
ON THE DURATION DOMAINS FOR THE INTERVAL TEMPORAL LOGIC

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The duration domains for the Interval Temporal Logic are characterized as the positive cones of the right-ordered groups.

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In 1985 B. Moszkowski [7] introduced a logical system called Interval Temporal Logic. Its semantics, proposed by B. Dutertre [2] in 1995, uses a kind of structures called *duration domains*. The same kind of structures have been used later also by D. Guelev for the semantics of other logical systems (cf., for example, [4, 5]). The structures in question can be defined as triples $(D, +, 0)$, where D is a set, $+$ is a binary operation in D , 0 is an element of D and the following five axioms are identically satisfied in D :

$$(D1) \quad (x + y) + z = x + (y + z),$$

$$(D2) \quad x + 0 = 0 + x = x,$$

$$(D3) \quad x + z = y + z \Rightarrow x = y, \quad z + x = z + y \Rightarrow x = y,$$

$$(D4) \quad x + y = 0 \Rightarrow x = y = 0,$$

$$(D5) \quad \exists z(x + z = y \vee y + z = x), \quad \exists z(z + x = y \vee z + y = x).$$

The aim of the present paper is to characterize the duration domains as the positive cones of the right-ordered groups. This will be done by proving theorems 1 and 2 below.

A *right-ordered group* (cf. [1, 6]) is a structure $(G, +, 0, -, \geq)$, where $(G, +, 0, -)$ is a group (not necessarily abelian), $+$, 0 , $-$ being respectively the binary group operation, the neutral element of the group and the unary operation of constructing the inverse element, and \geq is a linear ordering in G such that for all x, y, z in G the following implication holds:

$$x \geq y \Rightarrow x + z \geq y + z$$

(as in [6], we assume the orderings reflexive, although the orderings in [1] are assumed to be irreflexive). The *positive cone* of such a structure is the set of all elements x of G that satisfy the condition $x \geq 0$. If P is the positive cone of a right-ordered group $(G, +, 0, -, \geq)$, then the following three conditions are satisfied for all x and y in G :

- (P1) $x \in P \wedge -x \in P \Rightarrow x = 0$,
- (P2) $x \in P \wedge y \in P \Rightarrow x + y \in P$,
- (P3) $x \in P \vee -x \in P$.

Conversely, whenever $(G, +, 0, -)$ is a group and P is a subset of G with the properties (P1)–(P3), then a binary relation \geq in G exists such that $(G, +, 0, -, \geq)$ is a right-ordered group with positive cone P .

Theorem 1. *Let $(G, +, 0, -, \geq)$ be a right-ordered group and P be its positive cone. Let $+_P$ be the restriction of the operation $+$ to P^2 . Then $(P, +_P, 0)$ is a duration domain.*

Proof. The element 0 of G belongs to P by (P3), hence, taking into account also (P2), we may consider the structure $(P, +_P, 0)$. This structure obviously satisfies the axioms (D1)–(D3), and (D4) follows immediately from the property (P1). To verify (D5), suppose x and y are some elements of P . If we set $u = (-x) + y$, then the equalities $x + u = y$ and $y + (-u) = x$ hold, and, since some of the elements u and $-u$ belongs to P by (P3), this establishes the first statement of (D5). The second one can be established in a similar way. \square

Remark. Under the assumptions of the above theorem, if the considered group is not abelian, then the operation $+_P$ is not commutative.¹ In fact, let x and y be elements of G such that $x + y \neq y + x$. By (P3) some of the elements x and $-x$ belongs to P and also some of the elements y and $-y$ belongs to P . Therefore it is sufficient to establish the inequalities

$$x + (-y) \neq (-y) + x, \quad (-x) + y \neq y + (-x), \quad (-x) + (-y) \neq (-y) + (-x).$$

To prove the first one, we suppose the equality $x + (-y) = (-y) + x$ and get $y + (x + (-y)) + y = y + ((-y) + x) + y$, i.e. $y + x = x + y$. In a similar way we

¹ Since there are non-abelian right-ordered groups (examples of such groups can be found, for instance, in [1] and [3, ch. 2]), this implies the existence of a duration domain with non-commutative addition operation.

show the impossibility of the equality $(-x) + y = y + (-x)$. Finally, if we suppose that $(-x) + (-y) = (-y) + (-x)$, then we get $-((-x) + (-y)) = -((-y) + (-x))$, and this leads again to the contradictory equality $y + x = x + y$.

