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ESTIMATES FOR THE SINGULAR SOLUTIONS OF THE 3-D PROTTER'S PROBLEM

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For the wave equation we study boundary value problems, stated by Protter in 1952, as some three-dimensional analogues of Darboux problems on the plane. It is known that Protter's problems are not well posed and the solution may have singularity at the vertex O of a characteristic cone, which is a part of the domain's boundary $\partial\Omega$. It is shown that for n in $\mathbb{N}$ there exists a right-hand side smooth function from $C^n(\bar{\Omega})$, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}\backslash O)$, but it has a strong power-type singularity. It is isolated at the vertex O and does not propagate along the cone. The present article gives some necessary and sufficient conditions for the existence of a fixed order singularity. It states some exact a priori estimates for the solution.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions

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1. INTRODUCTION

We discuss some boundary value problems for the wave equation

$$\square u = u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \tag{1.1}$$

in a simply connected domain $\Omega \subset \mathbb{R}^3$. The domain

$$\Omega := \{(x_1, x_2, t) :\ 0 < t < 1/2,\ t < \sqrt{x_1^2 + x_2^2} < 1 - t\}$$

is bounded by two characteristic cones of (1.1)

$$S_1 = \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$
$$S_2 = \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}$$

and the circle $S_0 = \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$ centered at the origin $O(0,0,0)$. The following three-dimensional analogues of the plane Darboux problems are stated by M. Protter [27]:

Problems P1 and P2. To find a solution of the wave equation (1.1) in Ω, which satisfies one of the following boundary conditions:

P1 $u|_{S_0} = 0$ and $u|_{S_1} = 0$;

P2 $u_t|_{S_0} = 0$ and $u|_{S_1} = 0$.

The corresponding adjoint problems are:

Problems P1* and P2*. To find a solution of the wave equation (1.1) in Ω, which satisfies the corresponding boundary conditions:

P1* $u|_{S_0} = 0$ and $u|_{S_2} = 0$;

P2* $u_t|_{S_0} = 0$ and $u|_{S_2} = 0$.

For the recent known results concerning Protter's problems see [25] and references therein. For further publications in this area see [1, 2, 8, 13, 16, 19, 20].

Substituting the boundary condition on S_0 by $[u_t + \alpha u]|_{S_0} = 0$, one obtains Problem P_α, for which we refer to [11] and references therein. In the case of the wave equation, involving either lower order terms or some other type perturbations, Problem $P2$ in Ω has been studied in [1, 2, 3, 12]. On the other hand, Bazarbekov [5] gives another analogue of the classical Darboux problem in the same domain Ω. Some other statements of Darboux type problems can be found in [4, 6, 18] in bounded or unbounded domains different from Ω.

Protter [27] formulated and studied these three-dimensional analogues of the Darboux problem on the plane 50 years ago - in 1952. Nowadays, it is known that in contrast to the Darboux problem in $\mathbb{R}^2$ the $3-D$ Problems $P1$ and $P2$ are not well posed. The reason for this is that the adjoint homogeneous Problems $P1^*$ and $P2^*$ have smooth solutions and the linear space they generate is infinite dimensional as one could see in Tong Kwang-Chang [29], Popivanov, Schneider [24], Khe Kan Cher [20] and Popivanov, Popov [26].

Lemma 1.1. (*see* [13]) *Let ρ, φ and t be the polar coordinates in $\mathbb{R}^3$: $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$. Let us define the functions*

$$H_k^n(\rho, t) := \sum_{i=0}^{k} A_i^k \frac{t(\rho^2 - t^2)^{n-3/2-k-i}}{\rho^{n-2i}} \tag{1.2}$$

and

$$E_k^n(\rho, t) = \sum_{i=0}^{k} B_i^k \frac{(\rho^2 - t^2)^{n-1/2-k-i}}{\rho^{n-2i}},$$

where the coefficients are

$$A_i^k := (-1)^i \frac{(k-i+1)_i(n-1/2-k-i)_i}{i!(n-i)_i}$$

and

$$B_i^k := (-1)^i \frac{(k-i+1)_i(n+1/2-k-i)_i}{i!(n-i)_i}$$

with $(a)_i := a(a+1)...(a+i-1)$. *Then for* $n \in \mathbb{N}$, $n \geq 4$, *the functions*

$$V_k^{n,1}(\rho,t,\varphi) = H_k^n(\rho,t)\cos n\varphi \quad \text{and} \quad V_k^{n,2}(\rho,t,\varphi) = H_k^n(\rho,t)\sin n\varphi$$

are classical solutions of the homogeneous Problem $P1^*$ *for all* $k = 0, 1, ..., \left[\frac{n}{2}\right] - 2$, *and the functions*

$$W_k^{n,1}(\rho,t,\varphi) = E_k^n(\rho,t)\cos n\varphi \quad \text{and} \quad W_k^{n,2}(\rho,t,\varphi) = E_k^n(\rho,t)\sin n\varphi$$

are classical solutions of the homogeneous Problem $P2^*$
for all $k = 0, 1, ..., \left[\frac{n-1}{2}\right] - 1$.

A necessary condition for the existence of classical solution for Problem $P1$ (Problem $P2$) is the orthogonality of the right-hand side function f to all functions $V_k^{n,i}(\rho,t,\varphi)$ (respectively $W_k^{n,i}$). To avoid an infinite number of necessary conditions in the frame of classical solvability, we need to introduce some *generalized solutions* of Problems $P1$ and $P2$ with eventually singularity on the characteristic cone Σ_2, or only at its vertex O. Popivanov, Schneider in [24] and [25] give the following definition:

Definition 1.1. *A function* $u = u(x_1, x_2, t)$ *is called a generalized solution of* the Problem $P1$ in Ω if:

1) $u \in C^1(\bar{\Omega}\backslash O)$, $u\big|_{S_0\backslash O} = 0$, $u\big|_{S_1} = 0$;

2) the identity

$$\int_\Omega (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - f w)\, dx_1 dx_2 dt = 0$$

holds for all $w \in C^1(\bar{\Omega})$, $w = 0$ on S_0, and $w = 0$ in a neighborhood of S_2.

Garabedian [10] proved the uniqueness of a classical solution of Problem $P1$. Popivanov, Schneider [25] proved the uniqueness of a *generalized solution* in $C^1(\overline{\Omega}\backslash O)$. It is known (cf. Popivanov, Schneider [25], Aldashev [1]) that for every $n \in \mathbb{N}$, $n \geq 4$, there exists a smooth function

$$f_n(x_1, x_2, t) := f_n^1(\rho,t)\cos n\varphi \in C^{n-2}(\overline{\Omega}),$$

whose corresponding *generalized solution* of Problem $P1$ near the origin O behaves like r^{1-n}, where $r = (x_1^2 + x_2^2 + t^2)^{1/2}$ is the distance to the origin. The same phenomenon appears in the case of a more general boundary condition P_α (see [11]). These singularities of the *generalized solutions* do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation, whose right-hand side is sufficiently smooth in $\bar{\Omega}$, cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for operators of second order we refer to Hörmander [15, Chapter 24.5]. For some related results in the case of the plane Darboux problem see [23].

