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SOME APPLICATIONS OF THE FUNCTIONS OF THE SYSTEM $\Gamma_{\mathcal{B}_s}$ TO THE THEORY OF THE UNIFORMLY DISTRIBUTED SEQUENCES

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The authors use the $\mathcal{B}_s$-adic functions constructed in Cantor systems to show some of their applications to the theory of the uniformly distributed sequences. The notions of multidimensional modified integrals from these functions are introduced. The following results – the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion, are presented in the terms of the introduced integrals.

Keywords: function system $\Gamma_{\mathcal{B}_s}$, multidimensional modified integrals, LeVeque's inequality, Koksma's formula, Erdős-Turán-Koksma's inequality, integral Weyl's criterion, uniformly distributed sequences

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1. Introduction

Let $s \geq 1$ be a fixed integer, which will denote the dimension through the paper. Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence in the unit cube $[0,1)^s$. For an arbitrary integer $N \geq 1$ and a subinterval $J \subseteq [0,1)^s$ let us denote $A(J;N) = \#\{n\colon 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$. Let λ_s denote the Lebesgue measure on $[0,1)^s$. Following Kuipers and Niederreiter [15] we will remind that the sequence ξ is called uniformly distributed in $[0,1)^s$ if the limit equality $\lim\limits_{N\to\infty} \dfrac{A(J;N)}{N} = \lambda_s(J)$ holds for each subinterval J of $[0,1)^s$.

To assess the quality of the distribution of the points of sequences and nets we use special quantitative measures, called discrepancy and diaphony. Thus, to present

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the concepts of the extreme and the quadratic discrepancy, we will introduce some notations. For this purpose, let $\mathcal{J}$ denote a family of subintervals of $[0,1)^s$ of the form $J = \prod_{j=1}^{s} [u_j, v_j)$, where $0 \leq u_j < v_j \leq 1$ for $1 \leq j \leq s$. For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let us denote $[0, \mathbf{x}) = [0, x_1) \times \cdots \times [0, x_s)$.

Let $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. The extreme discrepancy $D(\xi_N)$ and the quadratic discrepancy $T(\xi_N)$ of the net ξ_N are defined respectively as

$$D(\xi_N) = \sup_{J \in \mathcal{J}} \left| \frac{A(J;N)}{N} - \lambda_s(J) \right|$$

and

$$T(\xi_N) = \left(\int_{[0,1]^s} \left| \frac{A([0,\mathbf{x});N)}{N} - x_1 \ldots x_s \right|^2 dx_1 \ldots dx_s \right)^{\frac{1}{2}}.$$

For each integer $N \geq 1$ the extreme and the quadratic discrepancy of the sequence ξ are defined as the corresponding discrepancies of the first N its elements.

Some classes of orthonormal function systems with very big success are used to solve many problems of the theory of the uniformly distributed sequences. Such classes are the trigonometric function system and systems constructed in b-adic number system as the Walsh function system and the Haar function system.

Also, function systems constructed in the so-called *Cantor systems* are used as a tool for studying the uniformly distributed sequences. Let us remind the concept of the Cantor systems, which are natural generalizations of the ordinary b-adic number system. The algebraic basis of these systems is given by the following explanations: Let $B = \{b_0, b_1, \ldots : b_i \geq 2 \text{ for } i \geq 0\}$ be an arbitrary sequence of integers, called bases. By using the sequence B, the so-called *generalized powers* are defined by the following recurrence manner: we put $B_0 = 1$ and for $i \geq 0$ define $B_{i+1} = B_i b_i$. The number system, which corresponds to the sequence B of bases and the sequence $\{B_0, B_1, \ldots\}$ of generalized powers usually is called B-adic Cantor system or system with variable bases. We will stick to the terminology B-adic number system.

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An arbitrary integer $k \in \mathbb{N}_0$ and a real $x \in [0,1)$ in the B-adic system have representations of the form

$$k = \sum_{i=0}^{\nu} k_i B_i \quad \text{and} \quad x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}},$$

where $k_i, x_i \in \{0, 1, \ldots, b_i - 1\}$ for $i \geq 0$ and $k_\nu \neq 0$. The representation of k is unique. Under the additional assumption that for infinitely many values i we have $x_i \neq b_i - 1$, than the B-adic representation of x is also unique.

The multidimensional version of the Cantor systems is given by the following manner: Let $\mathcal{B}_s = (B_1, \ldots, B_s)$ be an arbitrary set of sequences of bases, where $B_j = \left\{ b_0^{(j)}, b_1^{(j)}, \ldots \right\}$ for $1 \leq j \leq s$ and $\left\{ B_0^{(j)}, B_1^{(j)}, \ldots \right\}$ be the corresponding to B_j

sequence of the generalized powers. The number system, which correspond to the set $\mathcal{B}_s$, we will call $\mathcal{B}_s$-adic Cantor system.

Everywhere in the article we strictly will stick to the above concepts for B-adic and $\mathcal{B}_s$-adic Cantor systems.

