ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

# A GENERALIZATION OF REDFIELD'S MASTER THEOREM [1]

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Generalizations of Redfield's master theorem and the superposition theorem are proved by using decomposition of the tensor product of several induced monomial representations of the symmetric group $S_d$ into transitive constituents. As direct consequences, several corollaries concerning superpositions of graphs are obtained.

**Keywords**: monomial representations of the symmetric group, automorphism groups of superpositions of graphs

**2000 MSC**: 20C30, 05A15, 05C30

## INTRODUCTION

In the present paper we prove a generalization of Redfield's master theorem as a direct consequence of the decomposition of the tensor product of several induced monomial representations of the symmetric group into its transitive summands. The underlying permutation representations give rise to the original Redfield's group-reduced distributions, or, equivalently, to Read's equivalence relation of "$T$-similarity" and superpositions. The most important examples of superpositions are the superpositions of several graphs $\Gamma_1, \dots, \Gamma_k$, each on the same number of vertices. A superposition of $\Gamma_1, \dots, \Gamma_k$ is a graph that is obtained by superposing $\Gamma_m$ on the same set of vertices and by keeping their edges apart. The superposition theorem counts the number of superpositions of the graphs $\Gamma_m$ in terms of their

[1] Research partially supported by the Bulgarian Foundation of Scientific Research under Grant MM-1106/2001.

automorphism groups $W_m \leq S_d$. The corresponding generalization enables us to count those superpositions whose automorphism groups have certain properties in case one of the automorphism groups $W_m$, $m = 1, \ldots, k$, has an one-dimensional character of special type.

The paper is stratified as follows. In Section 1 we note that the tensor product of several induced monomial representations of the symmetric group $S_d$ is a monomial one. Then we discuss the corresponding permutation representation of $S_d$ and, in particular, show that there is a canonical bijection between the $S_d$-orbit space thus obtained and the set of Read's equivalence classes from [4, Sec. 3]. Section 2 contains two equivalent statements that generalize Redfield's master theorem and a generalization of the superposition theorem. In Section 3 we find the number of all superpositions with certain properties of several graphs.

## 1. TENSOR PRODUCT OF INDUCED MONOMIAL REPRESENTATIONS OF $S_d$

Throughout the paper we assume that $K$ is an algebraically closed field of characteristic zero and that all group characters are $K$-valued.

Let $R^d$ be the Abelian group consisting of all generalized characters of the symmetric group $S_d$, and let $\Lambda^d$ be the Abelian group of homogeneous degree $d$ symmetric functions with integer coefficients in a countable set of variables $x_0, x_1, x_2, \ldots$ If $u$, $v \in R^d$, we denote by $\langle u, v \rangle$ their (integer-valued) scalar product. According to [3, Ch. I, Sec. 4], we can define an integer-valued scalar product $\langle\ ,\ \rangle$ on the group $\Lambda^d$, such that the characteristic map $ch\colon R^d \to \Lambda^d$ (see [3, Ch. I, Sec. 7]) is an isometric isomorphism of Abelian groups.

Let $W \leq S_d$ be a permutation group and $\chi\colon W \to K$ be an one-dimensional character. We set

$$Z(\chi; p_1, \ldots, p_d) = \frac{1}{|W|} \sum_{\sigma \in W} \chi(\sigma) p_1^{c_1(\sigma)} \ldots p_d^{c_d(\sigma)},$$

where $p_s = \sum_{i=0}^{\infty} x_i^s$ are the power sums, and $c_s(\sigma)$ is the number of cycles of length $s$ in the cyclic decomposition of the permutation $\sigma$. The symmetric function $Z(\chi) = Z(\chi; p_1, \ldots, p_d)$ is said to be generalized cyclic index of the group $W$. For $\zeta \in S_d$, we denote by $\varrho(\zeta)$ the corresponding partition $(1^{c_1(\zeta)}, \ldots, d^{c_d(\zeta)})$ of the natural number $d$.