Further, we study the Problem $P1$ only in the case when the right-hand side function f is a trigonometric polynomial of order l:

$$f(x_1, x_2, t) = f_0^1(\rho, t) + \sum_{n=1}^{l} \left(f_n^1(\rho, t) \cos n\varphi + f_n^2(\rho, t) \sin n\varphi \right). \qquad (1.3)$$

In this case Popivanov, Schneider [25] proved the existence and uniqueness of a *generalized solution* $u(x_1, x_2, t)$ of the corresponding problem. We already know that $u(x_1, x_2, t)$ may have power type singularity at the origin. More precisely, there are solutions that have the growth of r^{1-l} at the point O. In this paper we will prove some existence and uniqueness results for Problems $P1$ and $P2$ and study the behavior of the *generalized solution* around the origin. Let us denote the weighted uniform norm

$$||f_n||_q^* := \sum_{|\alpha| \le q;\ i=1,2} \max_{0 \le t \le \rho \le 1-t} \left| r^{|\alpha|+1/2} D^\alpha f_n^i(\rho, t) \right|,$$

analogous to the weighted Sobolev norms in corner domains (see [21], [14]). Denote as usual $x_+^\alpha = x^\alpha$ for $x > 0$, and $x_+^\alpha = 0$ for $x \le 0$. Then the main results are:

Theorem 1.1. *Let us suppose that* $f(x_1, x_2, t) \in C^{(l-4)_+}(\overline{\Omega})$ *has the form* (1.3) *and that*

$$\int_\Omega V_k^{n,i}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt = 0 \qquad (1.4)$$

for all $n = 2, 3, ..., l$; $i = 1, 2$; $k = 0, 1, ..., \left[\frac{n}{2}\right] - 1$. *Then there exists an unique generalized solution* u *of Problem* $P1$. *Moreover,* $u \in C^{1+(l-4)_+}(\overline{\Omega} \backslash O)$ *and for every* $\varepsilon > 0$ *it holds the a priori estimate*

$$\begin{aligned} |u(x_1, x_2, t)| \le \quad & C_1 r^{1/4} ||f_0||_0^* + C_{2,\varepsilon} r^{-\varepsilon} ||f_1||_0^* \\ & + C_3 r^{1/2} |\ln r| \sum_{k=1}^{[l/2]} ||f_{2k}||_{(2k-4)_+}^* \\ & + C_4 \sum_{k=1}^{\left[\frac{l-1}{2}\right]} ||f_{2k+1}||_{(2k-3)_+}^*, \end{aligned} \qquad (1.5)$$

where $C_{2,\varepsilon}$ *depends on* ε, *but all the constants* C_1, $C_{2,\varepsilon}$, C_3 *and* C_4 *are independent on the function* f.

Theorem 1.1 gives an a priori estimate of the *generalized solution*. Now the next Theorem 1.2 provides an a priori estimate for the *generalized solution* and clarifies the significance of the above orthogonality conditions (1.4). In other worlds, for any couple (n, k) the corresponding condition "controls" one power-type singularity.

Theorem 1.2. *Let $s \in \mathbb{N}$ be such that $2 \le s \le l$. Suppose that $f(x_1, x_2, t) \in C^{(l-4)_+}(\overline{\Omega})$ has the form (1.3) and satisfies the orthogonality conditions (1.4) for any couple (n, k) such that $n, k \in \mathbb{N} \cup \{0\}$, $2 \le n \le l$, $n - 2k \ge s + 1$ and $i = 1, 2$. Then there exists an unique generalized solution u of Problem P1 such that: $u \in C^{1+(l-4)_+}(\overline{\Omega} \backslash O)$ and the estimate*

$$|u(x_1, x_2, t)| \le C r^{-(s-1)} \sum_{k=0}^{l} \|f_k\|^*_{(k-s-1)_+} \tag{1.6}$$

holds. If we suppose additionally that there are $m, p \in \mathbb{N} \cup \{0\}$ and $j = 1$ or 2 such that $2 \le m \le l$, $m - 2p = s$ and

$$\int_{\Omega} V_p^{m,j}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt \ne 0, \tag{1.7}$$

then in some neighborhood of the origin one has

$$\left| u_m^j|_{S_2}(x_1, x_2) \right| \ge c(x_1^2 + x_2^2)^{-(s-1)/2}, \quad c > 0, \tag{1.8}$$

where

$$u_m^1|_{S_2}(x_1, x_2) := \int_0^{2\pi} u|_{t=|x|} \cos m\varphi \, d\varphi, \quad u_m^2|_{S_2}(x_1, x_2) := \int_0^{2\pi} u|_{t=|x|} \sin m\varphi \, d\varphi.$$

To illustrate the dependence of the singularity of the *generalized solution* on the orthogonality assumptions, let us consider the following table:

Table 1. The orthogonality conditions and the order of singularity

	l	$l-1$	$l-2$	$l-3$	...	m		...	4	3	2
1	...	...	...	...	...	...		...	$V_1^{4,i}$	$\diamond$	$V_0^{2,i}$
2	...	...	...	...	...	...		...	$\diamond$	$V_0^{3,i}$	
3	...	...	...	...	...	...		...	$V_0^{4,i}$		
...	...	...	...	...	...	...		...			
$m-2p-1$	...	...	...	...	...	$V_p^{m,i}$	...				
						...					
...	...	...	...	...	...						
$l-4$	$\diamond$	$V_1^{l-1,i}$	$\diamond$	$V_0^{l-3,i}$							
$l-3$	$V_1^{l,i}$	$\diamond$	$V_0^{l-2,i}$								
$l-2$	$\diamond$	$V_0^{l-1,i}$									
$l-1$	$V_0^{l,i}$										

Observe that both $V_k^{n,1}$ and $V_k^{n,2}$ are located in the n-th column and $(n-2k-1)$-st row of Table 1. Thus, $V_0^{n,i}$ form the right most diagonal, the next one is empty – we put in the cells "diamonds" $\diamond$, $V_1^{n,i}$ constitute the third one, and so on. The first column designates the order of singularity of the *generalized solution.*

Theorem 1.1 asserts that the *generalized solution* of Problem $P1$ is bounded if the right-hand side f is orthogonal to all the functions $V_k^{n,i}$ from the table. Theorem 1.2 specifies that (if $l \geq 2$) the singularity of the *generalized solution* is no worse than r^{1-s} if f is orthogonal to $V_k^{n,i}$ from the triangle under the $(s-1)$-st row. In other words, the functions from the k-th row of the table are "responsible" for the *generalized solutions* with behavior r^{-k} near the origin O.

The present paper is a generalization, extension and improvement of the results obtained in [26]. It consists of an introduction and five consecutive sections. Section 2 is devoted to the solutions of the homogeneous adjoint Problems $P1^*$ and $P2^*$. In Section 3 are formulated the $2-D$ boundary Problems $P12$ and $P13$, shortly related to the $3-D$ Problem $P1$. The main technical results are established in Sections 4 and 5 – we study the behavior of solutions of $2-D$ Problems $P12$ and $P13$. In the last Section 6 we give proofs of Theorem 1.1 and Theorem 1.2 based on the results of the previous two sections.

2. PROPERTIES OF THE SOLUTIONS H_K^N AND E_K^N

First, we will present three different ways to introduce the solutions of the homogeneous adjoint Problems $P1^*$ and $P2^*$. The functions H_k^n and E_k^n could be found in Khe [20] in the form

$$t\rho^{n-2k-3}(1-t^2/\rho^2)^{n-2k-3/2}F(n-k,-k;3/2;t^2/\rho^2)$$

and

$$\rho^{n-2k-1}(1-t^2/\rho^2)^{n-2k-1/2}F(n-k,-k;1/2;t^2/\rho^2),$$

where F is the hypergeometric Gaus function.

On the other hand, one could obtain $H_k^n(\rho,t)$ and $E_k^n(\rho,t)$ by differentiation of $E_0^n(\rho,t)$ with respect to t.

