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SHARP ESTIMATES FOR THE FIFTH COEFFICIENT OF THE INVERSE FUNCTIONS OF THE TOTALLY MONOTONIC FUNCTIONS

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We find the minimum and the maximum of the fifth coefficient for the inverse functions of the totally monotonic functions.

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1. INTRODUCTION

Let T denote the class of totally monotonic functions

$$w = \varphi(z) = \int_0^1 \frac{z\, d\mu(t)}{1 - zt} = \sum_{n=1}^{\infty} a_n z^n, \quad z \notin [1, +\infty], \tag{1.1}$$

where $\mu(t)$ is a probability measure on $[0, 1]$ and

$$a_n = \int_0^1 t^{n-1}\, d\mu(t), \quad n = 1, 2, \ldots, \quad a_1 = 1. \tag{1.2}$$

In [1] it it noted that the largest common region of convergence of all Taylor series at the point $w = 0$ of the inverse functions $z = \psi(w)$ of the functions (1.1) is the disk $|w| < 1/2$. Let

$$z = \psi(w) = \sum_{n=1}^{\infty} b_n w^n, \quad |w| < \frac{1}{2}, \quad b_1 = 1, \tag{1.3}$$

be such series, where in [1] the coefficients b_n are determined explicitly by the coefficients a_n in (1.2). In [1] we found the minimum and the maximum of the coefficients b_2, b_3 and b_4 and conjectured that the extrema of all coefficiens b_n, $n = 2, 3, 4, \dots$, in (1.3) are attained only for the rational functions of the form

$$\varphi(z) = cz + \frac{(1-c)z}{1-z} \in T, \quad 0 \le c \le 1, \tag{1.4}$$

for suitable values of c, and, in addition,

$$b_{2m} \ge -1, \quad m = 1, 2, \dots, \tag{1.5}$$

and

$$b_{2m+1} \le 1, \quad m = 1, 2, \dots, \tag{1.6}$$

where the equalities in (1.5) and (1.6) hold only for the function

$$\psi(w) = \frac{w}{1+w} = \sum_{n=1}^{\infty} (-1)^{n-1} w^n, \tag{1.7}$$

inverse of the function (1.4) for $c = 0$, respectively.

Now we verify these conjectures for the fifth coefficient b_5 in (1.3) as well.

2. SHARP ESTIMATES FOR b_5

In [1, p. 41, Theorem 4] we have proved that the minimum (the maximum) of the coefficients b_n, $n \ge 2$, in (1.3) in the class T is attained only either in the subclass of functions (1.4) or in the subclass of functions

$$\varphi(z) = \sum_{k=1}^{p} \frac{c_k z}{1 - t_k z} \in T, \tag{2.1}$$

where

$$1 \le p \le m, \quad n = 2m, \quad m = 1, 2, \dots, \tag{2.2}$$

$$1 \le p \le m+1, \quad n = 2m+1, \quad m = 1, 2, \dots, \tag{2.3}$$

$$0 \le t_1 \le t_2 \le \dots \le t_p \le 1, \quad 0 \le c_k \le 1, \quad \sum_{k=1}^{p} c_k = 1, \tag{2.4}$$

and $t_1, t_2, \ldots, t_p$ are among the numbers 0 and 1 and the roots in the interval $0 \le t \le 1$ of the equation

$$P(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s}(s-1)t^{s-2} = 0, \quad n \ge 3. \tag{2.5}$$

The function

$$Q(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} t^{s-1}, \quad Q'(t) = P(t), \quad n \ge 3, \tag{2.6}$$

has equal values at any two adjacent points of the sequence $t_1, t_2, \ldots, t_p$ for $p \ge 2$, i.e.

$$Q(t_1) = Q(t_2) = \cdots = Q(t_p), \quad p \ge 2. \tag{2.7}$$

The equations (2.7) are necessary conditions for the extrema of b_n with respect to $c_1, c_2, \ldots, c_p$. In fact, b_n depends on $a_2, a_3, \ldots, a_n$, which by (2.1) are equal to

$$a_s = \sum_{k=1}^{p} c_k t_k^{s-1}, \quad 2 \le s \le n, \quad n \ge 3, \quad p \ge 2. \tag{2.8}$$

From (2.8) and the last equation in (2.4) we have

$$\frac{\partial a_s}{\partial c_k} = t_k^{s-1} - t_{k+1}^{s-1}, \quad 1 \le k \le p, \quad p \ge 2, \quad t_{p+1} = t_1. \tag{2.9}$$

Having in mind (2.9) and (2.6), we obtain the formula

$$\frac{\partial b_n}{\partial c_k} = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} \cdot \frac{\partial a_s}{\partial c_k} = Q(t_k) - Q(t_{k+1}), \tag{2.10}$$
$$n \ge 3, \quad 1 \le k \le p, \quad p \ge 2, \quad Q(t_{p+1}) = Q(t_1).$$

Since $\partial b_n / \partial c_k = 0$ at the extrema of b_n, formula (2.10) yields (2.7).