The tensor product of two finite-dimensional $K$-linear representations of $S_d$ with characters $u$ and $v$ has character $uv$. If $f = ch(u)$ and $g = ch(v)$, where $u$ and $v$ are generalized characters of $S_d$, one defines internal product $f * g$ of two symmetric functions $f, g \in \Lambda^d$ by $f * g = ch(uv)$. With respect to the internal product, the Abelian group $\Lambda^d$ becomes a commutative and associative ring such that the complete symmetric function $h_d = ch(1_{S_d})$ is an identity element (see [3, Ch. I, Sec. 7]).

Let $W$ be a subgroup of the symmetric group $S_d$ and let $\chi: W \to K$ be an one-dimensional character of $W$. The field $K$ has a natural structure of left $KW$-module given by $\sigma c = \chi(\sigma)c$, where $\sigma \in W$, $c \in K$. We denote by $K_\chi$ the corresponding one-dimensional $K$-linear representation of $W$. Let $I$ be a left transversal of $W$ in $S_d$. The induced monomial representation $Ind_W^{S_d}(\chi) = KS_d \otimes_{KW} K_\chi$ has a natural basis $(e_i)_{i \in I}$, $e_i = i \otimes 1$, as a $K$-linear space. Since for any $\zeta \in S_d$ and $i \in I$ there exist unique $j \in I$ and $\sigma \in W$ such that $\zeta i = j\sigma$, we obtain a group homomorphism $s: S_d \to S(I)$ defined by the formula

$$(s(\zeta)(i))^{-1}\zeta i \in W.$$

Moreover, the permutation group $s(S_d)$ is transitive on the set $I$. We have $\zeta e_i = \zeta(i \otimes 1) = (\zeta i) \otimes 1 = (j\sigma) \otimes 1 = j \otimes (\sigma 1)$. Therefore the action of $S_d$ on $Ind_W^{S_d}(\chi)$ is given by

$$\zeta e_i = \beta_i(\zeta) e_{s(\zeta)(i)},$$

where $\beta_i(\zeta) = \chi(\sigma) = \chi((s(\zeta)(i))^{-1}\zeta i)$.

For the rest of the paper we introduce the following notation:

$(W_m)_{m=1}^k$ is a finite family of subgroups of the symmetric group $S_d$;

$(\chi_m)_{m=1}^k$, $\chi_m: W_m \to K$, is a family of one-dimensional characters;

$I_m$, $(e_i)_{i \in I_m}$, $s_m: S_d \to S(I_m)$ and $(\beta_i^{(m)})_{i \in I_m}$, are the above ingredients for the induced monomial representation $Ind_{W_m}^{S_d}(\chi_m)$, where $m = 1, \ldots, k$.

The rule

$$\zeta(i_1, \ldots, i_k) = (s_1(\zeta)(i_1), \ldots, s_k(\zeta)(i_k)), \tag{1.1}$$

where $(i_1, \ldots, i_k) \in I_1 \times \cdots \times I_k$ and $\zeta \in S_d$, defines an action of the group $S_d$ on the set $I = I_1 \times \cdots \times I_k$.

We denote by $W^\circ$ the group $W$ with the opposite group structure. The Cartesian product of groups $S_d \times W_1^\circ \times \cdots \times W_k^\circ$ acts on the set $S_d \times \cdots \times S_d$ by virtue of the rules

$$(\zeta, w_1, \ldots, w_k)(a_1, \ldots, a_k) = (\zeta a_1 w_1, \ldots, \zeta a_k w_k) \tag{1.2}$$

and

$$(\zeta, w_1, \ldots, w_k).(a_1, \ldots, a_k) = (w_1^{-1} a_1 \zeta^{-1}, \ldots, w_k^{-1} a_k \zeta^{-1}). \tag{1.3}$$

The next obvious lemma follows from the definitions of the actions of the corresponding groups and paves the way for some combinatorial applications.