Lemma 2.1. (*see* [13, Theorem 4.2]) *The functions* $H_k^n(\rho,t)$ *and* $E_k^n(\rho,t)$, *defined in Lemma* 1.1, *satisfy*

$$\frac{\partial}{\partial t}H_k^n(\rho,t) = 2(n-k-1)E_{k+1}^n(\rho,t),$$

$$\frac{\partial}{\partial t}E_k^n(\rho,t) = 2(k-n+1/2)H_k^n(\rho,t)$$

and they represent some derivative of $E_0^n(\rho,t)$ *over* t:

$$H_k^n(\rho,t) = \frac{(-1)^{k+1}}{(2n-2k-1)_{2k+1}}\left(\frac{\partial}{\partial t}\right)^{2k+1}\left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right),$$

$$E_k^n(\rho,t) = \frac{(-1)^k}{(2n-2k)_{2k}} \left(\frac{\partial}{\partial t}\right)^{2k} \left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right).$$

Remark 2.1. This procedure of differentiating the function E_0^n (or $W_0^{n,i}$) with respect to t will (as in Lemma 1.1) produce solutions $V_k^{n,i}$ and $W_k^{n,i}$ of the equation (1.1), but for $k \geq n/2$ the smoothness at the point O will be lost.

Remark 2.2. The solutions of the adjoint Problem $P2^*$ ($P1^*$) given in Lemma 1.1 are not orthogonal. For example, one could check this out for $W_0^{n,i}$ and $W_1^{n,i}$. It is sufficient to show that

$$K := \int_0^{1/2} \int_t^{1-t} E_0^n(\rho,t) E_1^n(\rho,t) \rho d\rho dt \neq 0.$$

In fact, Lemma 2.1 implies $\partial^2 E_0^n/\partial t^2(\rho,t) = cE_1^n(\rho,t)$ for some constant c and therefore

$$\begin{aligned} cK &= \int_0^{1/2} \int_t^{1-t} E_0^n(\rho,t) \frac{\partial^2 E_0^n}{\partial t^2}(\rho,t)\rho\, d\rho dt \\ &= \int_{1/2}^{1} E_0^n(\rho,1-\rho) \frac{\partial E_0^n}{\partial t}(\rho,1-\rho)\,\rho\, d\rho - \int_0^{1/2} \int_t^{1-t} \left(\frac{\partial E_0^n}{\partial t}(\rho,t)\right)^2 \rho\, d\rho dt < 0, \end{aligned}$$

because $E_0^n(\rho,t) \geq 0$ and $\partial E_0^n/\partial t(\rho,t) \leq 0$ for $t \leq \rho$.

Remark 2.3. The functions $H_k^n(\rho,t)$ and $E_k^n(\rho,t)$ are linearly independent. Indeed, suppose that some linear combination of these functions is zero. Then from Lemma 2.1 it follows that E_0^n as a function of t is a solution (for a fixed ρ) of a homogeneous linear differential equation with constant coefficients. Therefore E_0^n must be a finite sum of quasi-polynomials of t, which obviously is not true.

A basic tool for our treatment of Protter problems are the Legendre functions P_ν (see (3.8) below). Some properties of the Legendre functions P_ν one can find in [9]. The next lemma plays a key role in the last section.

Lemma 2.2. *Let us denote*

$$h_k^\nu(\xi,\eta) = \int_\eta^\xi s^k P_\nu\left(\frac{\xi\eta+s^2}{s(\xi+\eta)}\right) ds.$$

If $\nu = n-1/2$, *then it hold:* (**a**) *for* $i = 0,1,\dots,\left[\frac{\nu-1}{2}\right]$

$$h_{\nu-2i-2}^\nu(\xi,\eta)\Big|_{\xi=\frac{\rho+t}{2};\eta=\frac{\rho-t}{2}} = c_i^n \rho^{1/2} H_i^n(\rho,t)$$

and (**b**) *for* $i = 0, 1, ..., \left[\frac{\nu}{2}\right]$

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^{\nu}_{\nu-2i}(\xi,\eta)\,\big|_{\xi=(\rho+t)/2;\eta=(\rho-t)/2} = d^n_i \rho^{1/2} E^n_i(\rho, t),$$

where the constants c^n_i, $d^n_i \neq 0$.

Proof. (**a**) We will calculate the integrals $h^{\nu}_k(\xi,\eta)$ using the Mellin transform, given by

$$f^*(s) = \int\limits_0^{\infty} t^{s-1} f(t)\,dt,$$

and the Mellin convolution "$\circ$", defined by

$$(f \circ g)(x) = \int\limits_0^{\infty} f\left(\frac{x}{t}\right) g(t)\frac{dt}{t}.$$

Recall that the relation between both is (see [22, formula (1.2)])

$$(f \circ g)^*(s) = f^*(s) g^*(s). \tag{2.1}$$

To apply the last formula to $h^{\nu}_k(\xi,\eta)$, let us introduce new variables x, y and z defined by

$$2\sqrt{x} = \sqrt{\frac{\xi}{\eta}} + \sqrt{\frac{\eta}{\xi}}; \quad 2\sqrt{y} = \frac{\sqrt{\xi\eta}}{s} + \frac{s}{\sqrt{\eta\xi}}; \quad z = \sqrt{\xi\eta}.$$

Then we have

$$\frac{\xi\eta + s^2}{s(\xi+\eta)} = \sqrt{\frac{y}{x}}\,; \; \frac{s}{\sqrt{\eta\xi}} = \sqrt{y} \pm \sqrt{y-1};$$
$$\frac{d(2\sqrt{y})}{ds} = \frac{d}{ds}\left(\frac{\sqrt{\xi\eta}}{s} + \frac{s}{\sqrt{\eta\xi}}\right) = \frac{s^2 - z^2}{s^2 z};$$
$$(\sqrt{y} + \sqrt{y-1})^2 - 1 = 2\sqrt{y-1}(\sqrt{y-1} + \sqrt{y});$$
$$(\sqrt{y} - \sqrt{y-1})^2 - 1 = 2\sqrt{y-1}(\sqrt{y-1} - \sqrt{y});$$

when $s = \sqrt{\xi\eta}$, we have $y = 1$, and $y = x$ for $s = \xi$ or η . Substituting in h^{ν}_k, we find

$$h^{\nu}_k(x,z) = \frac{z^{k+1}}{2}\int\limits_0^{\infty}\left(\frac{x}{y} - 1\right)^0_+ P_{\nu}\left(\sqrt{\frac{y}{x}}\right)(y-1)^{-1/2}_+\left((\sqrt{y} + \sqrt{y-1})^{k+1}\right.$$
$$\left.+(\sqrt{y} - \sqrt{y-1})^{k+1}\sqrt{y}\frac{dy}{y}\right.$$

Now we are ready to use (2.1) and formulae (11.13(4)), (2.10(4)) from [22] – applying the Mellin transform over x (here "$\mapsto$" means "transforms into"):

$$(x-1)^0_+ P_{\nu}\left(\sqrt{\frac{1}{x}}\right) \mapsto \frac{\Gamma(-s)\Gamma\left(\frac{1}{2} - s\right)}{\Gamma\left(\frac{1-\nu}{2} - s\right)\Gamma\left(1 + \frac{\nu}{2} - s\right)};$$

$$\sqrt{y}(y-1)_+^{-1/2}\left((\sqrt{y}+\sqrt{y-1})^{k+1} \quad + \quad (\sqrt{y}-\sqrt{y-1})^{k+1}\right)$$

$$\mapsto 2\pi\frac{\Gamma\left(\frac{1+k+1}{2}-s-\frac{1}{2}\right)\Gamma\left(\frac{1-k-1}{2}-s-\frac{1}{2}\right)}{\Gamma\left(1-s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s-\frac{1}{2}\right)}$$

and therefore

$$h_k^\nu(x,z)z^{-k-1}\mapsto\pi\frac{\Gamma\left(\frac{1+k}{2}-s\right)\Gamma\left(-\frac{1+k}{2}-s\right)}{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)}. \tag{2.2}$$