Theorem 2. *Any duration domain can be obtained in the way from Theorem 1 at a convenient choice of some right-ordered group $(G, +, 0, -, \geq)$.*

Proof. Let $(D, +, 0)$ be a duration domain. To each element s of $D \setminus \{0\}$ we make to correspond an object \bar{s} not belonging to D in such a way that $\bar{s} \neq \bar{t}$ whenever s and t are distinct elements of $D \setminus \{0\}$. Then we set

$$G = D \cup \{\bar{s} \mid s \in D \setminus \{0\}\},$$

and we define the inverse element of any element of G by setting $-0 = 0$ and

$$-s = \bar{s}, \quad -\bar{s} = s$$

for any s in $D \setminus \{0\}$. We extend the binary operation $+$ from D to G by stipulating the equalities

$$(z + s) + \bar{s} = z, \quad x + \overline{t + x} = \bar{t}, \quad \bar{s} + (s + z) = z, \quad \overline{y + t} + y = \bar{t}, \quad \bar{s} + \bar{t} = \overline{t + s}$$

for all x, y, z in D and all s, t in $D \setminus \{0\}$.² It follows immediately that

$$0 + \bar{t} = \bar{t} + 0 = \bar{t}, \quad s + \bar{s} = \bar{s} + s = 0$$

for all s, t in $D \setminus \{0\}$, hence

$$0 + u = u + 0 = u, \quad u + (-u) = (-u) + u = 0$$

for all u in G . If we denote the set D by P , then the properties (P1)–(P3) will be obviously present. Therefore the proof will be completed if we show that the operation $+$ in G is associative. This reduces to showing that for all p, q, r in D the following seven implications hold:

- (A1) $r \neq 0 \Rightarrow (p + q) + \bar{r} = p + (q + \bar{r})$,
- (A2) $q \neq 0 \Rightarrow (p + \bar{q}) + r = p + (\bar{q} + r)$,
- (A3) $q \neq 0 \wedge r \neq 0 \Rightarrow (p + \bar{q}) + \bar{r} = p + (\bar{q} + \bar{r})$,
- (A4) $p \neq 0 \Rightarrow (\bar{p} + q) + r = \bar{p} + (q + r)$,
- (A5) $p \neq 0 \wedge r \neq 0 \Rightarrow (\bar{p} + q) + \bar{r} = \bar{p} + (q + \bar{r})$,
- (A6) $p \neq 0 \wedge q \neq 0 \Rightarrow (\bar{p} + \bar{q}) + r = \bar{p} + (\bar{q} + r)$,
- (A7) $p \neq 0 \wedge q \neq 0 \wedge r \neq 0 \Rightarrow (\bar{p} + \bar{q}) + \bar{r} = \bar{p} + (\bar{q} + \bar{r})$.

Thus the remaining part of the proof decomposes into the verifications of (A1)–(A7), where p, q, r are arbitrary elements of D .

² To show that the above definition is a legitimate one, we use all axioms (D1)–(D5); in particular, the axiom (D4) is used for showing that it is not possible to have simultaneously two equalities $z + s = x$, $s = t + x$ or two equalities $s = y + t$, $s + z = y$, where $x, y, z \in D$, $s, t \in D \setminus \{0\}$, and the axiom (D5) is used for showing that the extension is defined for any pair of elements of G .