A set of functions constructed in Cantor systems was first of all proposed by Vilenkin [21] and independent of him was considered by Price [20]. The details are as follows: For an arbitrary $k \in \mathbb{N}_0$ and a real $x \in [0,1)$ with the B-adic representations $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where $k_i, x_i \in \{0,1,\ldots,b_i-1\}$ for $i \geq 0$, $k_\nu \neq 0$ and for infinitely many values of i we have $x_i \neq b_i - 1$, the function ${}_B\mathrm{vil}_k \colon [0,1) \to \mathbb{C}$ is defined as

$$ {}_B\mathrm{vil}_k(x) = \prod_{i=0}^{\nu} e^{2\pi\mathbf{i}\frac{k_i x_i}{b_i}}. $$

For an arbitrary vector $\mathbf{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th multidimensional Vilenkin function is defined as

$$ {}_{\mathcal{B}_s}\mathrm{vil}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\mathrm{vil}_{k_j}(x_j), \quad \mathbf{x} = (x_1,\ldots,x_j) \in [0,1)^s. $$

The set $\mathrm{Vil}_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\mathrm{vil}_{\mathbf{k}}(\mathbf{x}) \colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called Vilenkin function system. For the system Vil_B usually the name multiplicative system is used, see Agaev et al. [1].

Quite recently, Hellekalek and Niederraiter [13] introduced the so-called **b**-adic function system $\Gamma_{\mathbf{b}}$, where $\mathbf{b} = (b_1,\ldots,b_s)$ is a vector of not necessary distinct integers $b_j \geq 2$. Some applications of the system $\Gamma_{\mathbf{b}}$ to the theory of the uniformly distributed sequences were presented. Petrova [19] generalized the construction of the functions of the system $\Gamma_{\mathbf{b}}$ to functions considered in $\mathcal{B}_s$-adic Cantor systems. We will recall this constructive principle.

Definition 1.1. For an arbitrary integer $k \geq 0$ and a real number $x \in [0,1)$, which in B-adic system have the representations of the form

$$ k = \sum_{i=1}^{\nu} k_i B_i \quad \text{and} \quad x = \sum_{i=1}^{\infty} \frac{x_i}{B_{i+1}}, $$

where $k_i, x_i \in \{0,1,\ldots,b_i-1\}$ for $i \geq 0$, $k_\nu \neq 0$ and for infinitely many values of i we have that $x_i \neq b_i - 1$, the k-th B-adic function ${}_B\gamma_k \colon [0,1) \to \mathbb{C}$ is defined as

$$ {}_B\gamma_k(x) = e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\frac{k_1}{B_2}+\cdots+\frac{k_\nu}{B_{\nu+1}}\right)(x_0B_0+x_1B_1+\cdots+x_\nu B_\nu)}. $$

The multidimensional functions constructed in $\mathcal{B}_s$-adic Cantor systems are defined in the following definition.

Definition 1.2. For an arbitrary vector $\mathbf{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th multidimensional $\mathcal{B}_s$-adic function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} \colon [0,1)^s \to \mathbb{C}$ is defined as

$$ {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j), \quad \mathbf{x} = (x_1,\ldots,x_s) \in [0,1)^s. $$

The set $\Gamma_{\mathcal{B}_s} = \{_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called *$\mathcal{B}_s$-adic function system* and is proved that it is a complete orthonormal basis of the space $L_2([0,1)^s)$.

The inequality of LeVeque [17] is an upper bound of the extreme discrepancy of an arbitrary one-dimensional net in the terms of the functions of the trigonometric function system. This is one-dimensional result and to the authors is unknown multidimensional version of this inequality.

The Erdős-Turán-Koksma's inequality, which has a long history, gives upper bounds of the extreme discrepancy in the terms of the functions of some classes of orthonormal function systems. First, Erdős and Turán [3] and Koksma [14] presented this inequality in the terms of the functions of the trigonometric function system. Niederreiter [18] in its monograph "Random Number Generator and Quasi-Monte Carlo Methods" systematizes and generalizes the results related to the Erdős-Turán-Koksma's inequality, which are based on the trigonometric function system. The form of the Erdős-Turán-Koksma's inequality for the extreme discrepancy, which is based on the Walsh and the Haar functions in base b have been given by Hellekalek [10–12]. Grozdanov and Stoilova [8] presented the form of this inequality in the terms of the functions of an arbitrary complete orthonormal function system constructed in Cantor systems.

Grozdanov [6] presented the form of the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma' inequality and the integral Weyl criterion in the terms of the modified integrals from the Vilenkin functions. Also, the form of the LeVeque's inequality, the Koksma's formula and the integral Weyl criterion in the terms of the modified integrals from the Haar functions constructed in Cantor systems, were presented.

There are examples of sequences constructed in b-adic number system, see Faure [4], which are generalizations of the classical multidimensional Halton's sequence. The next step of a generalization of the Halton's sequence was developed by Bednařík, Lertchoosakul, Markes and Trojovský [2]. They consider sequences constructed in Cantor systems, which are natural generalization of the Halton's and Faure's sequences.

Bednařik et al. [2] studied the extreme discrepancy of this sequence. Also, Lertchoosakul and Nair [16] realized investigations related to the generalized sequence of Halton. Grozdanov and Sevdinova [7] studied the $(\Gamma_b; \alpha; \gamma)$-diaphony of the Van der Corput sequence constructed in Cantor systems.

In such a way, we have objects constructed in Cantor systems. To study of these sequences and nets, we need appropriate analytical tools, usually some complete orthonormal function systems constructed in the same Cantor systems. It is clear that two such systems are these of the Vilenkin and the Haar functions.

The purpose of our investigation is to show some applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of the uniformly distributed sequences. Some classical results of the quantitative theory of the uniformly distributed sequences, as the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion in the terms of the modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$, will be shown.

The rest of the article is organized as following. In Section 2 the notions of multidimensional modified integrals from the functions of the system $\Gamma_{\mathcal{B}_s}$ are introduced. In Section 3 some useful preliminary statements are given. In Section 4 the main results of the article are presented and proved.

2. Multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$

We will introduce the following useful symbol. For arbitrary and fixed integers $b \geq 2$ and $\beta \in \{1, 2, \ldots, b-1\}$ let us denote $\Delta(b;\beta) = \sum_{h=0}^{b-1} he^{2\pi \mathbf{i} \frac{\beta}{b} h}$. Then, the following equality holds

$$\Delta(b;\beta) = -\frac{b}{2}\left(1 + \mathbf{i} \coth \frac{\pi}{b}\beta\right).$$

Following Fine [5], for an arbitrary integer $k \geq 0$ and a real $x \in [0, 1)$ we will consider the integral from the function ${}_B\gamma_k(x)$, thus let us denote

$${}_BJ_k(x) = \int_0^x {}_B\gamma_k(t)\, dt.$$

We will introduce the concept of the multidimensional modified integrals from the functions of the $\mathcal{B}_s$-adic system. For this purpose, let us denote $S = \{1, 2, \ldots, s\}$. For an arbitrary integer $0 \leq u \leq s$ let

$$A_u = \{\alpha_1, \ldots, \alpha_u \colon 1 \leq \alpha_1 < \cdots < \alpha_u \leq s\}$$

be an arbitrary subset of S. Obviously we have $C_s^u = \dfrac{s!}{u!(s-u)!}$ choices of the subsets A_u. Let us define $C_{s-u} = S \setminus A_u$ and to denote

$$C_{s-u} = \{\beta_1, \ldots, \beta_{s-u} \colon 1 \leq \beta_1 < \cdots < \beta_{s-u} \leq s\}.$$

In the case when $u = 0$ we will think that $A_0 = \varnothing$ and $C_s = S$. When $u = s$ we have that $A_s = S$ and $C_0 = \varnothing$.

Let us assume that $u = 0$. To the value $u = 0$ we can think that corresponds the s-dimensional vector $\mathbf{k} = \mathbf{0}$. Let us introduce the notion of *modified integral* from the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{0}}(\mathbf{x})$ as

$${}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}) = \prod_{j=1}^{s}(1 - x_j) - \frac{1}{2^s}, \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s.$$

Now, let us assume that $1 \leq u \leq s$ is a fixed index. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ let us introduce the following assumption:

(*C*) *Exactly u of number coordinates of* $\mathbf{k}$ *are nonzero and let they be* $k_{\alpha_1}, \ldots, k_{\alpha_u}$.

In addition, for $1 \leq j \leq u$ the coordinate $k_{\alpha_j} = k_{g_j}^{(\alpha_j)} B_{g_j}^{(\alpha_j)} + k'_{\alpha_j}$, where $g_j \geq 0$, $0 \leq k'_{\alpha_j} \leq B_{g_j}^{(\alpha_j)} - 1$ and $k_{g_j}^{(\alpha_j)} \in \{1, \ldots, b_{g_j}^{(\alpha_j)} - 1\}$.

Now, we are able to introduce the notion of *modified integral of rang* u from the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})$ as

$$
{}_{\mathcal{B}_s}J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}) = (-1)^u \prod_{j=1}^{u} {}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j}) \cdot \prod_{j=1}^{s-u}(1 - x_{\beta_j})
$$

$$
- \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \ldots b_{g_j}^{(\alpha_j)}} \right)} - 1} \cdot \Delta \left(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)} \right) \right] \frac{1}{2^{s-u}},
$$

$\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$. In the case when $u = s$, we assume that $\prod_{j=1}^{s-u}(1 - x_{\beta_j}) = 1$.

3. Preliminary statements

To prove the main results of our investigation we need of some useful preliminary statements. For this purpose, in Lemmas 3.1, 3.2, 3.3 and 3.4 we will use an arbitrary one-dimensional B-adic Cantor system, thus let $B = \{b_0, b_1, \ldots : b_i \geq 2 \text{ for } i \geq 0\}$ be an arbitrary sequence of bases and $\{B_0, B_1, \ldots\}$ be the corresponding to B sequence of generalized powers. Let us start with the following lemma, where we will present some properties of the functions of the system Γ_B.

Lemma 3.1. *Let $k \geq 1$ be an arbitrary integer with the B-adic representation*

$$
k = k_g B_g + k_{g-1} B_{g-1} + \cdots + k_1 B_1 + k_0,
$$

where $g \geq 0$ for $0 \leq i \leq g$, $k_i \in \{0, 1, \ldots, b_i - 1\}$ and $k_g \neq 0$. Then the following holds:

(i) *For each integers h and μ, and a real x such that $0 \leq h \leq B_g - 1$, $0 \leq \mu \leq b_g - 1$ and $x \in \left[\frac{h}{B_g} + \frac{\mu}{B_{g+1}}, \frac{h}{B_g} + \frac{\mu+1}{B_{g+1}} \right)$ the equality holds*

$$
{}_B\gamma_k(x) = {}_B\gamma_k\left(\frac{h}{B_g}\right) \cdot e^{2\pi \mathbf{i} \frac{k_g}{b_g} \mu};
$$

(ii) *For each integer p such that $0 \leq p \leq B_g - 1$ the equality holds*

$$
\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\, dx = 0;
$$

(iii) *Additionally let us denote $k' = k_{g-1}B_{g-1} + \cdots + k_1 B_1 + k_0$. Then, the equality holds*

$$
{}_B\gamma_k(x) = {}_B\gamma_{k_g B_g}(x) {}_B\gamma_{k'}(x) \text{ for all } x \in [0,1).
$$