To find the inverse image of the right-hand side, denote as usual by I_{0+}^α the Riemann-Liouville fractional integral (derivative) of order α:

$$I_{0+}^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int\limits_0^x f(t)(x-t)^{\alpha-1}dt,\ \alpha>0;$$

$$I_{0+}^0 f(x):=f(x);$$

$$I_{0+}^{-\alpha}f(x):=\left(I_{0+}^\alpha\right)^{-1}f(x):=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}f\right)(x)\ \ ,\ \alpha>0.$$

Under these notations, the right-hand side of (2.2) is the Mellin transform of the function

$$\frac{\pi}{\Gamma\left(\frac{\nu-k+1}{2}\right)}x^{\frac{\nu+1}{2}}I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right).$$

Indeed, using (2.2.(4)), (1.10) and (1.4) from [22], for $k<\nu-1$ we find

$$x^{\frac{\nu+1}{2}}I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right)\mapsto\Gamma\left(\frac{\nu-k+1}{2}\right)\frac{\Gamma\left(-\frac{1+k}{2}-s\right)\Gamma\left(\frac{1+k}{2}-s\right)}{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)}.$$

Hence, taking $k=\nu-2i-2$, for some constants $C_{\nu,i}$ we have

$$h_{\nu-2i-2}^\nu(x,z)z^{-\nu+2i+1}=C_{\nu,i}x^{\frac{\nu+1}{2}}\left(\frac{d}{dx}\right)^i\left(x^{-\nu+i-1/2}(x-1)_+^{i+1/2}\right).$$

Let us now return to the variables ρ and t :

$$\begin{aligned}
z^{\nu-2i-1}x^{\frac{\nu+1}{2}} \quad & (d/dx)^i\left(x^{-\nu+i-1/2}(x-1)^{i+1/2}\right)\\
&=(\xi\eta)^{\frac{\nu-2i-1}{2}}\sum_{j=0}^{i}d_j'\left(\frac{(\xi+\eta)^2}{\xi\eta}\right)^{-\nu/2+j}\left(\frac{(\xi-\eta)^2}{\xi\eta}\right)^{i+1/2-j}\\
&=(\rho^2-t^2)^{\frac{\nu-2i-1}{2}}\sum_j d_j''\left(\frac{\rho^2}{(\rho^2-t^2)}\right)^{-\nu/2+j}\left(\frac{t^2}{(\rho^2-t^2)}\right)^{i+1/2-j}\\
&=\sum_{j+l\le i}d_{j,l}t(\rho^2-t^2)^{\nu-i-1-(j+l)}\rho^{-\nu+2(j+l)}\\
&=\sum_{j=0}^{i}a_j^i\frac{t(\rho^2-t^2)^{n-3/2-j-i}}{\rho^{n-2j-1/2}}.
\end{aligned}$$

In order to determine the coefficients a_j^i, one could notice that from the definition of h_k^ν the function

$$\rho^{-1/2} h^\nu_{\nu-2i-2}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) \sin n\varphi$$

satisfies the wave equation. Therefore, after substituting in the equation, we find $a_j^i = a_0^i A_j^i$ and

$$\rho^{-1/2} h^\nu_{\nu-2i-2}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) = a_i^\nu H_i^n(\rho, t).$$

(b) Let us find the functions $(\partial/\partial\xi - \partial/\partial\eta)\, h^\nu_{\nu-2i}(\xi,\eta)$, where $i = 0, 1, \ldots, \left[\frac{\nu}{2}\right]$. Notice that for $i \geq 1$, due to Lemma 2.1 and **(a)**,

$$\rho^{-1/2}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_{\nu-2i}(\xi,\eta) = C_i^n \frac{\partial}{\partial t} H^n_{i-1}(\rho,t) = C_i^{n\prime} E_i^n(\rho,t).$$

Only the case $i = 0$ has been omitted, i.e. we need to calculate

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_\nu(\xi,\eta).$$

Therefore we consider the function

$$\frac{\partial}{\partial t} z^{k+1} x^{\frac{\nu+1}{2}} I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right),$$

where $k = \nu$ and

$$x = \frac{\rho^2}{\rho^2 - t^2}; \quad 2z = (\rho^2 - t^2)^{1/2}.$$

That is

$$\frac{\partial}{\partial t} z^{\nu+1} x^{\frac{\nu+1}{2}} I^1_{0+}\left(x^{-\nu-\frac{3}{2}}(x-1)_+^{-1/2}\right) = \frac{\partial}{\partial t} z^{\nu+1} x^{\frac{\nu+1}{2}} \int_0^x \tau^{-\nu-\frac{3}{2}}(\tau-1)_+^{-1/2} d\tau$$

$$= \frac{\partial}{\partial t}\rho^{\nu+1} 2^{-\nu-1}\int_1^x \tau^{-\nu-\frac{3}{2}}(\tau-1)^{-1/2} d\tau = 2^{-\nu-1}\rho^{\nu+1} x^{-\nu-\frac{3}{2}}(x-1)^{-1/2}\frac{\partial x}{\partial t}$$

$$= 2^{-\nu-1}\rho^{\nu+1}\frac{(\rho^2-t^2)^{\nu+3/2}}{\rho^{2\nu+3}}\frac{(\rho^2-t^2)^{1/2}}{t}\frac{2t\rho^2}{(\rho^2-t^2)^2} = 2^{-\nu}\frac{(\rho^2-t^2)^\nu}{\rho^\nu}.$$

Hence for $\nu = n - 1/2$ we conclude that

$$\rho^{-1/2}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_\nu(\xi,\eta) = \rho^{-1/2} C_n \frac{(\rho^2-t^2)^{n-1/2}}{\rho^{n-1/2}} = C_n E_0^n(\rho,t). \qquad \square$$

3. SOLVING PROBLEM P1

In terms of Theorems 1.1 and 1.2 it is sufficient to study the Problem $P1$ only when the right-hand side f of the wave equation is simply

$$f(\rho,t,\varphi) = f_n^1(\rho,t)\cos n\varphi + f_n^2(\rho,t)\sin n\varphi, \quad n \in \mathbb{N}\cup\{0\}.$$

Then we seek solutions of the wave equation of the same form:

$$u(\rho, t, \varphi) = u_n^1(\rho, t)\cos n\varphi + u_n^2(\rho, t)\sin n\varphi.$$

Thus Problem P1 reduces to the following one:

Problem P12. To solve the equation

$$(u_n)_{\rho\rho} + \frac{1}{\rho}(u_n)_\rho - (u_n)_{tt} - \frac{n^2}{\rho^2}u_n = f_n \tag{3.1}$$

in $\Omega_0 = \{0 < t < 1/2;\ t < \rho < 1 - t\} \subset \mathbb{R}^2$ with the boundary conditions

P12 $\quad u_n(\rho, 0) = 0$ for $0 < \rho \leq 1$ and $u_n(\rho, 1-\rho) = 0$ for $1/2 \leq \rho \leq 1$.

