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GENERALIZED TURÁN'S GRAPH THEOREM

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Let G be an n-vertex graph and there is a vertex of G which is contained in maximum number of p-cliques, but is not contained in $(s+1)$-clique, where $2 \leq p \leq \min(s, n)$. Then the number of p-cliques of G is less than the number of p-cliques in the n-vertex S-partite Turán's graph $T_s(n)$ or $G = T_s(n)$.

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One of the fundamental results in graph theory is the theorem of P. Turán, proved in 1941, [5]. It generalizes a result of Mantel from 1906, [4], saying that if a graph on n vertices has more than $n^2/4$ edges, then this graph necessarily contains a triangle.

Turán's theorem was significantly generalized by Zykov in 1949, [6]. This generalization, unlike Turán's theorem, is not so popular. In this article we present a method to prove Zykov's theorem and its extension, used by us for solving similar problems (see [1], [2] and [3]). Let us fix some notations. We consider graphs $G = (V, E)$, where V is the set of vertices and $E \subseteq \binom{V}{2}$ is the set of edges. If $\{u, v\} \in E$, we say that the vertices u and v are adjacent. We call a *p-clique* of G a set of p vertices, each two of which are adjacent. The number of p-cliques of the graph G will be denoted by $c_p(G)$, and the number of p-cliques containing a vertex v by $c_p(v)$.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_s = (V_s, E_s)$ be graphs such that $V_i \cap V_j = \emptyset$, $i \neq j$. We denote by $G_1 + G_2 + \cdots + G_s$ the graph $G = (V, E)$ with

$$V = V_1 \cup V_2 \cup \cdots \cup V_s \quad \text{and} \quad E = E_1 \cup E_2 \cup \cdots \cup E_s \cup E',$$

where E' consists of all 2-element subsets $\{u, v\}$, $u \in V_i$, $v \in V_j$, $i \neq j$.

Consider a graph with n vertices. If each two of them are adjacent, we denote this graph by K_n, and if no two are adjacent – by $\overline{K}_n$. The graph $\overline{K}_{n_1} + \cdots + \overline{K}_{n_s}$ will be denoted by $K(n_1, \ldots, n_s)$. Obviously, $K(n_1, \ldots, n_s)$ is a complete s-partite graph. If $n_1 + \cdots + n_s = n$ and $|n_i - n_j| \leq 1$ for all i, j, then $K(n_1, \ldots, n_s)$ is denoted by $T_s(n)$ and is called s-partite n-vertex Turán's graph. Clearly, $T_s(n) = K_n$ for $s \geq n$.

Turán's theorem. ([5]) *Let s and n be positive integers and G be an n-vertex graph without $(s+1)$-cliques. Then*

$$c_2(G) \leq c_2\left(T_s(n)\right)$$

and $c_2(G) = c_2\left(T_s(n)\right)$ only if $G = T_s(n)$.

Zykov's theorem. ([6]) *Let p, s and n be positive integers and G be an n-vertex graph without $(s+1)$-cliques. Then:*

(a) $c_p(G) \leq c_p\left(T_s(n)\right)$;

(b) *if $c_p(G) = c_p\left(T_s(n)\right)$ and $2 \leq p \leq \min(n, s)$, then $G = T_s(n)$.*

A special case of Zykov's theorem is the following

Lemma. *Let p, s and n be positive integers and $2 \leq p \leq \min(n, s)$. Then*

$$c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) \leq c_p(T_s(n))$$

for each s-tuple $(n_1, n_2, \ldots, n_s)$ of nonnegative integers n_i such that $n_1 + n_2 + \cdots + n_s = n$. The equality is possible only if $K(n_1, n_2, \ldots, n_s) = T_s(n)$.

Proof. Suppose that $n_1, n_2, \ldots, n_s$ are such that $c_p\Big(K(n_1, n_2, \ldots, n_s)\Big)$ is maximal. Let also $n_1 = \max\{n_1, n_2, \ldots, n_s\}$ and $n_2 = \min\{n_1, n_2, \ldots, n_s\}$.

For $2 \leq p \leq \min(s, n)$ we have

$$\begin{aligned} c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) &= \sum\{n_{i_1} \ldots n_{i_p} \,|\, 1 \leq i_1 < i_2 < \cdots < i_p \leq s\} \\ &= n_1 n_2 M + (n_1 + n_2) N + P, \end{aligned}$$

where M, N and P do not depend on n_1 and n_2 and $M > 0$. Hence

$$c_p\Big(K(n_1 - 1, n_2 + 1, n_3, \ldots, n_s)\Big) - c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) = M(n_1 - n_2 - 1).$$

The maximality of $c_p\Big(K(n_1, n_2, \ldots, n_s)\Big)$ implies $n_1 - n_2 \leq 1$. From this inequality it follows $K(n_1, n_2, \ldots, n_s) = T_s(n)$.

Proof of Zykov's theorem. Let v_0 be a vertex of the graph G which is contained in a maximum number of p-cliques, i. e. $c_p(v) \le c_p(v_0)$ for each vertex v. Denote by A the set of vertices v of G, $v \ne v_0$, such that both v and v_0 are contained in some p-clique of the graph G, and by B the set of the remaining vertices of G. Let $\langle A\rangle$ be the subgraph of G generated by A (the vertex set of $\langle A\rangle$ is A and two vertices are adjacent in $\langle A\rangle$ if and only if they are adjacent in G).

