3. *The hypersurface* (3.1), (3.2), *which has a boundary* $\partial S = {}^S\Gamma_1 \cup {}^S\Gamma_{21} \cup {}^S\Gamma_4$ *is rigid if the part* $^S\Gamma_1 \cup {}^S\Gamma_{21}$ *of* ∂S *is fixed.*

4. AN EXAMPLE

The hypersurface

$$S: x^{n+1} = \sum_{\alpha=1,3,\dots,n-3,n-1} [(x^{\alpha+1})^3 + x^{\alpha+1}(x^{\alpha})^2] + (x^1)^2,\ x = (x^1,\dots,x^n) \in G$$

is of the class (3.1), (3.2) since

$f_{\beta\beta\beta} = 6, f_{\beta\beta\beta}f_{\beta\alpha\alpha} - f^2_{\beta\beta\alpha} = 12,\ \alpha = 1,3,\dots,n-3,n-1,\ \beta = \alpha+1.$

It is not from Beez-Killing's class. Its type number τ for example at the point $O(0,...,0)$ is 1.

Let $n = 4$ and G be a 4-dimensional cube, i.e.

$$S: x^5 = (x^2)^3 + (x^4)^3 + x^2(x^1)^2 + x^4(x^3)^2 + (x^1)^2, \tag{4.1}$$

$$G = \{(x^1,...,x^4) \in \mathbb{R}^4 : -1 \le x^i \le 1,\ i = 1,...,4\}.$$

We have $U(\xi^1(x),\dots,\xi^4(x),\zeta(x)),\ x = (x^1,\dots,x^4)$, and:

(a) $f_{22} = 6x^2, f_{12} = 2x^1, f_{44} = 6x^4, f_{34} = 2x^3$;

(b) $H^{12} = 6x^2(\widetilde{n}^1)^2 - 4x^1\widetilde{n}^1\widetilde{n}^2 + (2x^2+2)(\widetilde{n}^2)^2$,
$H^{34} = 6x^4(\widetilde{n}^3)^2 - 4x^3\widetilde{n}^3\widetilde{n}^4 + 2x^4(\widetilde{n}^4)^2$;

(c) $\partial G = \sum_{i=1}^{4} \partial G_i^{\pm}$,
$\partial G_1^{\pm} : x^1 = \pm 1, -1 \le x^2, x^3, x^4 \le 1,$
$\partial G_2^{\pm} : x^2 = \pm 1, -1 \le x^1, x^3, x^4 \le 1,$
$\partial G_3^{\pm} : x^3 = \pm 1, -1 \le x^1, x^2, x^4 \le 1,$
$\partial G_4^{\pm} : x^4 = \pm 1, -1 \le x^1, x^2, x^3 \le 1;$

(d) $\widetilde{n}|_{\partial G_1^{\pm}}(\pm 1,0,0,0),\quad \widetilde{n}|_{\partial G_2^{\pm}}(0,\pm 1,0,0),$
$\widetilde{n}|_{\partial G_3^{\pm}}(0,0,\pm 1,0),\quad \widetilde{n}|_{\partial G_4^{\pm}}(0,0,0,\pm 1).$

From 1) - 4) and (a) - (d) we find

$$\partial G_1^{\pm} \subset \Gamma_2^{12},\ \ \partial G_1^{\pm}\backslash\theta_1^{\pm} \subset \Gamma_2^{34},\ \ \theta_1^{\pm} \subset \Gamma_4^{34},\ \ \theta_1^{\pm} : \begin{cases} x^1 = \pm 1, \\ x^4 = 0, \\ -1 \le x^2,\ x^3 \le 1; \end{cases} \tag{4.2}$$

$$\partial G_2^{\pm} \subset \Gamma_1^{12},\ \ \partial G_2^{\pm}\backslash\theta_2^{\pm} \subset \Gamma_2^{34},\ \ \theta_2^{\pm} \subset \Gamma_4^{34},\ \ \theta_2^{\pm} : \begin{cases} x^2 = \pm 1, \\ x^4 = 0, \\ -1 \le x^1,\ x^3 \le 1; \end{cases} \tag{4.3}$$

$$\partial G_3^{\pm}\backslash\theta_3^{\pm} \subset \Gamma_2^{12},\ \ \theta_3^{\pm} \subset \Gamma_4^{12},\ \ \theta_3^{\pm} : \begin{cases} x^3 = \pm 1, \\ x^2 = 0, \\ -1 \le x^1,\ x^4 \le 1 \end{cases},\ \partial G_3^{\pm} \subset \Gamma_2^{34}; \tag{4.4}$$

$$\partial G_4^{\pm}\backslash\theta_4^{\pm} \subset \Gamma_2^{12},\ \ \theta_4^{\pm} \subset \Gamma_4^{12},\ \ \theta_4^{\pm} : \begin{cases} x^4 = \pm 1, \\ x^2 = 0, \\ -1 \le x^1,\ x^3 \le 1 \end{cases},\ \partial G_4^{\pm} \subset \Gamma_1^{34}; \tag{4.5}$$

We note that:

a$_1$) if $\zeta|_{\partial G_1^\pm} = const$ then $\zeta_2 = \zeta_4 = 0$ since $\partial G_1^\pm || Ox^2, Ox^4$;

b$_1$) if $\zeta|_{\partial G_2^\pm} = const$ then $\zeta_4 = 0$ since $\partial G_2^\pm || Ox^4$;

c$_1$) if $\zeta|_{\partial G_3^\pm} = const$ then $\zeta_2 = \zeta_4 = 0$ since $\partial G_3^\pm || Ox^2, Ox^4$;

d$_1$) if $\zeta|_{\partial G_4^\pm} = const$ then $\zeta_2 = 0$ since $\partial G_4^\pm || Ox^2$;

From Theorem 3.1, (4.2) - (4.5) and a$_1$) - d$_1$) we obtain

Proposition 4.1. *The hypersurface S (4.1) in $\mathbb{R}^5$ is rigid under inf. b. with sliding (or generalized sliding)along its boundary ∂S with respect to the hyperplane $\gamma = Ox^1x^2x^3x^4$.*

Proposition 4.2. *The hypersurface S (4.1) in $\mathbb{R}^5$ is rigid if its boundary ∂S is fixed.*

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