**Lemma 1.1.** *The following four statements hold:*

(i) *two $k$-tuples $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ are in the same $S_d$-orbit in $I$ if and only if there exist $\zeta \in S_d$ and $w_m \in W_m$ such that $j_m = \zeta i_m w_m$ for $m = 1, \ldots, k$;*

(ii) *the stabilizer of the $k$-tuple $(i_1, \ldots, i_k) \in I$ in the symmetric group $S_d$ is the intersection $i_1 W_1 i_1^{-1} \cap \ldots \cap i_k W_k i_k^{-1}$;*

(iii) *the inclusion $I_1 \times \cdots \times I_k \subset S_d \times \cdots \times S_d$ induces a bijection between the orbit space $S_d \backslash I$ and the orbit space $S_d \times W_1^\circ \times \cdots \times W_k^\circ \backslash S_d \times \cdots \times S_d$ with respect to the action (1.2);*

(iv) *the inversion*

$$S_d \times \cdots \times S_d \to S_d \times \cdots \times S_d,\ (a_1, \ldots, a_k) \mapsto (a_1^{-1}, \ldots, a_k^{-1}),$$

*is an isomorphism of the actions* (1.2) *and* (1.3) *of the group* $S_d \times W_1^\circ \times \cdots \times W_k^\circ$.

**Remark 1.** (i) The set of the orbits of the action (1.2) coincides with the factor-set of $S_d \times \cdots \times S_d$ with respect to the equivalence relation "$T$-similarity", defined in [4, Sec. 3]. Therefore, according to Lemma 1.1, (iii), there is a bijection between the orbit space $S_d \backslash I$ and the set of all distinct superpositions of $k$ graphs with $d$ vertices each (multiple edges and loops allowed), see [4, Sec. 4]. Moreover, the stabilizer $i_1 W_1 i_1^{-1} \cap \ldots \cap i_k W_k i_k^{-1}$ is the automorphism group of the superposition $(i_1, \ldots, i_k)$.

(ii) The orbits of the action (1.3) are Redfield's group-reduced distributions, determined in [5, p. 434]. Lemma 1.1, (iii), (iv), yields that there is a bijection between the orbit space $S_d \backslash I$ and the set of all distinct group-reduced distributions.

**Proposition 1.1.** *The tensor product*

$$Ind_{W_1}^{S_d}(\chi_1) \otimes_K \cdots \otimes_K Ind_{W_k}^{S_d}(\chi_k) \tag{1.4}$$

*is a monomial* $K$*-linear representation of* $S_d$ *with basis* $(e_i = e_{i_1} \otimes \cdots \otimes e_{i_k})_{i \in I}$, *the action of* $S_d$ *being given by the rule*

$$\zeta e_i = \beta_i(\zeta) e_{s(\zeta)(i)},$$

*where* $\beta_i(\zeta) = \beta_{i_1}^{(1)}(\zeta) \ldots \beta_{i_k}^{(k)}(\zeta)$.

*Proof.* It is clear that the family $(e_i)_{i \in I}$ is a basis for the $K$-linear space (1.4). We have $\zeta e_i = \zeta e_{i_1} \otimes \cdots \otimes \zeta e_{i_k} = \beta_{i_1}^{(1)}(\zeta) \ldots \beta_{i_k}^{(k)}(\zeta) e_{s_1(\zeta)(i_1)} \otimes \cdots \otimes e_{s_k(\zeta)(i_k)} = \beta_i(\zeta) e_{s(\zeta)(i)}$. In particular, (1.4) is a monomial representation of $S_d$. □

Due to [1, Lemma 1], the characteristic of the tensor product (1.4) is the internal product $Z(\chi_1) * \cdots * Z(\chi_k)$. We set

$$C(W_1, \ldots, W_k) =$$

$$\{(\sigma_1, \ldots, \sigma_k) \in W_1 \times \cdots \times W_k \mid c_s(\sigma_1) = \cdots = c_s(\sigma_k),\ s = 1, \ldots, d\}.$$

Obviously, $((1), \ldots, (1)) \in C(W_1, \ldots, W_k)$.