Proof. (i) Let us use the B-adic representation $\dfrac{h}{B_g} = 0.h_0h_1\dots h_{g-1}$, where for $0 \le i \le g-1$, $h_i \in \{0,1,\dots,b_i-1\}$. Then, an arbitrary $x \in \left[\dfrac{h}{B_g} + \dfrac{\mu}{B_{g+1}}, \dfrac{h}{B_g} + \dfrac{\mu+1}{B_{g+1}}\right)$ has the B-adic representation of the form $x = 0.h_0h_1\dots h_{g-1}\mu x_{g+1}\dots$. According to Definition 1.1 we have that

$$\begin{aligned}{}_B\gamma_k(x) &= e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\dots+\frac{k_g}{B_{g+1}}\right)(h_0+h_1B_1+\dots+h_{g-1}B_{g-1}+\mu B_g)}\\ &= e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\dots+\frac{k_g}{B_{g+1}}\right)(h_0+h_1B_1+\dots+h_{g-1}B_{g-1})}\cdot e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} = {}_B\gamma_k\left(\frac{h}{B_g}\right)\cdot e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu}.\end{aligned}$$

(ii) If $p = 0$, then the statement obviously is true. Now, let us assume that $p \ge 1$. In this case, we will use the presentation

$$\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\,dx = \sum_{h=0}^{p-1}\sum_{\mu=0}^{b_g-1}\int_{\frac{h}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{h}{B_g}+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_k(x)\,dx.$$

We will use the part (i) of the Lemma and the above equality to obtain that

$$\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\,dx = \frac{1}{B_{g+1}}\sum_{h=0}^{p-1} {}_B\gamma_k\left(\frac{h}{B_g}\right)\sum_{\mu=0}^{b_g-1} e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} = 0.$$

(iii) For an arbitrary real $x \in [0,1)$ we will use the B-adic representation $x = 0.x_0x_1\dots$, where for infinitely many values of i we have that $x_i \ne b_i - 1$. Then, according to Definition 1.1, we consecutively obtain that

$$\begin{aligned}{}_B\gamma_k(x) &= e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\dots+\frac{k_{g-1}}{B_g}+\frac{k_g}{B_{g+1}}\right)(x_0+x_1B_1+\dots+x_gB_g)}\\ &= e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\dots+x_gB_g)}\cdot e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\dots+\frac{k_{g-1}}{B_g}\right)(x_0+x_1B_1+\dots+x_{g-1}B_{g-1})}\\ &= {}_B\gamma_{k_gB_g}(x)\,{}_B\gamma_{k'}(x).\end{aligned}$$

The lemma is finally proved. □

Lemma 3.2. *For arbitrary integers $g \ge 0$ and $\kappa_g \in \{1,\dots,b_g-1\}$, and a real $x \in [0,1)$ with the B-adic representation $x = 0.x_0x_1x_2\dots$, where for infinitely many values of i we have that $x_i \ne b_i - 1$, let us define the function*

$$\delta_{\kappa_g}(x) = \begin{cases}\displaystyle\sum_{\mu=0}^{x_g-1} e^{2\pi\mathbf{i}\frac{\kappa_g}{b_g}\mu}, & \text{if } x_g \ne 0,\\ 0, & \text{if } x_g = 0.\end{cases}$$

Let $k \ge 1$ be an arbitrary integer of the form $k = k_gB_g + k'$, where $g \ge 0$, $0 \le k' \le B_g - 1$ and $k_g \in \{1,\dots,b_g-1\}$. Then, the modified integral ${}_BJ_k(x)$ from *the function* ${}_B\gamma_k(x)$ *satisfies the equality*

$$ {}_BJ_k(x) = \frac{1}{B_{g+1}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0 + x_1 B_1 + \cdots + x_{g-1}B_{g-1})} \delta_{k_g}(x) {}_B\gamma_{k'}(x) + {}_B\gamma_k(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}} $$

for all $x \in [0,1)$.

Proof. We will use the statements of Lemma 3.1 to obtain that for an arbitrary and fixed real $x \in [0,1)$ the presentations hold

$$ \begin{aligned} {}_BJ_k(x) &= \int_0^x {}_B\gamma_k(t)\,dt = \int_0^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}} {}_B\gamma_k(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_k(t)\,dt \\ &= \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t) {}_B\gamma_{k'}(t)\,dt = {}_B\gamma_{k'}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt \\ &= {}_B\gamma_{k'}(x) \left[\int_0^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}} {}_B\gamma_{k_gB_g}(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt \right] \\ &= {}_B\gamma_{k'}(x) \int_0^x {}_B\gamma_{k_gB_g}(t)\,dt = {}_B\gamma_{k'}(x) {}_BJ_{k_gB_g}(x). \end{aligned} $$

Hence, the equality holds

$$ {}_BJ_k(x) = {}_B\gamma_{k'}(x) {}_BJ_{k_gB_g}(x) \text{ for all } x \in [0,1). \tag{3.1} $$

Now, we will calculate the integral ${}_BJ_{k_gB_g}(x)$. Lemma 3.1 (ii) gives us that

$$ {}_BJ_{k_gB_g}(x) = \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt. $$

First of all, let us assume that $x_g \neq 0$. Then, from the above equality we obtain that

$$ \begin{aligned} {}_BJ_{k_gB_g}(x) &= \sum_{\mu=0}^{x_g-1} \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_{k_gB_g}(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^x {}_B\gamma_{k_gB_g}(t)\,dt \\ &= e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \sum_{\mu=0}^{x_g-1} e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu+1}{B_{g+1}}} dt \\ &\qquad + {}_B\gamma_{k_gB_g}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^x dt \\ &= \frac{1}{B_{g+1}} e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \sum_{\mu=0}^{x_g-1} e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} + {}_B\gamma_{k_gB_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \end{aligned} \tag{3.2} $$