Theorem 2.1. *The coefficient b_5 in* (1.3) *satisfies the sharp inequalities*

$$-0.1317545\ldots = 14c^4 - 35c^3 + 30c^2 - 10c + 1 \le b_5 \le 1, \tag{2.11}$$

where

$$c = 0.294997\ldots, \quad 0.294997 < c < 0.294998, \tag{2.12}$$

is the smallest positive root of the equation

$$56c^3 - 105c^2 + 60c - 10 = 0, \tag{2.13}$$

and the equalities hold only for the following extremal functions: on the left-hand side of (2.11) — *for the inverse function of the function* (1.4) *for* c *determined by* (2.11)–(2.13), *and on the right-hand side of* (2.11) — *for the function* (1.7).

Proof. In terms of the coefficients a_2, a_3, a_4, a_5 in (1.2), the coefficient b_5 in (1.3) has the following explicit form (see Theorem 3 and its proof in [1]):

$$b_5 = -a_5 + 6a_2a_4 + 3a_3^2 - 21a_2^2a_3 + 14a_2^4. \tag{2.14}$$

According to our general theorem, expressed by means of (2.1)–(2.7) applied to $n = 5$ and (2.14), the only possible extremal functions for b_5 are the functions of the form (1.4), and

$$\varphi(z) = \frac{z}{1-tz} \in T, \quad 0 \le t \le 1, \tag{2.15}$$

$$\varphi(z) = cz + \frac{(1-c)z}{1-tz} \in T, \quad 0 < c < 1, \quad 0 < t < 1, \tag{2.16}$$

$$\varphi(z) = \frac{cz}{1-tz} + \frac{(1-c)z}{1-z} \in T, \quad 0 < c < 1, \quad 0 < t < 1, \tag{2.17}$$

$$\varphi(z) = \frac{cz}{1-t_1z} + \frac{(1-c)z}{1-t_2z} \in T, \quad 0 < c < 1, \quad 0 < t_1 < t_2 < 1, \tag{2.18}$$

$$\varphi(z) = c_1z + \frac{c_2z}{1-tz} + \frac{c_3z}{1-z} \in T, \tag{2.19}$$
$$0 < c_{1,2,3} < 1, \quad c_1 + c_2 + c_3 = 1, \quad 0 < t < 1$$

(in general t is different for each function), with the corresponding equations

$$P(t) = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3 - 21a_2^2)2t + 18a_2t^2 - 4t^3 = 0 \tag{2.20}$$

in t and functions

$$Q(t) = \left(6a_4 - 42a_2a_3 + 56a_2^3\right)t + \left(6a_3 - 21a_2^2\right)t^2 + 6a_2t^3 - t^4 \tag{2.21}$$

with $Q'(t) = P(t)$. For the latter and for the corresponding functions (1.4), (2.16)–(2.19) we have the equations

$$Q(0) = Q(1), \tag{2.22}$$
$$P(t) = 0, \quad Q(0) = Q(t), \tag{2.23}$$
$$P(t) = 0, \quad Q(t) = Q(1), \tag{2.24}$$
$$P(t_1) = 0, \quad P(t_2) = 0, \quad Q(t_1) = Q(t_2), \tag{2.25}$$
$$P(t) = 0, \quad Q(0) = Q(t), \quad Q(t) = Q(1), \quad Q(1) = Q(0). \tag{2.26}$$

(i) First, we examine the function (1.4). From it we find the Taylor coefficients

$$a_2 = 1 - c, \quad a_3 = 1 - c, \quad a_4 = 1 - c, \quad a_5 = 1 - c. \tag{2.27}$$