For any $\sigma = (\sigma_1, \ldots, \sigma_k) \in C(W_1, \ldots, W_k)$ we define $c_s(\sigma) = c_s(\sigma_1) = \cdots = c_s(\sigma_k)$ for $s = 1, \ldots, d$. Moreover, we set $z_\sigma = \prod_{s=1}^d s^{c_s(\sigma)} c_s(\sigma)!$.

The next proposition links the present definition of internal product to Read's one from [4, Subsec. 3.3].

**Proposition 1.2.** *It holds*

$$Z(\chi_1) * \cdots * Z(\chi_k)$$

$$= \frac{1}{|W_1| \ldots |W_k|} \sum_{\sigma \in C(W_1, \ldots, W_k)} z_\sigma^{k-1} \chi_1(\sigma_1) \ldots \chi_k(\sigma_k) p_1^{c_1(\sigma)} \ldots p_d^{c_d(\sigma)}.$$

*Proof.* The proof is an immediate consequence of [3, Ch. I, Sec. 7, (7.12)]. □

## 2. REDFIELD'S ANSATZ

In this section we generalize the Redfield's master theorem and the superposition theorem.

**Theorem 2.1.** *It holds*

$$Ind_{W_1}^{S_d}(\chi_1) \otimes_K \cdots \otimes_K Ind_{W_k}^{S_d}(\chi_k)$$

$$\simeq \oplus_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} Ind^{S_d}_{\omega_1 W_1 \omega_1^{-1}\cap\ldots\cap\omega_k W_k \omega_k^{-1}}(\psi_{(\omega_1,\ldots,\omega_k)}),$$

*where $T(W_1,\ldots,W_k)$ is a system of distinct representatives of the $S_d$-orbits in the Cartesian product $I = I_1 \times \cdots \times I_k$ with respect to the action* (1.1) *of $S_d$, and $\psi_{(\omega_1,\ldots,\omega_k)}$ is the one-dimensional character of the group $\omega_1 W_1 \omega_1^{-1} \cap \ldots \cap \omega_k W_k \omega_k^{-1}$, which is the restriction of the expression*

$$\beta_\omega(\zeta) = \chi_1((s_1(\zeta)(\omega_1))^{-1}\zeta\omega_1)\ldots\chi_k((s_k(\zeta)(\omega_k))^{-1}\zeta\omega_k)$$

*from Proposition* 1.1.

*Proof.* Due to Proposition 1.1, the tensor product (1.4) is a monomial representation of $S_d$, so it gives an induced monomial representation on each $S_d$-orbit in the set $I$ and (1.4) is the direct sum of these transitive constituents. Now, Lemma 1.1, (ii), finishes the proof. □

Transferring this result by virtue of the characteristic map *ch* on the Abelian group $\Lambda^d$, we obtain a direct generalization of the Redfield's master theorem.

**Theorem 2.2.** *It holds*

$$Z(\chi_1) * \cdots * Z(\chi_k) = \sum_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} Z(\psi_{(\omega_1,\ldots,\omega_k)}).$$

Following R. C. Read, if $A$ is a polynomial in several variables $p_1,\ldots,p_d$, we denote by $N(A)$ the sum of its coefficients.

**Theorem 2.3.** *The number of the elements $\omega \in T(W_1,\ldots,W_k)$ such that $\psi_{(\omega_1,\ldots,\omega_k)} = 1$ on the stabilizer $\omega_1 W_1 \omega_1^{-1} \cap \ldots \cap \omega_k W_k \omega_k^{-1}$ is*

$$N(Z(\chi_1) * \cdots * Z(\chi_k)).$$

*Proof.* Applying the operation $N$ on the two sides of the equality from Theorem 2.2, we obtain

$$N(Z(\chi_1) * \cdots * Z(\chi_k)) = \sum_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} N(Z(\psi_{(\omega_1,\ldots,\omega_k)})).$$