Let us now introduce new coordinates

$$\xi = \frac{\rho + t}{2}\ ;\ \ \eta = \frac{\rho - t}{2}, \tag{3.2}$$

and set

$$v(\xi, \eta) = \rho^{1/2}u_n(\rho, t)\ ;\ \ g(\xi, \eta) = \rho^{1/2}f_n(\rho, t). \tag{3.3}$$

Denoting $\nu = n - \dfrac{1}{2}$, one transforms Problem $P12$ into

Problem P13. To find a solution $v(\xi, \eta)$ of the equation

$$v_{\xi\eta} - \frac{\nu(\nu+1)}{(\xi+\eta)^2}v = g \tag{3.4}$$

in the domain $D = \{0 < \xi < 1/2;\ 0 < \eta < \xi\}$ with the following boundary conditions:

P13 $\quad v(\xi, \xi) = 0$ for $\xi \in (0, 1/2)$ and $v(1/2, \eta) = 0$ for $\eta \in (0, 1/2)$.

Problems $P12$ and $P13$ have been introduced in [25], although the change of coordinates $\xi = 1 - \rho - t$ and $\eta = 1 - \rho + t$ is used there instead of (3.2). Of course, because the solution of Problem $P1$ may be singular, the same is true for the solutions of $P12$ and $P13$. For that reason, Popivanov and Schneider [25] have defined and proved the existence and uniqueness of generalized solutions of Problems $P12$ and $P13$, which correspond to the *generalized solution* of Problem $P1$. Further, by "solution" of Problem $P12$ or $P13$ we will mean exactly this unique generalized solution.

Remark 3.1. Notice that even when the right-hand side function $f_n(\rho, t)$ belongs to $C^k(\overline{\Omega}_0)$, the corresponding function $g(\xi, \eta) = \rho^{1/2}f_n(\rho, t)$ in (3.4) belongs to $C^k(\overline{\Omega}_0\backslash O)$, but its derivative may not be continuous at the origin O. At the same time, when the solution $v(\xi, \eta)$ of Problem $P13$ is bounded, the solution $u_n(\rho, t) = \rho^{-1/2}v(\xi, \eta)$ of Problem $P12$ may be singular.

Nevertheless, we will solve Problem $P13$ instead of Problem $P1$. We can construct the solution of the Problem $P13$ using two different methods. First, following Popivanov, Schneider [25], one could use the equivalent integral equation

$$U(\xi_0, \eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}}\int\limits_{\eta=\eta_0}^{\eta=\xi_0}\left(\frac{\nu(\nu+1)}{(\xi+\eta)^2}U(\xi, \eta) + g(\xi, \eta)\right)d\eta d\xi \tag{3.5}$$

and construct the solution as a limit of a sequence of successive approximations $U^{(k)}$ defined by

$$U^{(0)}(\xi_0,\eta_0)=\int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}}\int\limits_{\eta=\eta_0}^{\eta=\xi_0} g(\xi,\eta)d\eta d\xi, \tag{3.6}$$

$$U^{(k+1)}(\xi_0,\eta_0)=\int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}}\int\limits_{\eta=\eta_0}^{\eta=\xi_0}\left(\frac{\nu(\nu+1)}{(\xi+\eta)^2}U^{(k)}(\xi,\eta)+g(\xi,\eta)\right)d\eta d\xi. \tag{3.7}$$

On the other hand, notice that the function

$$R(\xi_1,\eta_1,\xi,\eta)=P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1)+2\xi_1\eta_1+2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right) \tag{3.8}$$

is a Riemann function for the equation (3.4) (Copson [7]). Therefore, we can construct the function v as a solution of a Goursat's problem in D with boundary conditions $v(1/2,\eta)=0$ and $v(\xi,0)=\varphi(\xi)$ with some unknown function $\varphi(\xi)$, which will be determined later:

$$v(\xi,\eta)=\varphi(\xi)+\int\limits_{\xi}^{\frac{1}{2}}\varphi(\xi_1)\frac{\partial}{\partial\xi_1}R(\xi_1,0,\xi,\eta)d\xi_1-\int\limits_{\xi}^{\frac{1}{2}}\int\limits_{0}^{\eta}R(\xi_1,\eta_1,\xi,\eta)g(\xi_1,\eta_1)d\eta_1 d\xi_1. \tag{3.9}$$

Now, following Aldashev [1], the boundary condition $v(\xi,\xi)=0$ gives the equation

$$G(\xi)=\varphi(\xi)+\int\limits_{\xi}^{\frac{1}{2}}\varphi(\xi_1)\frac{\partial}{\partial\xi_1}P_\nu\left(\frac{\xi}{\xi_1}\right)d\xi_1=-\int\limits_{\xi}^{\frac{1}{2}}\varphi'(\xi_1)P_\nu\left(\frac{\xi}{\xi_1}\right)d\xi_1 \tag{3.10}$$

for the function $\varphi(\xi)$, where

$$G(\xi)=\int\limits_{\xi}^{\frac{1}{2}}\int\limits_{0}^{\xi}P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)g(\xi_1,\eta_1)d\eta_1 d\xi_1. \tag{3.11}$$

According to formulae (35.17), (35.28) from [28], the integral equation (3.10) is invertible and we have

$$\frac{d}{d\xi}\varphi(\xi)=-\xi\frac{d^2}{d\xi^2}\xi\int\limits_{\xi}^{\frac{1}{2}}P_\nu\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1^2}d\xi_1=G'(\xi)+\frac{d}{d\xi}\int\limits_{\xi}^{\frac{1}{2}}P'_\nu\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}d\xi_1.$$

Finally, due to $\varphi(1/2)=G(1/2)=0$, it follows

$$\varphi(\xi)=G(\xi)+\int\limits_{\xi}^{\frac{1}{2}}P'_\nu\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}d\xi_1. \qquad \Box \tag{3.12}$$

Using the sequence $\{U^{(k)}\}$ defined by (3.6), (3.7), we have the following

Lemma 4.1. *Let $0 < \varepsilon < 1$ and suppose that $|\nu(\nu+1)| < \varepsilon(1+\varepsilon)$. Then the solution $U(\xi,\eta)$ of the integral equation (3.5) in D satisfies the estimate*

$$|U(\xi,\eta)| \le C(\xi-\eta)(\xi+\eta)^{-\varepsilon} \sup_D |g|,$$

where the constant C depends only on ν and ε.

Proof. The key point in the proof is the estimate for the integrals:

$$\begin{aligned}
I_\varepsilon &:= \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} (\xi-\eta)(\xi+\eta)^{-\varepsilon-2} d\eta d\xi \\
&= \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left((\xi+\eta)^{-\varepsilon-1} - 2\eta(\xi+\eta)^{-\varepsilon-2}\right) d\eta d\xi \\
&= \int\limits_{\eta_0}^{\xi_0} \left(\frac{1}{\varepsilon}(\xi_0+\eta)^{-\varepsilon} - \frac{1}{\varepsilon}(\frac{1}{2}+\eta)^{-\varepsilon} - \frac{2\eta}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1} + \frac{2\eta}{1+\varepsilon}(\frac{1}{2}+\eta)^{-\varepsilon-1}\right) d\eta \\
&\le \int\limits_{\eta_0}^{\xi_0} \left(\frac{1}{\varepsilon}(\xi_0+\eta)^{-\varepsilon} - \frac{2\eta}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1}\right) d\eta \\
&= \int\limits_{\eta_0}^{\xi_0} \left(\frac{1-\varepsilon}{\varepsilon(1+\varepsilon)}(\xi_0+\eta)^{-\varepsilon} + \frac{2\xi_0}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1}\right) d\eta \\
&= -\frac{1}{\varepsilon(1+\varepsilon)}(\xi_0+\eta_0)^{1-\varepsilon} + \frac{2\xi_0}{\varepsilon(1+\varepsilon)}(\xi_0+\eta_0)^{-\varepsilon} \\
&= \frac{1}{\varepsilon(1+\varepsilon)}(\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon}.
\end{aligned}$$