Now, let us assume that $x_g = 0$. In this case, the equalities hold

$$ {}_BJ_{k_gB_g}(x) = {}_B\gamma_{k_gB_g}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^{x} dt = {}_B\gamma_{k_gB_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \tag{3.3} $$

By using the introduced function $\delta_{\kappa_g}(x)$, equalities (3.2) and (3.3) can be rewritten as

$$ {}_BJ_{k_gB_g}(x) = \frac{1}{B_{g+1}} e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})}\delta_{k_g}(x) + {}_B\gamma_{k_gB_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \tag{3.4} $$

Equalities (3.1) and (3.4) give us that the equality

$$ {}_BJ_k(x) = \frac{1}{B_{g+1}} e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})}\delta_{k_g}(x)\,{}_B\gamma_{k'}(x) + {}_B\gamma_k(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}} $$

holds for all $x \in [0, 1)$. The Lemma is finally proved. □

Lemma 3.3. *For each integer $k \geq 1$ with the B-adic representation $k = \sum_{i=0}^{g} k_iB_i$, where $g \geq 0$ for $0 \leq i \leq g$, $k_i \in \{0, \ldots, b_i - 1\}$ and $k_g \neq 0$, the equality holds*

$$ \int_0^1 {}_BJ_k(x)\,dx = -\frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_g}{b_g}} - 1}{e^{2\pi\mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\cdots b_g}\right)} - 1} \cdot \Delta(b_g; k_g) $$

with the introduced in Section 2 symbol $\Delta(b_g; k_g)$.

Proof. By using the result of Lemma 3.2 we obtain that

$$ \int_0^1 {}_BJ_k(x)\,dx = \frac{1}{B_{g+1}} \int_0^1 e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})}\delta_{k_g}(x)\,{}_B\gamma_{k'}(x)\,dx + \sum_{h=g+1}^{\infty} \frac{1}{B_{h+1}} \int_0^1 x_h\,{}_B\gamma_k(x)\,dx. \tag{3.5} $$

We will calculate the integrals in the right side of equality (3.5). For this purpose, let us denote $k' = \sum_{i=0}^{g-1} k_iB_i$. Hence, we consecutively have that

$$ \int_0^1 e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})}\delta_{k_g}(x)\,{}_B\gamma_{k'}(x)\,dx $$

$$= \sum_{t_0=0}^{b_0-1} \cdots \sum_{t_g=0}^{b_g-1} \int_{\frac{t_0}{B_1}+\cdots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\cdots+\frac{t_g+1}{B_{g+1}}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \delta_{k_g}(x)_B\gamma_{k'}(x)\,dx$$

$$= \frac{1}{B_{g+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \cdots$$

$$\sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1}b_g}\right)t_{g-1}} \sum_{t_g=0}^{b_g-1} \delta_{k_g}(t_g). \quad (3.6)$$

It is easy to calculate that

$$\sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \cdots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1}b_g}\right)t_{g-1}}$$

$$= \frac{e^{2\pi \mathbf{i}\frac{k_g}{b_g}}-1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)}-1}. \quad (3.7)$$

By using the definition of the function $\delta_{k_g}(x)$, we consecutively have that

$$\sum_{t_g=0}^{b_g-1} \delta_{k_g}(t_g) = \sum_{t_g=1}^{b_g-1}\sum_{\mu=0}^{t_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}\mu} = \sum_{h=0}^{b_g-1}(b_g-1-h)e^{2\pi \mathbf{i}\frac{k_g}{b_g}h}$$

$$= (b_g-1)\sum_{h=0}^{b_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}h} - \sum_{h=0}^{b_g-1} he^{2\pi \mathbf{i}\frac{k_g}{b_g}h} = -\sum_{h=0}^{b_g-1} he^{2\pi \mathbf{i}\frac{k_g}{b_g}h} = -\Delta(b_g;k_g). \quad (3.8)$$

From equalities (3.6), (3.7) and (3.8) we obtain that

$$\int_0^1 e^{2\pi \mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \delta_{k_g}(x)_B\gamma_{k'}(x)\,dx$$

$$= -\frac{1}{B_{g+1}} \cdot \frac{e^{2\pi \mathbf{i}\frac{k_g}{b_g}}-1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)}-1} \cdot \Delta(b_g;k_g). \quad (3.9)$$

Now, we will calculate the second integral in the right hand side of equality (3.5). Thus, let $h \geq g+1$ be a fixed index. We will use the presentations

$$\int_0^1 x_{h\,B}\gamma_k(x)\,dx = \sum_{t_0=0}^{b_0-1} \cdots \sum_{t_g=0}^{b_g-1} \cdots \sum_{t_h=0}^{b_h-1} \int_{\frac{t_0}{B_1}+\cdots+\frac{t_h}{B_{h+1}}}^{\frac{t_0}{B_1}+\cdots+\frac{t_h+1}{B_{h+1}}} x_{h\,B}\gamma_k(x)\,dx$$

$$= \frac{1}{B_{h+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \cdots$$

$$\sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}t_g} \cdots \sum_{t_h=0}^{b_h-1} t_h = 0. \quad (3.10)$$

The equalities (3.5), (3.9) and (3.10) finally prove the statement of the lemma. □

Lemma 3.4. *For each integer $k \geq 1$ with the B-adic representation $k = \sum_{i=0}^{g} k_i B_i$, where $g \geq 0$ for $0 \leq i \leq g$, $k_i \in \{0, 1, \dots, b_i - 1\}$ and $k_g \neq 0$, the equality holds*

$$\int_0^1 x_B \gamma_k(x)\, dx = \frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_g}{b_g}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0} + \dots + \frac{k_g}{b_0 \dots b_g}\right)} - 1} \cdot \Delta(b_g; k_g)$$

with the introduced in Section 2 symbol $\Delta(b_g; k_g)$.