From (2.27) and (2.14) we obtain that

$$\begin{aligned} b_5 &= 14c^4 - 35c^3 + 30c^2 - 10c + 1 \\ &= 14(c-1)\left(c-\frac{1}{2}\right)\left(c-\frac{7+\sqrt{21}}{14}\right)\left(c-\frac{7-\sqrt{21}}{14}\right) := b_5(c), \quad 0 \le c \le 1. \end{aligned} \tag{2.28}$$

It follows from (2.28) that the derivative equation

$$b_5'(c) = 56c^3 - 105c^2 + 60c - 10 = 0 \tag{2.29}$$

has three real roots

$$\begin{aligned} c' &= 0.294997\ldots, \quad 0.294997 < c' < 0.294998, \\ c'' &= 0.652\ldots, \quad 0.652 < c'' < 0.653, \\ c''' &= 0.9270\ldots, \quad 0.9270 < c''' < 0.9271 \end{aligned} \tag{2.30}$$

for which

$$\begin{aligned} \min b_5(c') &= -0.1317545\ldots, \\ \max b_5(c'') &= 0.062235\ldots, \\ \min b_5(c''') &= -0.03281\ldots \end{aligned} \tag{2.31}$$

In addition,

$$b_5(0) = 1, \quad b_5(1) = 0. \tag{2.32}$$

The derivative equation (2.29) follows from formula (2.10) and equation (2.22) as well. In fact, we have

$$b_5'(c) = Q(0) - Q(1) = 56c^3 - 105c^2 + 60c - 10 = 0 \tag{2.33}$$

by (2.10) for $n = 5$, $k = 1$, $p = 2$, $c_1 = c$, $t_1 = 0$, $t_2 = 1$, and (2.21) for the values (2.27) and (2.22).

(ii) Second, we examine the function (2.15). Converting (2.15) or by means of the coefficients of (2.15) and (2.14), we obtain

$$b_5 = t^4, \quad 0 \le t \le 1, \quad \min b_5 = 0, \quad \max b_5 = 1. \tag{2.34}$$

(iii) Third, we examine the function (2.16). From (2.16) we find the coefficients

$$a_2 = (1-c)t, \quad a_3 = (1-c)t^2, \quad a_4 = (1-c)t^3, \quad a_5 = (1-c)t^4. \tag{2.35}$$

From (2.20), (2.21) and (2.23) we obtain

$$\frac{1}{3t}\left[P(t) - \frac{1}{t}Q(t)\right] = 2a_3 - 7a_2^2 + 4a_2t - t^2 = 0. \tag{2.36}$$

It follows from (2.35)–(2.36) that

$$7(1-c)^2 - 6(1-c) + 1 = 0. \tag{2.37}$$

From (2.37) and (2.35) we find

$$a_2^{\pm} = \frac{3 \pm \sqrt{2}}{7} t, \quad a_3^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^2, \quad a_4^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^3, \quad a_5^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^4, \tag{2.38}$$

respectively. Now (2.38) and (2.14) yield

$$b_5^{\pm} = t^4 \frac{-13 \pm 16\sqrt{2}}{343}, \quad 0 < t < 1, \tag{2.39}$$

respectively. Equations (2.39) lead to the corresponding boundaries

$$\inf b_5^{+} = 0, \quad \sup b_5^{+} = 0.0280682\ldots, \quad 0 < t < 1, \tag{2.40}$$

$$\inf b_5^{-} = -0.10387\ldots, \quad \sup b_5^{-} = 0, \quad 0 < t < 1. \tag{2.41}$$

(iv) Fourth, we examine the function (2.17). From (2.17) we find the coefficients

$$a_2 = c(t-1) + 1,\ a_3 = c(t^2 - 1) + 1,\ a_4 = c(t^3 - 1) + 1,\ a_5 = c(t^4 - 1) + 1. \tag{2.42}$$

From (2.20), (2.21) and (2.24) we obtain

$$\begin{aligned} &\frac{1}{t-1}\left\{P(t) - \frac{1}{t-1}[Q(t) - Q(1)]\right\} \\ &= 6a_3 - 21a_2^2 + 6a_2(2t+1) - 3t^2 - 2t - 1 = 0. \end{aligned} \tag{2.43}$$

It follows from (2.42)–(2.43) that

$$21(1-t)^2c^2 - 6(1-t)(5-3t)c + (3t^2 + 2t + 16) = 0. \tag{2.44}$$

The discriminant of the equation (2.44) in c is

$$3(1-t)^2[2t(3t-52) - 37] < 0, \quad 0 < t < 1. \tag{2.45}$$

From (2.45) we conclude that the equation (2.44) has no real roots for c, and hence, the function (2.17) is not extremal for b_5.