For the function $U^{(0)}$ we find

$$\sup_\Omega |U^{(0)}| \le \frac{1}{2}(\xi_0-\eta_0) \sup_D |g| \le (\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|.$$

Now, because of

$$\left|(U^{(k+1)} - U^{(k)})(\xi_0,\eta_0)\right| \le \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \frac{|\nu(\nu+1)|}{(\xi+\eta)^2} \left|(U^{(k)} - U^{(k-1)})(\xi,\eta)\right| d\eta d\xi,$$

if take $\alpha := \dfrac{|\nu(\nu+1)|}{\varepsilon(1+\varepsilon)}$, we have by induction

$$\left|(U^{(k)} - U^{(k-1)})(\xi_0,\eta_0)\right| \le \alpha^k (\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|.$$

Indeed, using the calculations of I_ε, we find that

$$\begin{aligned}\left|(U^{(k+1)} - U^{(k)})(\xi_0,\eta_0)\right| \le & \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \frac{|\nu(\nu+1)|}{(\xi+\eta)^{2+\varepsilon}} \alpha^k (\xi-\eta) \sup_D |g| d\eta d\xi \\ & \le \alpha^{k+1}(\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|.\end{aligned} \tag{4.1}$$

Finally, we arrive at the estimate

$$\begin{aligned}|U^{(k+1)}(\xi_0,\eta_0)| \le & \sum_{i=0}^{k} \left|(U^{(i+1)} - U^{(i)})(\xi_0,\eta_0)\right| \\ & \le \sum_{i=0}^{k} \alpha^{i+1}(\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g| \\ & = \frac{(1-\alpha^{k+2})}{(1-\alpha)}(\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|,\end{aligned}$$

which, together with (4.1), shows that if $\alpha < 1$, the sequence $\{U^{(k)}\}$ uniformly converges to a solution $U(\xi,\eta)$ of

$$U(\xi_0,\eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left(\frac{\nu(\nu+1)}{(\xi+\eta)^2} U(\xi,\eta) + g(\xi,\eta) \right) d\eta d\xi.$$

Even more, it holds the estimate

$$|U(\xi,\eta)| \le \frac{1}{(1-\alpha)}(\xi-\eta)(\xi+\eta)^{-\varepsilon} \sup_D |g|. \qquad \square$$

Now, for $n = 0$ or 1 we have $\nu = -1/2$ or $1/2$, respectively, and we will apply Lemma 4.1 with suitable ε.

Theorem 4.1. *For the solution $u_n(\rho,t)$ of Problem P12 with right-hand side function $f_n \in C(\overline{\Omega}_0)$ the following a priori estimate holds:*

(i) *for the case $n = 0$*

$$|u_0(\rho,t)| \le C\rho^{1/4} \max_{\overline{\Omega}_0} |\rho^{1/2} f_0|$$

with constant C, independent of the function $f_0(\rho,t)$;

(ii) *if $n = 1$, then for every $\delta > 0$ there exists a constant C_δ, independent of the function $f_1(\rho,t)$, such that*

$$|u_1(\rho,t)| \le C_\delta \rho^{-\delta} \max_{\overline{\Omega}_0} |\rho^{1/2} f_1|.$$

Proof. Notice that when $n = 0$ and $n = 1$ we have $|\nu(\nu+1)| = 1/4$ and $|\nu(\nu+1)| = 3/4$, respectively. Therefore one could apply Lemma 4.1 for the solution

$v(\xi,\eta) = U(\xi,\eta)$ of Problem $P13$ with $\varepsilon = 1/4$ and $\varepsilon = 1/2+\delta$, respectively. The assertion follows from the relation (3.3):

$$|u(\rho,t)| = \left|\rho^{-1/2} v\left(\frac{\rho+t}{2},\frac{\rho-t}{2}\right)\right| \le C'\rho^{1-\varepsilon-1/2}\max_{\overline{D}}|g| \le C\rho^{1/2-\varepsilon}\max_{\overline{\Omega}_0}|\rho^{1/2}f|. \square$$

5. THE CASE $n \ge 2$

In this case, unlike the above approach, the behavior of the solution is studied in [26], using the properties of the Riemann function (3.8), given by Legendre functions P_ν. Let us remind some of these results here. For the function $v(\xi,\eta)$, defined by (3.9), (3.12) and (3.11), it is not hard to see that $v(\xi,0) = \varphi(\xi)$ may blow up when ξ tends to 0. Nevertheless, one could control the growth of

$$I(\xi) := \int_{\xi}^{\frac{1}{2}} P'_\nu\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}d\xi_1 \tag{5.1}$$

with the help of the following lemma:

Lemma 5.1. (*see* [26, Lemma 3.1]) *Let $\nu > 1$ be a real number with integral part $[\nu]$ and fractional part $\{\nu\} = \nu - [\nu] \neq 0$. Suppose that $G \in C^{[\nu-2]_+}(0,1/2]$, $|G^{(k)}(\xi)| \le A\xi^{1-k}$ for $k = 0,1,...,[\nu-2]_+$ for some constant A, and*

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2}G(\xi)d\xi = 0 \quad \textit{for} \;\; i = 0,1,...,\left[\frac{\nu-1}{2}\right]. \tag{5.2}$$

Then the function $I(\xi)$, defined by (5.1), is $C^{[\nu]-1}(0,1/2]$. More precisely, there is a constant C, independent of $G(\xi)$, such that:

(i) $|I(\xi)| \le CA\xi$ *if $[\nu]$ is an odd number;*

(ii) $|I(\xi)| \le CA\xi^{1-\{\nu\}}$ *if $[\nu]$ is an even number.*

Besides, Lemma 3.2 from [26] asserts that each of the orthogonality conditions (5.2) actually "controls" one power-type singularity of the function $I(\xi)$:

Lemma 5.2. *Let $\nu > 1, \{\nu\} \neq 0$, and p be a nonnegative integer, $p \le [(\nu-1)/2]$. Suppose that $G \in C^{(2p-1)_+}(0,1/2]$, $|G^{(k)}(\xi)| \le A\xi^{2p-[\nu]+1-k}$ for $k = 0,1,...,(2p-1)_+$ for some constant A, and*

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2}G(\xi)d\xi = 0 \qquad \textit{for } i = 0,1,...,p-1. \tag{5.3}$$

Then the estimate

$$|I(\xi)| \le C_1 A\xi^{-(\nu-2p-1)}$$

holds for some constant C_1, independent of $G(\xi)$. Moreover, if

$$\int_0^{\frac{1}{2}} \xi^{\nu-2p-2} G(\xi) d\xi \neq 0, \tag{5.4}$$

then

$$|I(\xi)| \geq C_2 \xi^{-(\nu-2p-1)}$$

for $C_2 > 0$ and sufficiently small ξ.