Proof. We will use the following presentations:

$$\int_0^1 x_B \gamma_k(x)\, dx = \sum_{t_0=0}^{b_0-1} \sum_{t_1=0}^{b_1-1} \cdots \sum_{t_{g-1}=0}^{b_{g-1}-1} \sum_{t_g=0}^{b_g-1} \int_{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\dots+\frac{t_g+1}{B_{g+1}}} x_B \gamma_k(x)\, dx$$

$$= \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\dots+\frac{k_g}{b_1 \dots b_g}\right) t_1} \dots$$

$$\sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \int_{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}+\frac{1}{B_{g+1}}} x\, dx$$

$$= \sum_{t_0=1}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\dots+\frac{k_g}{b_1 \dots b_g}\right) t_1} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}}$$

$$\times \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \left\{ \frac{1}{B_{g+1}} \left(\frac{t_0}{B_1} + \frac{t_1}{B_2} + \dots + \frac{t_g}{B_{g+1}} \right) + \frac{1}{2B_{g+1}^2} \right\}$$

$$= \frac{1}{B_{g+1}} \left[\frac{1}{B_1} \sum_{t_0=0}^{b_0-1} t_0 \cdot e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \right.$$

$$+ \dots + \frac{1}{B_g} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} t_{g-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$\left. + \frac{1}{B_{g+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} t_g e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \right]$$

$$+ \frac{1}{2B_{g+1}^2} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$= \frac{1}{B_{g+1}^2} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} t_g e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$= \frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_g}{b_g}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0} + \cdots + \frac{k_g}{b_0 \ldots b_g}\right)} - 1} \cdot \Delta(b_g; k_g).$$

The lemma is finally proved. □

In the next lemma we will use an arbitrary $\mathcal{B}_s$-adic Cantor system. Thus, let $\mathcal{B}_s = (B_1, \ldots, B_s)$, where for $1 \le j \le s$ $B_j = \{b_0^{(j)}, b_1^{(j)}, \ldots : b_i^{(j)} \ge 2 \text{ for } i \ge 0\}$ be a sequence of bases and $\{B_0^{(j)}, B_1^{(j)}, \ldots\}$ be the corresponding to B_j sequence of generalized powers.

Lemma 3.5. *Let $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by $N \ge 1$ points in $[0,1)^s$ and for $0 \le n \le N-1$ let us denote $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$. For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let $g(\xi_N; \mathbf{x}) = \dfrac{1}{N} A([0,\mathbf{x}); \xi_N) - x_1 \ldots x_s$ be the so-called local discrepancy of the net ξ_N.*

Then, for an arbitrary vector $\mathbf{k} \in \mathbb{N}_0^s$ the Fourier's coefficient

$$\widehat{g}(\xi_N; \mathbf{k}) = \int_{[0,1]^s} g(\xi_N; \mathbf{x})\, {}_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}$$

of the function $g(\xi_N; \mathbf{x})$ with respect to the functions of the system $\Gamma_{\mathcal{B}_s}$ satisfies the following presentations:

(i) *If $\mathbf{k} = \mathbf{0}$, then the equality $\widehat{g}(\xi_N; \mathbf{0}) = \dfrac{1}{N} \sum\limits_{n=0}^{N-1} \left[\prod\limits_{j=1}^{s} (1 - {x_n}^{(j)}) - \dfrac{1}{2^s} \right]$ holds;*

(ii) *Let $\mathbf{k} \ne \mathbf{0}$ and for some u, such that $1 \le u \le s$, the vector $\mathbf{k}$ satisfies the presented in Section 2 assumption (C). Then, the equality holds*

$$\overline{\widehat{g}}(\xi_N; \mathbf{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ (-1)^u \prod_{j=1}^{u} B_{\alpha_j} J_{k_{\alpha_j}}(x_n^{(\alpha_j)}) \prod_{j=1}^{s-u} (1 - x_n^{(\beta_j)}) \right.$$
$$\left. - \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \ldots b_{g_j}^{(\alpha_j)}}\right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \frac{1}{2^{s-u}} \right\}$$

with the additional condition, that in the case when $u = s$, we have

$$\prod_{j=1}^{s-u} (1 - x_n^{(\beta_j)}) = 1.$$

Proof. For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ the local discrepancy $g(\xi_N; \mathbf{x})$ of the net ξ_N satisfies the presentation

$$g(\xi_N; \mathbf{x}) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} 1_{[x_n^{(j)}, 1)}(x_j) - x_1 \ldots x_s \right],$$

where $1_{[x_n^{(j)},1)}(x_j)$, $0 \le n \le N-1$, $1 \le j \le s$, is the characteristic function over the interval $[{x_n}^{(j)}, 1)$.