Given a group $G \leq S_d$ and an one-dimensional character $\psi: G \to K$, we have $N(Z(\psi)) = \langle \psi, 1_G \rangle_G$, where $\langle \ , \ \rangle$ is the standard scalar product of functions on the group $G$ (see [3, Ch. I, Sec. 7]). Since $\langle \psi, 1_G \rangle_G = 0$ when $\psi \neq 1_G$, and $\langle \psi, 1_G \rangle_G = 1$ when $\psi = 1_G$, the proof is done. □

**Remark 2.** When $\chi_m = 1_{W_m}$ for $m = 1, \ldots, k$, Theorem 2.2 (respectively, Theorem 2.3) turns into the Redfield's master theorem (respectively, turns into the superposition theorem).

## 3. GRAPHICAL COROLLARIES

Here is how the above machinery applies to the graph theory. Combining Theorem 2.3 and Remark 1, we establish Theorem 3.1 and several graphical corollaries of it. In accordance with Remark 1, these statements can also be formulated in the language of Redfield's ranges and the associated range-groups, see [5, p. 434].

Let $\Gamma_1, \ldots, \Gamma_k$ be graphs with $d$ vertices (loops and multiple edges allowed) and let $W_1 \leq S_d, \ldots, W_k \leq S_d$ be their automorphism groups, respectively. Let $\chi_m: W_m \to K$ be an one-dimensional character of $W_m$, $m = 1, \ldots, k$. Suppose that $\chi_2 = 1_{W_2}, \ldots, \chi_k = 1_{W_k}$, and set $W = W_1$, $\chi = \chi_1$.

**Theorem 3.1.** *Let $\mathcal{G}$ be a set of subgroups of the symmetric group $S_d$, which is closed with respect to conjugations. Let $H \leq W$ be the kernel of the character $\chi$. Let us assume that the set of all subgroups of $W$, which belong to $\mathcal{G}$, coincides with the set of all subgroups of $H$. Then the number of those superpositions of the graphs $\Gamma_1, \ldots, \Gamma_k$, whose automorphism groups belong to $\mathcal{G}$, is $N(Z(\chi) * Z(1_{W_2}) * \cdots * Z(1_{W_k}))$.*

*Proof.* For any subgroup $H' \leq W$ we have $H' \in \mathcal{G}$ if and only if $\chi_{|H'} = 1_{H'}$. The automorphism group $A_\omega = \omega_1 W_1 \omega_1^{-1} \cap \ldots \cap \omega_k W_k \omega_k^{-1}$ of any superposition $\omega = (\omega_1, \ldots, \omega_k)$ is a subgroup of $\omega_1 W \omega_1^{-1}$. Obviously, the subgroups of $\omega_1 W \omega_1^{-1}$ from $\mathcal{G}$ are exactly the subgroups of the kernel $\omega_1 H \omega_1^{-1}$ of the one-dimensional character $\chi(\omega_1^{-1} \zeta \omega_1)$ of the group $\omega_1 W \omega_1^{-1}$. On the other hand, the one-dimensional character $\psi_\omega(\zeta)$ of $A_\omega$ from Theorem 2.2 is the restriction of $\chi(\omega_1^{-1} \zeta \omega_1)$. Thus, $A_\omega \in \mathcal{G}$ if and only if $\psi_\omega(\zeta)$ is identically 1 on $A_\omega$. Therefore Theorem 2.3 implies the result. □

Given a cyclic group of order $b$ and a divisor $a$ of $b$, let $\varrho^{(a)}$ be an one-dimensional character of this cyclic group, whose kernel has order $a$. If $\mathcal{G}$ is the set of all cyclic subgroups of $S_d$ of order that divides $a$, Theorem 3.1 yields