(v) Fifth, we examine the function (2.18). From it we find the coefficients

$$\begin{aligned} a_2 &= c(t_1 - t_2) + t_2, \quad a_3 = c(t_1^2 - t_2^2) + t_2^2, \\ a_4 &= c(t_1^3 - t_2^3) + t_2^3, \quad a_5 = c(t_1^4 - t_2^4) + t_2^4. \end{aligned} \tag{2.46}$$

On the other hand, from (2.20), (2.21) and (2.25) we obtain

$$\frac{1}{2(t_1 - t_2)}\left\{P(t_1) + P(t_2) - \frac{2}{t_1 - t_2}[Q(t_1) - Q(t_2)]\right\} = 3a_2 - t_1 - t_2 = 0, \tag{2.47}$$

$$a_2 = \frac{t_1 + t_2}{3}.$$

Further, from (2.21) and (2.25) we get

$$\frac{1}{2(t_1 - t_2)}[P(t_1) - P(t_2)] = 6a_3 - 21a_2^2 + 9a_2(t_1 + t_2) - 2(t_1^2 + t_1t_2 + t_2^2) = 0. \tag{2.48}$$

It follows from (2.47) and (2.48) that

$$a_3 = \frac{2t_1^2 + t_1t_2 + 2t_2^2}{9}. \tag{2.49}$$

Finally, from (2.20), the first equation in (2.25), (2.47) and (2.49) we obtain

$$a_4 = \frac{2(t_1 + t_2)(7t_1^2 - 4t_1t_2 + 7t_2^2)}{81}. \tag{2.50}$$

Now, identifying the both expressions of a_2 in (2.46) and (2.47), we find

$$c = \frac{t_1 - 2t_2}{3(t_1 - t_2)}. \tag{2.51}$$

Having in mind (2.51), the identification of the corresponding expressions of a_3 and a_4 in (2.46), (2.49) and (2.50) leads to the system of equations

$$t_1^2 - 4t_1t_2 + t_2^2 = 0, \quad 13t_1^2 - 46t_1t_2 + 13t_2^2 = 0. \tag{2.52}$$

Setting $t_2 = kt_1$ in (2.52), we obtain the equations

$$k^2 - 4k + 1 = 0, \quad 13k^2 - 46k + 13 = 0. \tag{2.53}$$

But equations (2.53) have no common root, whence it follows that the function (2.18) is not extremal for b_5.

(vi) Sixth, we examine the function (2.19). From (2.19) we find the coefficients

$$a_2 = c_2t + c_3, \quad a_3 = c_2t^2 + c_3, \quad a_4 = c_2t^3 + c_3, \quad a_5 = c_2t^4 + c_3. \tag{2.54}$$

On the other hand, from (2.21) and (2.26) we have $Q(0) = 0$.

$$\frac{1}{t}Q(t) = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3 - 21a_2^2)t + 6a_2t^2 - t^3 = 0, \tag{2.55}$$

$$\frac{1}{t-1}[Q(t) - Q(1)] = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3 - 21a_2^2)(t + 1) \tag{2.56}$$

$$+ 6a_2(t^2 + t + 1) - (t + 1)(t^2 + 1) = 0,$$

$$Q(1) = (6a_4 - 42a_2a_3 + 56a_2^3) + (6a_3 - 21a_2^2) + 6a_2 - 1 = 0. \tag{2.57}$$

Subtracting (2.57) from (2.56), we obtain

$$(6a_3 - 21a_2^2)t + 6a_2(t^2 + t) - t^3 - t^2 - t = 0. \tag{2.58}$$

If we add (2.58) to (2.56) and subtracting this sum from (2.20), then we find

$$a_2 = \frac{2t+1}{6} \cdot \tag{2.59}$$

It follows from (2.58)–(2.59) that

$$a_3 = \frac{16t^2+4t+7}{72} \cdot \tag{2.60}$$

From (2.57) and (2.60) we find

$$a_4 = \frac{224t^3+48t^2+42t+91}{1296} \cdot \tag{2.61}$$

Now the identification of the corresponding expressions of a_2 and a_3 in (2.54) and (2.59)–(2.60) leads to the values

$$c_2 = \frac{16t^2-20t-5}{72t(t-1)}, \quad c_3 = \frac{8t^2+8t-7}{72(t-1)} \cdot \tag{2.62}$$