In other words, one has to impose some conditions on the right-hand side g of Problem $P13$ to secure certain behavior of the solution v. In fact, the definition (3.11) of the function $G(\xi)$ gives the equality (see [26])

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2} G(\xi) d\xi = \int_0^{\frac{1}{2}} \int_0^{\xi_1} \left(\int_{\eta_1}^{\xi_1} \xi^{\nu-2i-2} P_\nu \left(\frac{\xi_1 \eta_1 + \xi^2}{\xi(\xi_1 + \eta_1)} \right) d\xi \right) g(\xi_1, \eta_1) d\eta_1 d\xi_1.$$

It shows that one needs orthogonality of g to the functions $h^\nu_{\nu-2i-2}$ defined in Lemma 2.2. As a result we are able to prove the following

Theorem 5.1. (*see* [26, Theorem 4.1]) *Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that $g \in C^{(n-4)_+}(\overline{D} \backslash O)$, $|D^\alpha g(\xi, \eta)| \leq A\xi^{-|\alpha|}$ for $|\alpha| \leq (n-4)_+$, and the orthogonality conditions*

$$\int_0^{\frac{1}{2}} \int_0^{\xi} h^\nu_{\nu-2i-2}(\xi, \eta) g(\xi, \eta) d\eta d\xi = 0 \quad \text{for } i = 0, 1, ..., \left[\frac{\nu-1}{2}\right] \tag{5.5}$$

are satisfied with $\nu = n - 1/2$. Then the solution $v(\xi, \eta)$ of Problem $P13$ belongs to $C^{(n-4)_+ + 1}(\overline{D} \backslash O)$ and satisfies the following estimates:

(i) *if n is an even number, then*

$$|v(\xi, \eta)| \leq CA\xi |\ln \xi|;$$

(ii) *if n is an odd number, then*

$$|v(\xi, \eta)| \leq CA\xi^{1/2}.$$

In both cases the constant C does not depend on the function $g(\xi, \eta)$.

In the same way one gets the following theorem, which corresponds to Lemma 5.2:

Theorem 5.2. (*see* [26, Theorem 4.2]) *Let p be a nonnegative integer and $p \leq [(\nu-1)/2]$. Suppose that the function $g \in C^{(2p-2)_+}(\overline{D} \backslash O)$, $|D^\alpha g(\xi, \eta)| \leq A\xi^{-|\alpha|}$ for $|\alpha| \leq (2p-2)_+$ and*

$$\int_0^{\frac{1}{2}} \int_0^{\xi} h^\nu_{\nu-2i-2}(\xi, \eta) g(\xi, \eta) d\eta d\xi = 0 \quad \text{for } i = 0, 1, ..., p-1. \tag{5.6}$$

Then the estimate

$$|v(\xi,\eta)| \le CA\xi^{-(\nu-2p-1)} \tag{5.7}$$

holds for some constant C, independent of the function $g(\xi,\eta)$. Moreover, the condition

$$\int_0^{\frac{1}{2}}\int_0^{\xi} h^{\nu}_{\nu-2p-2}(\xi,\eta)g(\xi,\eta)d\eta d\xi \ne 0 \tag{5.8}$$

implies that the lower estimate

$$|v(\xi,0)| \ge c\xi^{-(\nu-2p-1)} \tag{5.9}$$

holds for some constant $c > 0$ and sufficiently small ξ.

All these preparations and Lemma 2.2 lead us to the following estimate for the solution of Problem $P12$:

Theorem 5.3. *Suppose that $n \in \mathbb{N}$, $n \ge 2$, $f_n(\rho,t) \in C^{(n-4)+}(\overline{\Omega}_0\backslash O)$, and there is a constant A such that $|\rho^{|\alpha|+1/2}D^{\alpha}f_n(\rho,t)| \le A$ for $|\alpha| \le (n-4)_+$. Let also there hold the orthogonality conditions*

$$\int_0^{\frac{1}{2}}\int_t^{1-t} H_i^n(\rho,t)f_n(\rho,t)\rho d\rho dt = 0 \quad \text{for } i = 0,1,...,\left[\frac{n}{2}\right]-1. \tag{5.10}$$

Then the solution $u_n(\rho,t)$ of Problem $P12$ satisfies:
(i) $|u_n(\rho,t)| \le CA\rho^{1/2}|\ln\rho|$ *if n is an even number;*
(ii) $|u_n(\rho,t)| \le CA$ *if n is an odd number.*
In both cases the constant C does not depend on the function $f_n(\rho,t)$.

Proof. Let us define the function $g(\xi,\eta) = \rho^{1/2}f_n(\rho,t)$, where $\xi = (\rho+t)/2$, $\eta = (\rho-t)/2$. Then the estimates for f_n imply that g satisfies $|D^{\alpha}g(\xi,\eta)| \le CA\xi^{-|\alpha|}$ for $|\alpha| \le (n-4)_+$. The orthogonality conditions, due to Lemma 2.2, yield

$$\int_0^{\frac{1}{2}}\int_0^{\xi} h^{\nu}_{\nu-2i-2}(\xi,\eta)g(\xi,\eta)\,d\eta d\xi = C\int_0^{\frac{1}{2}}\int_t^{1-t} H_i^n(\rho,t)f_n(\rho,t)\rho d\rho dt = 0.$$

Now Theorem 5.1 gives the required estimates for the solution $v(\xi,\eta)$ of Problem $P13$, given by (3.9), and $u = \rho^{-1/2}v$ is the solution of Problem $P12$. □

Using the similar arguments, we get the corresponding result for the case when not all of the orthogonality conditions (5.10) are fulfilled:

Theorem 5.4. *Let $n,q \in \mathbb{N}\cup\{0\}$, $n \ge 2$, $q \le \left[\frac{n}{2}\right]-1$. Suppose that the function $f_n \in C^{(2q-2)+}(\overline{\Omega}_0\backslash O)$, $|D^{\alpha}f_n(\rho,t)| \le A\rho^{-|\alpha|-1/2}$ for $|\alpha| \le (2q-2)_+$ and*

$$\int_0^{\frac{1}{2}}\int_t^{1-t} H_i^n(\rho,t)f_n(\rho,t)\rho d\rho dt = 0 \quad \text{for } i = 0,1,...,q-1. \tag{5.11}$$

Then for the solution $u_n(\rho, t)$ of Problem P12 the upper estimate

$$|u_n(\rho, t)| \le CA\rho^{-(n-2q-1)} \tag{5.12}$$

holds, where the constant C is independent of $f_n(\rho, t)$. If we suppose also that

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_q^n(\rho, t) f_n(\rho, t)\rho d\rho dt \neq 0, \tag{5.13}$$

then the lower estimate

$$|u_n(\rho, \rho)| \ge c\rho^{-(n-2q-1)}$$

holds for $c > 0$ and sufficiently small ρ.

Proof. We again define $g(\xi, \eta) = \rho^{1/2} f_n(\rho, t)$ and we prove the theorem applying Theorem 5.2 for $v = \rho^{1/2} u$ and g instead of Theorem 5.1 as in the proof of Theorem 5.3. □

Finally, Theorems 5.3 and 5.4 show that every solution of Problem $P12$ is a linear combination of at most $\left[\frac{n}{2}\right]$ fixed singular solutions:

Lemma 5.3. *For $n \ge 2$ there exist $\left[\frac{n}{2}\right]$ functions $v_n^i(\rho, t)$, $i = 0, ..., \left[\frac{n}{2}\right] - 1$, such that for every generalized solution $u_n(\rho, t)$ of Problem P12 with some right-hand side function $f_n \in C^{(n-4)+}(\overline{\Omega}_0)$ the equality*

$$u_n(\rho, t) = \sum_{i=0}^{\left[\frac{n}{2}\right]-1} c_i v_n^i(\rho, t) + w(\rho, t)$$

holds with some constants c_i and some bounded function $w(\rho, t)$ dependent on $u_n(\rho, t)$.