Each p-clique of G is either entirely contained in A or has at least one vertex in B. Hence

$$c_p(G) \le c_p(\langle A\rangle) + \sum_{v\in B} c_p(v), \tag{1}$$

with equality if and only if each p-clique of G has at most one vertex in B. Obviously, $c_p(v_0) = c_{p-1}(\langle A\rangle)$ for $p \ge 2$, and since $c_p(v) \le c_p(v_0)$ for each vertex v,

$$c_p(v) \le c_{p-1}(\langle A\rangle) \quad \text{for each vertex } v \text{ in } B \text{ and } p \ge 2. \tag{2}$$

If $k = |A|$ and $p \ge 2$, it follows from (1) and (2) that

$$c_p(G) \le c_p(\langle A\rangle) + (n-k)c_{p-1}(\langle A\rangle). \tag{3}$$

Equality holds in (3) if and only if it holds in (1) and (2), that is, when there are no p-cliques with more than one vertex in B, and each vertex of B is adjacent to the vertices of each $(p-1)$-clique of $\langle A\rangle$. In the special case $p = s = 2$, equality occurs in (3) if and only if $G = K(k, n-k)$.

We prove the inequality (a) by induction on s. The base $s = 1$ is clear, since in this case $G = \overline{K}_n$.

For the inductive step, assume that $s \ge 2$. Suppose first that $p = 1$. Then $c_1(G) = c_1(T_s(n)) = n$. Let $p \ge 2$. If $c_p(v_0) = 0$, then $c_p(G) = 0$ and (a) is obvious. Let $c_p(v_0) > 0$, i. e. $A \ne \emptyset$. Note that $\langle A\rangle$ does not contain s-cliques, since G does not contain $(s+1)$-cliques. Applying the inductive hypothesis for $\langle A\rangle$, we conclude that if $|A| = k$, then

$$c_p(\langle A\rangle) \le c_p(T_{s-1}(k)), \tag{4}$$

$$c_{p-1}(\langle A\rangle) \le c_{p-1}(T_{s-1}(k)). \tag{5}$$

It follows from (3) – (5) that

$$c_p(G) \le c_p(T_{s-1}(k)) + (n-k)c_{p-1}(T_{s-1}(k)). \tag{6}$$

Set $\Gamma = \overline{K}_{n-k} + T_{s-1}(k)$. Clearly,

$$c_p(\Gamma) = c_p(T_{s-1}(k)) + (n-k)c_{p-1}(T_{s-1}(k)). \tag{7}$$

The lemma, applied to the graph Γ, yields

$$c_p(\Gamma) \le c_p(T_s(n)). \tag{8}$$

The inequality (a) now follows from (6) – (8).

Passing on to (b), let G be a graph with n vertices without $(s+1)$-cliques, $2 \le p \le \min(s,n)$ and

$$c_p(G) = c_p(T_s(n)). \tag{9}$$

We prove the equality $G = T_s(n)$ by induction on s. Note first that the equality (9) implies equalities in (3) – (6) and (8). By the assumption $2 \le p \le \min(s,n)$, the minimal admissible value of s is 2.

The base of the induction is then $s = 2$; in this case $p = 2$. Let G be a graph with n vertices without 3-cliques satisfying (9) for $p = 2$. Then there is equality in (3) and, as pointed out above, $G = K(k, n-k)$. In view of this, $c_2(K(k,n-k)) = c_2(T_s(n))$. The lemma implies $K(k,n-k) = T_s(n)$ and so $G = T_s(n)$.

Assume now $s \ge 3$ and that (b) holds for graphs without s-cliques. We start the inductive step by noting that $k \ge p-1$. Indeed, it follows from $p \le \min(n,s)$ that $c_p(T_s(n)) > 0$, and (9) implies $c_p(G) > 0$. Thus $c_p(v_0) = c_{p-1}(\langle A \rangle) > 0$, which clearly yields $k = |A| \ge p-1$.

Now we prove that

$$\langle A \rangle = T_{s-1}(k). \tag{10}$$

The cases $p \ge 3$ and $p = 2$ will be treated separately. Let $p \ge 3$. Then $2 \le p-1$. Also, $p-1 \le \min(s-1,k)$. By the inductive hypothesis the equality in (5) implies (10). We are left with the case $p = 2$. If $k \ge 2$, then $p = 2 \le \min(s-1,k)$. So, by the inductive hypothesis, the equality in (4) implies (10). If $k = 1$, (10) holds trivially, because $\langle A \rangle = T_{s-1}(1) = K_1$.

Based on (10), we prove that $G = \Gamma$. It follows from $p-1 \le \min(s-1,k)$ that each vertex of $T_{s-1}(k)$ is a vertex of a $(p-1)$-clique. Since there is equality in (3), we conclude that each vertex of A is adjacent to each vertex of B. On the other hand, B does not contain adjacent vertices. Otherwise, two such vertices, together with $(p-2)$-clique of A, would form a p-clique containing two vertices of B, contradicting the fact that there is equality in (3). It follows from this argument and (10) that $G = \Gamma$.

By the lemma the equality in (8) yields $\Gamma = T_s(n)$, and so $G = T_s(n)$.

The proof of Zykov's theorem is complete. Instead of $c_{s+1}(G) = 0$ we have used the weaker condition $c_{s+1}(v_0) = 0$. Hence, this proof, actually, establishes the following stronger statement:

Theorem. *Let p, s and n be positive integers and $2 \le p \le \min(s,n)$. Let v_0 be a vertex of an n-vertex graph G such that $c_p(v_0) = \max\{c_p(v) \mid v \in G\}$ and v_0 is not contained in an $(s+1)$-clique. Then the inequality (a) and the statement (b) of the theorem of Zykov hold.*

In conclusion, let us note that a direct counting argument for the p-cliques of $T_s(n)$ gives

$$c_p(T_s(n)) = \sum_{t=0}^{p} \binom{\nu}{t} \binom{s-t}{p-t} k^{p-t},$$

where $n = ks + \nu$, $0 \leq \nu < s$.

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