From (2.54) and (2.62) we find

$$a_4 = \frac{16t^3-4t^2-t+7}{72} \tag{2.63}$$

and

$$a_5 = \frac{16t^4-4t^3-9t^2-t+7}{72} \cdot \tag{2.64}$$

The identification of (2.61) and (2.63) yields the equation

$$64t^3-120t^2-60t+35=0. \tag{2.65}$$

The equation (2.65) has three real roots lying in the intervals $(-\infty, 0)$, $(0,1)$ and $(1,+\infty)$, respectively, where the root in the interval $(0,1)$ is

$$t' = 0.3668\ldots, \quad 0.3668 < t' < 0.3669. \tag{2.66}$$

The value (2.14) for (2.59)–(2.60) and (2.63)–(2.64) is

$$b_5 = \frac{128t^4-320t^3-240t^2+280t-91}{5184} := b_5(t). \tag{2.67}$$

From (2.66)–(2.67) we obtain

$$b_5(t') = -0.0065704\ldots \tag{2.68}$$

Now comparison of (2.30)–(2.32), (2.34), (2.40)–(2.41) and (2.68) leads to (2.11)–(2.13), which completes the proof of Theorem 2.1.

For the next coefficients b_6, b_7, ... we can proceed in the same way. In accordance with our conjecture, for the functions (1.4) we can expect that the functions (2.1)–(2.4) different from the functions (1.4) are not extremal for b_n, $n \geq 6$.

3. AN EXPLICIT FORM OF THE COEFFICIENTS OF THE INVERSE FUNCTIONS OF THE FUNCTIONS (1.4)

The function (1.4) can be rewritten in the form

$$w = \varphi_c(z) := \frac{z(1-cz)}{1-z} = z + (1-c)\sum_{n=2}^{\infty} z^n \in T, \quad |z| < 1, \quad 0 \le c \le 1 \tag{3.1}$$

The branch of the two-valued inverse function of (3.1) determined by the values $z = 0$ for $w = 0$ is the function

$$\begin{aligned} z = \psi_c(w) &= \frac{1 + w - \sqrt{1 - 2(2c-1)w + w^2}}{2c} \\ &\equiv \frac{2w}{1 + w + \sqrt{1 - 2(2c-1)w + w^2}} \end{aligned} \tag{3.2}$$

with $\sqrt{1} = 1$, analytic and univalent in the w-plane cut along the two two-times-describable rays

$$w = \varphi_c(1 + iy) = 2c - 1 + i\left(cy + \frac{1-c}{y}\right), \quad -\infty \le y \le +\infty,$$

connecting the branch points

$$w_{1,2} = \varphi_c(z_{1,2}) = 2c - 1 \pm 2i\sqrt{c(1-c)}, \quad z_{1,2} = 1 \pm i\frac{\sqrt{c(1-c)}}{c},$$

through the point at infinity, which correspond to the equations

$$\frac{\partial \varphi_c(z_{1,2})}{\partial z} = 0.$$

According to our earlier results for the univalence of the class T of functions (1.1), their derivatives $\varphi'(z)$ vanish on the straight line $\operatorname{Re} z = 1$ only for the functions (1.4) with $0 < c < 1$ (see [2, pp. 417–418, Theorem 1; Eq. (4) contains a misprint where an inequality sign is reversed], [3, p. 120, Theorem 1], [4] and [5]). Hence the image of the half-plane $\operatorname{Re} z \le 1$ by each function (1.1) of the class T except the functions (1.4) for $0 < c < 1$ has exterior points.

Theorem 3.1. *The inverse function* (3.2) *has the Taylor expansion*

$$z = \psi_c(w) = \sum_{n=1}^{\infty} b_n(c) w^n, \quad b_1(c) = 1, \quad |w| < 1, \tag{3.3}$$

where

$$b_n(c) = \sum_{\nu=0}^{n-1} \frac{(-1)^{n-1-\nu}}{\nu+1} \binom{2\nu}{\nu} \binom{n-1+\nu}{n-1-\nu} c^\nu, \quad n = 1, 2, \ldots \tag{3.4}$$

Proof. By the first representation in (3.2) we obtain

$$z = \psi_c(w) = \frac{1}{2c}(1+w)\left(1 - \sqrt{1 - \frac{4cw}{(1+w)^2}}\right). \tag{3.5}$$