For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$ the Fourier's coefficient of the function $g(\xi_N; \mathbf{x})$ satisfies the equality

$$\overline{\widehat{g}}(\xi_N; \mathbf{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} \int_0^1 1_{[{x_n}^{(j)},1)}(x_j) {}_{B_j}\gamma_{k_j}(x_j)\, dx_j - \prod_{j=1}^{s} \int_0^1 x_j\, {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \right]. \tag{3.11}$$

(i) Let us assume that $\mathbf{k} = \mathbf{0}$. Then, by using equality (3.11) we obtain that

$$\begin{aligned} \widehat{g}(\xi_N; \mathbf{0}) &= \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} \int_0^1 1_{[x_n^{(j)},1)}(x_j)\, dx_j - \prod_{j=1}^{s} \int_0^1 x_j\, dx_j \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} (1 - {x_n}^{(j)}) - \frac{1}{2^s} \right]. \end{aligned}$$

(ii) Let us assume that $u = s$, thus for each $1 \le j \le s$ we have that $k_j \ne 0$. Lemma 3.1 (ii) gives us the following

$$\begin{aligned} \int_0^1 1_{[x_n^{(j)},1)}(x_j) {}_{B_j}\gamma_{k_j}(x_j)\, dx_j &= \int_{x_n^{(j)}}^1 {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \\ &= \int_0^1 {}_{B_j}\gamma_{k_j}(x_j)\, dx_j - \int_0^{x_n^{(j)}} {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \\ &= -\int_0^{x_n^{(j)}} {}_{B_j}\gamma_{k_j}(x_j)\, dx_j = -{}_{B_j}J_{k_j}(x_n^{(j)}). \end{aligned} \tag{3.12}$$

By using the statement of Lemma 3.4, for $1 \le j \le s$ we have that

$$\int_0^1 x_j\, {}_{B_j}\gamma_{k_j}(x_j)\, dx_j = \frac{1}{\left[B_{g_j+1}^{(j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(j)}}{b_{g_j}^{(j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(j)}}{b_0^{(j)}} + \dots + \frac{k_{g_j}^{(j)}}{b_0^{(j)} \dots b_{g_j}^{(j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(j)}; k_{g_j}^{(j)}). \tag{3.13}$$

From equalities (3.11), (3.12) and (3.13) we obtain the presentation

$$\begin{aligned} \overline{\widehat{g}}(\xi_N; \mathbf{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \Bigg[&(-1)^s \prod_{j=1}^{s} {}_{B_j}J_{k_j}(x_n^{(j)}) \\ &- \prod_{j=1}^{s} \frac{1}{\left[B_{g_j+1}^{(j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(j)}}{b_{g_j}^{(j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(j)}}{b_0^{(j)}} + \dots + \frac{k_{g_j}^{(j)}}{b_0^{(j)} \dots b_{g_j}^{(j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(j)}; k_{g_j}^{(j)}) \Bigg]. \end{aligned}$$

Let us assume that $1 \leq u \leq s-1$. By using equality (3.11) and the statements of Lemma 3.1 (ii) and Lemma 3.4 we obtain that

$$\overline{\overline{g}}(\xi_N;\mathbf{k}) = \frac{1}{N}\sum_{n=0}^{N-1}\left[\prod_{j=1}^{u}\int_0^1 1_{[x_n^{(\alpha_j)},1)}(x_{\alpha_j})\, {}_{B_{\alpha_j}}\gamma_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j}\prod_{j=1}^{s-u}\int_0^1 1_{[x_n^{(\beta_j)},1)}(x_{\beta_j})\,dx_{\beta_j}\right.$$
$$\left. - \prod_{j=1}^{u}\int_0^1 x_{\alpha_j}\, {}_{B_{\alpha_j}}\gamma_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j}\prod_{j=1}^{s-u}\int_0^1 x_{\beta_j}\,dx_{\beta_j}\right]$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\left\{(-1)^u\prod_{j=1}^{u}{}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_n^{(\alpha_j)})\prod_{j=1}^{s-u}(1-x_n^{(\beta_j)})\right.$$
$$\left. - \prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\dots b_{g_j}^{(\alpha_j)}}\right)}-1}\cdot\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right]\frac{1}{2^{s-u}}\right\}.$$

The last finally proves the lemma. □

4. The main results

Now, we are able to present the main results of our article. In Theorem 4.1 we will use an arbitrary B-adic Cantor system. Thus, let $B = \{b_0, b_1, \dots : b_i \geq 2 \text{ for } i \geq 0\}$ be a sequence of bases and $\{B_0, B_1, \dots\}$ be the corresponding to B sequence of generalized powers. The following theorem holds.

Theorem 4.1 (The LeVeque's inequality). *Let $\xi_N = \{x_0, \dots, x_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)$. Then the extreme discrepancy $D(\xi_N)$ of the net ξ_N satisfies the inequality*

$$D^3(\xi_N) \leq 12\sum_{g=0}^{\infty}\sum_{k_g=1}^{b_g-1}\sum_{k=k_gB_g}^{(k_g+1)B_g-1}\left|\frac{1}{N}\sum_{k=0}^{N-1}{}_BJ_{g,k_g,k}(x_n)\right|^2.$$

Proof. Following Kuipers and Niederreiter [15, Ch. 2, Theorem 2.4], let us denote $S(\xi_N) = \sum\limits_{n=0}^{N-1}\left(x_n - \frac{1}{2}\right)$. The following inequality

$$D^3(\xi_N) \leq \frac{12}{N^2}\int_0^1 [R(\xi_N;x) + S(\xi_N)]^2\,dx \tag{4.1}$$

is proved, where $R(\xi_N;x) = \sum\limits_{n=0}^{N-1}\left[1_{[x_n,1)} - x\right]$.