**Corollary 3.1.** *If the group $W = W_1$ is cyclic of order $b$ and if $a$ is a divisor of $b$, then the number of all superpositions of $\Gamma_1, \ldots, \Gamma_k$, which have a cyclic automorphism group of order dividing $a$, is*

$$N(Z(\varrho^{(a)}) * Z(1_{W_2}) * \cdots * Z(1_{W_k})).$$

In the particular case $a = 1$, we obtain

**Corollary 3.2.** *If the permutation group $W = W_1$ is cyclic, then the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, having a trivial automorphism group, is*

$$N(Z(\varrho) * Z(1_{W_2}) * \cdots * Z(1_{W_k})),$$

*where $\varrho: W \to K$ is an injective one-dimensional character of $W$.*

Now, let $\mathcal{G}$ be the set of all subgroups of $S_d$, consisting of even permutations. Then Theorem 3.1 implies

**Corollary 3.3.** *The number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism group consist of even permutations, is*

$$N(Z(\varepsilon) * Z(1_{W_2}) * \cdots * Z(1_{W_k})),$$

*where $\varepsilon: W_1 \to K$ is the restriction of the alternating character of $S_d$ on $W_1$.*

Let $r$ be a natural number. We suppose that:

(a) $W = W_1$ has a normal solvable subgroup $R$ of order $r$ such that the factor-group $W/R$ is cyclic of order relatively prime to $r$.

Then the group $W$ itself is solvable. According to the generalized Sylow theorems (cf [2, Ch. 9, Theorem 9.3.1]), $R$ is the only subgroup of $W$ of order $r$. Moreover, any subgroup of $W$ of order that divides $r$ is contained in $R$.

Denote by $\pi$ an one-dimensional character of $W$ with kernel $R$. If $\mathcal{G}$ is the set of all subgroups of $S_d$ of order dividing $r$, we obtain

**Corollary 3.4.** *If the group $W = W_1$ satisfies condition* (a), *then*

$$N(Z(\pi) * Z(1_{W_2}) * \cdots * Z(1_{W_k}))$$

*is the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism groups are of order dividing $r$.*

Now, we formulate an important version of Corollary 3.4. Let $q$ be a prime number. Suppose that:

(b) the group $W = W_1$ has a normal $q$-subgroup $R$ such that the factor-group $W/R$ is cyclic of order relatively prime to $q$.

In accordance with Sylow theorems (see [2, Ch. 4, Theorems 4.2.1 - 4.2.3]), $R$ is the only Sylow $q$-subgroup of $W$. Moreover, any $q$-subgroup of $W$ is contained in $R$.

Denote by $\iota$ an one-dimensional character of $W$ with kernel $R$. If $\mathcal{G}$ is the set of all $q$-subgroups of $S_d$, then we get

**Corollary 3.5.** *If the group $W = W_1$ satisfies condition* (b), *then*

$$N(Z(\iota) * Z(1_{W_2}) * \cdots * Z(1_{W_k}))$$

*is the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism group is a $q$-group.*

**Remark 3.** Examples of abstract groups $W = W_1$ which satisfy the hypothesis (a) (respectively, the hypothesis (b)) can be obtained by constructing a semi-direct product of a solvable group $R$ of order $r$ (respectively, a $q$-group $R$) with a cyclic group $C$ of order relatively prime to $r$ (respectively, relatively prime to $q$). The Schur-Zassenhaus' theorem (see [6, Ch. IV, Sec. 8, IV.7.c]) asserts that there are no other examples. In the symmetric group $S_d$, it is enough to choose $R \leq S_d$ and $C \leq S_d$ with the above-mentioned properties so that $RC = CR$, $R \cap C = \{(1)\}$ and $R$ is a normal subgroup of the group $W_1 = RC$.

**Acknowledgments.** The author would like to thank prof. R. Robinson who noted that Corollary 3.1 could be obtained by using Case 2 of Theorem from [5], p. 449, where Redfield has counted the superpositions by automorphism group when one of the range groups is cyclic. The author would like to thank also prof. F. Harary for his moral support and the referee for the helpful remarks which improved the readability of the paper.

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*Received on November 13, 2001*

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