Verification of (A1). Let $r \neq 0$. By axiom (D5), there is some element z of D such that $q = z + r$ or $r = z + q$. We choose such a z and we could assume that $z \neq 0$ in the second case, since if $z = 0$, then the second case is covered by the first one. If $q = z + r$, then

$$p + (q + \bar{r}) = p + ((z + r) + \bar{r}) = p + z, \quad (p + q) + \bar{r} = ((p + z) + r) + \bar{r} = p + z.$$

Consider now the case when $r = z + q$ and $z \neq 0$. Then

$$p + (q + \bar{r}) = p + (q + \overline{z + q}) = p + \bar{z}$$

and it is natural to apply the axiom (D5) again for choosing an element z' of G such that either $p = z' + z$ or $z = z' + p$, z' being distinct from 0 in the second case. If $p = z' + z$, then

$$(p + q) + \bar{r} = (z' + z + q) + \bar{r} = (z' + r) + \bar{r} = z', \quad p + (q + \bar{r}) = (z' + z) + \bar{z} = z'.$$

Otherwise, i.e. when $z = z' + p$ and $z' \neq 0$, we have

$$(p + q) + \bar{r} = (p + q) + \overline{z' + (p + q)} = \bar{z}', \quad p + (q + \bar{r}) = p + \overline{z' + p} = \bar{z}'.$$

Verification of (A2). Let $q \neq 0$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. Choosing such a z , we shall have

$$(p + \bar{q}) + r = ((z + q) + \bar{q}) + r = z + r$$

in the first case and

$$(p + \bar{q}) + r = (p + \overline{z + p}) + r = \bar{z} + r$$

in the second one. By the same axiom, there is an element z' of D such that either $r = q + z'$ or $q = r + z'$, z' being distinct from 0 in the second case. Choosing such a z' , we shall have

$$p + (\bar{q} + r) = p + (\bar{q} + (q + z')) = p + z'$$

in the first case and

$$p + (\bar{q} + r) = p + (\overline{r + z'} + r) = p + \bar{z}'$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $p = z + q$, $r = q + z'$. Then

$$(p + \bar{q}) + r = z + q + z', \quad p + (\bar{q} + r) = z + q + z'.$$

Combination 1.2: $p = z + q$, $q = r + z'$, $z' \neq 0$. Then

$$p + (\bar{q} + r) = ((z + r) + z') + \bar{z}' = z + r = (p + \bar{q}) + r.$$

Combination 2.1: $q = z + p$, $z \neq 0$, $r = q + z'$. Then

$$(p + \bar{q}) + r = \bar{z} + (z + (p + z')) = p + z' = p + (\bar{q} + r).$$

Combination 2.2: $q = z + p$, $z \neq 0$, $q = r + z'$, $z' \neq 0$. Then $z + p = r + z'$. By axiom (D5), there is an element z'' of D such that either $r = z + z''$ or $z = r + z''$, z'' being distinct from 0 in the second case. In the first case we get

$$p = z'' + z', \quad (p + \bar{q}) + r = \bar{z} + (z + z'') = z'', \quad p + (\bar{q} + r) = (z'' + z') + \bar{z}' = z''.$$

In the second one we have

$$z'' + p = z', \quad (p + \bar{q}) + r = \overline{r + z''} + r = \bar{z}'', \quad p + (\bar{q} + r) = p + \overline{z'' + p} = \bar{z}'.$$

Verification of (A3). Let $q \neq 0$, $r \neq 0$. Then $p + (\bar{q} + \bar{r}) = p + \overline{r + q}$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. In the first case we get

$$(p + \bar{q}) + \bar{r} = ((z + q) + \bar{q}) + \bar{r} = z + \bar{r}, \quad p + (\bar{q} + \bar{r}) = z + (q + \overline{r + q}) = z + \bar{r}.$$

In the second one we have

$$(p + \bar{q}) + \bar{r} = (p + \overline{z + p}) + \bar{r} = \bar{z} + \bar{r} = \overline{r + z}, \quad p + (\bar{q} + \bar{r}) = p + \overline{(r + z) + p} = \overline{r + z}.$$

Verification of (A4). Similar to the verification of (A1).