Proof. Let $u_n(\rho, t)$ be the generalized solution of Problem $P12$ with some right-hand side function $f_n(\rho, t) \in C^{(n-4)+}(\overline{\Omega}_0)$. In general, u_n has a singularity at the origin O. Let k be an integer, $0 \le k \le \left[\frac{n}{2}\right] - 1$, and $f_n^{(k)}(\rho, t)$ be the projection of f_n on the linear space $L_{k,n}$ of functions, orthogonal to the functions H_i^n for $i = 0, 1, ..., k$:

$$L_{k,n} := \left\{ f(\rho, t) \ : \ \int_0^{\frac{1}{2}} \int_t^{1-t} H_i^n(\rho, t) f(\rho, t)\rho d\rho dt = 0 \ \text{ for } i = 0, 1, ..., k \right\}.$$

Then $f_n - \sum_{i=0}^{k} \alpha_i^k H_i^n = f_n^{(k)} \in L_{k,n}$ with some constants α_i^k such that

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho, t) f_n^{(k)}(\rho, t)\rho d\rho dt = 0, \quad j = 0, ..., k,$$

i.e.

$$\sum_{i=0}^{k} \alpha_i^k \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) H_i^n(\rho,t) \rho d\rho dt = \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) f_n(\rho,t) \rho d\rho dt$$

for $j = 0, 1, ..., k$. This system has an unique solution for constants α_i^k . To show this, suppose that the rank of the $(k \times k)$-matrix with elements $\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n H_i^n \rho d\rho dt$ is less than k. Then there are numbers β_0, β_1,...,β_k such that at least one is not zero and

$$\sum_{i=0}^{k} \beta_i \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) H_i^n(\rho,t) \rho d\rho dt = 0 \quad \text{for } j = 0, 1, ..., k$$

or

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) \left(\sum_{i=0}^{k} \beta_i H_i^n(\rho,t) \right) \rho d\rho dt = 0 \quad \text{for } j = 0, 1, ..., k.$$

Therefore $\sum\limits_{i=0}^{k} \beta_i H_i^n = 0$ in Ω_0, which is impossible, because the functions H_i^n are linearly independent in view of Remark 2.3.

Let us denote by $v_n^i(\rho,t)$ the solution of the Problem $P12$ with a right-hand side function $H_i^n(\rho,t)$, and by $u_n^{(k)}(\rho,t)$ the solution with a right-hand side function $f_n^{(k)}(\rho,t)$. Then we have for u_n the representation

$$u_n = \sum_{i=0}^{k} \alpha_i^k v_n^i + u_n^{(k)}.$$

When $k < \left[\frac{n}{2}\right] - 1$, Theorem 5.4 gives the estimate

$$|u_n^{(k)}(\rho,t)| \leq C\rho^{-(n-2k-3)},$$

while for $k = k_0 := \left[\frac{n}{2}\right] - 1$ Theorem 5.3 shows that the function $w := u_n^{(k_0)}$ is at least bounded. □

6. PROOF OF THE MAIN RESULTS

We are now ready to prove Theorem 1.1 and Theorem 1.2 (see Introduction).

Proof of Theorem 1.1. When the right-hand side function has the form

$$f = f_0^1(\rho,t) + \sum_{n=1}^{l} \left(f_n^1(\rho,t) \cos n\varphi + f_n^2(\rho,t) \sin n\varphi \right),$$

one could construct the unique generalized solution $u(x_1, x_2, t)$ as

$$u = u_0^1(\rho, t) + \sum_{n=1}^{l} \left(u_n^1(\rho, t) \cos n\varphi + u_n^2(\rho, t) \sin n\varphi \right), \tag{6.1}$$

where the functions u_n^i are solutions of Problem $P12$ with right-hand side function $f_n^i \in C^{(l-4)_+}(\overline{\Omega}_0)$. First of all, we use Theorem 4.1 to estimate the functions u_0^1, u_1^1 and u_1^2:

$$|u_0^1(\rho, t)| \le C\rho^{1/4} \max_{\overline{\Omega}_0} |\rho^{1/2} f_0^1| \le Cr^{1/4} ||f_0||_0^*,$$

$$|u_1^i(\rho, t)| \le C_\delta \rho^{-\delta} \max_{\overline{\Omega}_0} |\rho^{1/2} f_1^i| \le C'_\delta r^{-\delta} ||f_1||_0^*.$$

For the case $n \ge 2$ we apply Theorem 5.3 with the constant $A = ||f_n||_{(n-4)_+}^*$. Because of the identity

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_k^n(\rho, t) f_n^i(\rho, t) \rho d\rho dt = c \int_\Omega V_k^{n,i}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt = 0, \tag{6.2}$$

the orthogonality conditions (5.10) are fulfilled. Therefore, if n is an even number,

$$|u_n^i| \le C_n A \rho^{1/2} |\ln \rho| \le C'_n r^{1/2} |\ln r| ||f_n||_{(n-4)_+}^*,$$

while

$$|u_n^i| \le C_n A = C_n ||f_n||_{(n-4)_+}^*$$

if n is an odd number. Finally, summing up all these inequalities, we find

$$\begin{aligned} |u| \le \ & |u_0^1| + \sum_{n=1}^{l} \left(|u_n^1| + |u_n^2| \right) \le C_1 r^{1/4} ||f_0||_0^* + C_{2,\delta} r^{-\delta} ||f_1||_0^* \\ & + C_3 r^{1/2} |\ln r| \sum_{k=1}^{[l/2]} ||f_{2k}||_{(2k-4)_+}^* + C_4 \sum_{k=1}^{\left[\frac{l-1}{2}\right]} ||f_{2k+1}||_{(2k-3)_+}^*. \qquad \square \end{aligned}$$

Remark 6.1. Notice that the orthogonality condition $\int_\Omega V_k^{n,i} f dx_1 dx_2 dt = 0$ for a function f with the representation (1.3) is imposed only on the function $f_n^i(\rho, t)$ and has no influence over the other functions $f_m^j(\rho, t)$ from (1.3) with indices $(m, j) \ne (n, i)$.

Proof of Theorem 1.2. We use again the representation (6.1). For $n = 0$ and 1 we have respectively

$$|u_0^1| \le C\rho^{1/4} ||f_0||_0^* \le C' r^{-s+1} ||f_0||_0^*$$

and

$$|u_1^i| \le C_{1/2} \rho^{-1/2} ||f_1||_0^* \le C'' r^{-s+1} ||f_1||_0^*,$$

because $s \ge 2$. For $n \ge 2$ we apply Theorem 5.4 with $q = \left[\dfrac{n-s+1}{2} \right]_+$ and $A = ||f_n||_{2q-2}^*$. Now, the identity (6.2) shows that the orthogonality conditions hold for $n-2k \geq s+1$, i.e. for all $k \geq 0$ such that $q-1 \geq \left[\frac{n-s+1}{2}\right]-1 \geq \left[\frac{2k+2}{2}\right]-1=k$. Therefore

$$|u_n^i| \leq CA\rho^{-n+2q+1} \leq C'\rho^{-s+1}||f_n||^*_{(2q-2)_+} \leq C''r^{-s+1}||f_n||^*_{(n-s-1)_+},$$

because depending of parity of n we have $-n+2\left[\frac{n-s+1}{2}\right]+1=-s+1$ or $-s+2$, and $2\left[\frac{n-s+1}{2}\right]-2=n-s-1$ or $n-s-2$. These give the required upper estimate for the solution u. For the second part of the theorem, let us notice that when $m-2p=s$, the corresponding number q is $q=\left[\frac{m-s+1}{2}\right]=p$. Thus, the lower estimate in Theorem 5.4 gives

$$|u_m^j|_{t=\rho}| \geq c\rho^{-m+2q+1}=c\rho^{-s+1},$$

which completes the proof. □

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