It follows for sufficiently small values of $|w|$ that

$$\begin{aligned} 1 - \sqrt{1 - \frac{4cw}{(1+w)^2}} &= 2\sum_{\nu=1}^{\infty} \frac{1}{\nu}\binom{2\nu-2}{\nu-1} c^\nu w^\nu (1+w)^{-2\nu} \\ &= 2\sum_{\nu=1}^{\infty} \frac{1}{\nu}\binom{2\nu-2}{\nu-1} c^\nu \sum_{n=\nu}^{\infty} (-1)^{n-\nu}\binom{\nu+n-1}{n-\nu} w^n \\ &= 2\sum_{n=1}^{\infty} w^n \sum_{\nu=1}^{n} \frac{(-1)^{n-\nu}}{\nu}\binom{2\nu-2}{\nu-1}\binom{\nu+n-1}{n-\nu} c^\nu. \end{aligned} \tag{3.6}$$

Now (3.5)–(3.6) lead to (3.3)–(3.4), which completes the proof of Theorem 3.1.

In particular, for $c = 1$ and $c = 1/2$, the coefficients of (3.2) are $b_n(1) = 0$, $n \geq 2$, and

$$b_{2n}\left(\frac{1}{2}\right) = \frac{(-1)^n}{n2^{2n-1}}\binom{2n-2}{n-1}, \quad b_{2n+1}\left(\frac{1}{2}\right) = 0, \quad n \geq 1,$$

respectively, which compared with (3.4) yield the corresponding identities.

Formula (2.10) for $n \geq 3$, $k = 1$, $p = 2$, $c_1 = c$, $t_1 = 0$, $t_2 = 1$ is reduced to the formula

$$b'_n(c) = Q(0) - Q(1),$$

where $b'_n(c)$, $Q(0) = 0$ and $Q(1)$ are determined by (3.4) and (2.6), respectively (for $n = 5$ this formula is noted in (2.33)).

Let

$$m_n = \min_T b_n, \quad M_n = \max_T b_n, \quad n = 2, 3, \ldots, \tag{3.7}$$

where b_n, $n \geq 2$, are those in (1.3).

If the conjecture for the function (1.4) is true, then

$$m_n = \min_{0 \leq c \leq 1} b_n(c), \quad M_n = \max_{0 \leq c \leq 1} b_n(c), \quad n = 2, 3, \ldots, \tag{3.8}$$

where $b_n(c)$, $n \geq 2$, are those in (3.4).

For $n = 2, 3, 4, 5$ it follows from (3.4) that

$$\begin{aligned} b_2(c) &= -1 + c, \quad b_3(c) = 1 - 3c + 2c^2, \\ b_4(c) &= -1 + 6c - 10c^2 + 5c^3, \\ b_5(c) &= 1 - 10c + 30c^2 - 35c^3 + 14c^4. \end{aligned} \tag{3.9}$$

By formulas (3.7)–(3.8) applied to the polynomials (3.9) we obtain the explicit values of m_n and M_n for $n = 2, 3, 4, 5$ as follows:

$$m_2 = -1 \ (c = 0), \quad M_2 \ (c = 1); \tag{3.10}$$

$$m_3 = -\frac{1}{8} \ \left(c = \frac{3}{4}\right), \quad M_3 = 1 \ (c = 0); \tag{3.11}$$

$$m_4 = -1 \ (c = 0), \tag{3.12}$$

$$M_4 = \frac{5 + 4\sqrt{10}}{135} = 0.13073415\ldots \ \left(c = \frac{10 - \sqrt{10}}{15} = 0.45584816\ldots\right);$$

$$m_5 = -0.1317545\ldots \ (c = 0.294997\ldots), \quad M_5 = 1 \ (c = 0); \tag{3.13}$$

where m_n and M_n for $n = 2, 3, 4, 5$ are realized only by the functions (1.4) (or (3.1)–(3.2)) for the values of c indicated in the parentheses and (2.12)–(2.13), respectively.

The equations (3.10)–(3.12) for $n = 2, 3, 4$ and the equations (3.13) for $n = 5$ are proved in [1] and Theorem 2.1 above, respectively.

For $n = 6, 7, \ldots$ the values of $m_6, m_7, \ldots, M_6, M_7, \ldots$ can be obtained by the conjectural formulas (3.8) applied to the polynomials (3.4) for $n = 6, 7, \ldots$

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