Verification of (A5). Let $p \neq 0$, $r \neq 0$. By Axiom (D5), there is an element z of D such that either $q = p + z$ or $p = q + z$, z being distinct from 0 in the second case. Choosing such a z , we shall have

$$(\bar{p} + q) + \bar{r} = (\bar{p} + (p + z)) + \bar{r} = z + \bar{r}$$

in the first case and

$$(\bar{p} + q) + \bar{r} = (\overline{q + z} + q) + \bar{r} = \bar{z} + \bar{r} = \overline{r + z}$$

in the second one. By the same axiom, there is an element z' of D such that either $q = z' + r$ or $r = z' + q$, z' being distinct from 0 in the second case. Choosing such a z' , we shall have

$$\bar{p} + (q + \bar{r}) = \bar{p} + ((z' + r) + \bar{r}) = \bar{p} + z'$$

in the first case and

$$\bar{p} + (q + \bar{r}) = \bar{p} + (q + \overline{z' + q}) = \bar{p} + \bar{z}' = \overline{z' + p}$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $q = p + z$, $q = z' + r$. Then $p + z = z' + r$. By Axiom (D5), there is an element z'' of D such that either $z = z'' + r$ or $r = z'' + z$, z'' being distinct from 0 in the second case. In the first case we get

$$p + z'' = z', \quad (\bar{p} + q) + \bar{r} = (z'' + r) + \bar{r} = z'', \quad \bar{p} + (q + \bar{r}) = \bar{p} + (p + z'') = z''.$$

In the second one we have

$$p = z' + z'', \quad (\bar{p} + q) + \bar{r} = z + \overline{z'' + z} = \bar{z}'', \quad \bar{p} + (q + \bar{r}) = \overline{z' + z''} + z' = \bar{z}'.$$

Combination 1.2: $q = p + z$, $r = z' + q$, $z' \neq 0$. Then

$$(\bar{p} + q) + \bar{r} = z + \overline{(z' + p)} + z = \overline{z' + p} = \bar{p} + (q + \bar{r}).$$

Combination 2.1: $p = q + z$, $z \neq 0$, $q = z' + r$. Then

$$\bar{p} + (q + \bar{r}) = \overline{z' + (r + z)} + z' = \overline{r + z} = (\bar{p} + q) + \bar{r}.$$

Combination 2.2: $p = q + z$, $z \neq 0$, $r = z' + q$, $z' \neq 0$. Then

$$(\bar{p} + q) + \bar{r} = \overline{z' + q + z}, \quad \bar{p} + (q + \bar{r}) = \overline{z' + q + z}.$$

Verification of (A6). Similar to the verification of (A3).

Verification of (A7). Let $p \neq 0$, $q \neq 0$, $r \neq 0$. Then

$$(\bar{p} + \bar{q}) + \bar{r} = \overline{q + p} + \bar{r} = \overline{r + q + p}, \quad \bar{p} + (\bar{q} + \bar{r}) = \bar{p} + \overline{r + q} = \overline{r + q + p}. \quad \square$$

APPENDIX

The proof of Theorem 2 makes use of the existence of some set that has the same cardinality as $D \setminus \{0\}$ and does not meet D . The existence of such a set can be obtained as a particular case of the statement that for any sets A and B there is a set having the same cardinality as A and not meeting B . This statement follows immediately from certain facts of the cardinal arithmetic, but some of them in the final analysis are based on the Axiom of Choice. Here is a direct proof of the statement without using that axiom. Let

$$C = (A \times \mathcal{P}(B)) \cap B,$$

where $\mathcal{P}(B)$ is the set of the subsets of B . Let f be the projection mapping of C into $\mathcal{P}(B)$ defined by the equality

$$f(x, Y) = Y.$$

Since C is a subset of B , the range of f is a proper subset of $\mathcal{P}(B)$ (as the well-known diagonal argument shows, the set $\{z \in C \mid z \notin f(z)\}$ is an element of $\mathcal{P}(B)$ not belonging to the range of f). If Y_0 is an element of $\mathcal{P}(B) \setminus \text{range}(f)$, then the set $A \times \{Y_0\}$ does not meet B , and clearly $A \times \{Y_0\}$ has the same cardinality as A .

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