For the integral in inequality (4.1) we will use the following presentations

$$\int_0^1 \left[R(\xi_N;x)+S(\xi_N)\right]^2 dx = \int_0^1 R^2(\xi_N;x)\,dx + 2S(\xi_N)\sum_{n=0}^{N-1}\int_0^1 \left[1_{[0,1)}(x)-x\right] dx$$

$$+ S^2(\xi_N) = \int_0^1 R^2(\xi_N;x)\,dx - S^2(\xi_N). \quad (4.2)$$

From (4.1) and (4.2) we obtain the inequality

$$D^3(\xi_N) \le \frac{12}{N^2}\left[\int_0^1 R^2(\xi_N;x)\,dx - S^2(\xi_N)\right].$$

To the integral of the above inequality, we will apply the Parseval's formula. For this purpose, the Fourier's coefficients of the function $R(\xi_N;x)$ were calculated in Lemma 3.5. In this way, we obtain that

$$D^3(\xi_N) \le 12\sum_{g=0}^{\infty}\sum_{k_g=1}^{b_g-1}\sum_{k=k_gB_g}^{(k_g+1)B_g-1}\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_BJ_{g,k_g,k}(x_n)\right|^2.$$

Theorem 4.1 is finally proved. □

In Theorems 4.2, 4.3 and 4.4 we will use an arbitrary $\mathcal{B}_s$-adic Cantor system. Thus, let $\mathcal{B}_s = (B_1,\dots,B_s)$, where for $1\le j\le s$ $B_j = \{b_0^{(j)}, b_1^{(j)},\dots : b_i^{(j)} \ge 2 \text{ for } i\ge 0\}$ be a sequence of bases and $\{B_0^{(j)}, B_1^{(j)},\dots\}$ be the corresponding to B_j sequence of generalized powers.

Theorem 4.2 (The Koksma's formula). *Let $\xi_N = \{\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\}$ be an arbitrary net composed by $N\ge 1$ points in $[0,1)^s$. Then, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the equality*

$$T^2(\xi_N) = \left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)\right|^2 + \sum_{u=1}^{s}\sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\prod_{j=1}^{u}\sum_{g_j=0}^{\infty}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2.$$

Proof. By using the Parseval's formula, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the presentation

$$T^2(\xi_N) = \int_{[0,1]^s} g^2(\xi_N;\mathbf{x})\,d\mathbf{x} = |\widehat{g}(\xi_N;\mathbf{0})|^2 + \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} |\widehat{g}(\xi_N;\mathbf{k})|^2.$$

In Lemma 3.5 the Fourier's coefficients of the local discrepancy $g(\xi_N;\mathbf{x})$ were calculated. We put the obtained results in the above equality and obtain that

$$T^2(\xi_N) = \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) \right|^2 + \sum_{u=1}^{s} \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{\infty} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_n),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

Theorem 4.2 is finally proved. □

Theorem 4.3 (The Erdős-Turán-Koksma's inequality). *Let us assume that the coordinate sequences of the $\mathcal{B}_s$-adic system are bounded from above, i.e., there exists an absolute integer constant C, such that for $1 \le j \le s$ and each $i \ge 0$, the inequality $b_i^{(j)} \le C$ holds. Let us denote $b = \min_{1\le j\le s} \min_{i\ge 0} b_i^{(j)}$, and let us define the constant*

$$K(C;s) = \begin{cases} 9/4, & \text{if } C = 2, \\ \left[1 + \left(C^2/8\right)^s\right]^2, & \text{if } C \ge 3. \end{cases}$$

Let $M > 1$ be an arbitrary integer. Let $\xi_N = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. Then, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the inequality

$$T^2(\xi_N) \le K(C;s) \left[1 + (C-1)\frac{b}{b-1}\right]^s \cdot \frac{1}{b^M} + \left(\frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) \right)^2$$

$$+ \sum_{u=1}^{s} \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{M-1} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}\cdot B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)\cdot B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

Proof. Let M be as in the condition of the theorem and let us denote

$$A(M;s;\xi_N) = \sum_{u=1}^{s} \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S} \sum_{\substack{(g_1,\dots,g_u)\in\mathbb{N}_0^u \\ \text{there is at least} \\ \text{one index } \delta, \\ 1\le\delta\le u, \\ \text{such that } g_\delta \ge M}} \prod_{j=1}^{u} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

The result of Theorem 4.2 gives us that the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the presentation

$$T^2(\xi_N) = \left(\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)\right)^2 + \sum_{u=1}^{s}\sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\prod_{j=1}^{u}\sum_{g_j=0}^{M-1}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2 + A(M;s;\xi_N). \quad (4.3)$$

We will obtain un upper bound of the quantity $A(M;s;\xi_N)$. Without loss of the community of our consideration, let us assume that $g_1 \geq M$, $g_2 \geq 0, \dots, g_u \geq 0$. In this way, we obtain that

$$A(M;s;\xi_N) \leq \sum_{u=1}^{s}\sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\sum_{g_2=0}^{\infty}\cdots\sum_{g_u=0}^{\infty}\prod_{j=1}^{u}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2. \quad (4.4)$$

We will prove that for each parameters $(g_1,\dots,g_u)$, $(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)})$ and $(k_{\alpha_1},\dots,k_{\alpha_u})$ the inequality holds

$$\left|{}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\right| \leq \left[1+\left(\frac{C^2}{8}\right)^u\right]\prod_{j=1}^{u}\frac{1}{B_{g_j}^{(\alpha_j)}}. \quad (4.5)$$

Really, by using the definition of the notion of modified integral, we obtain that

$$\left|{}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\right| \leq \prod_{j=1}^{u}\left|{}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j})\right| \times \prod_{j=1}^{s-u}(1-x_{\beta_j})$$

$$+\prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\left|\frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\dots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\dots b_{g_j}^{(\alpha_j)}}\right)}-1}\right|\cdot\left|\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right|\right]\cdot\frac{1}{2^{s-u}}. \quad (4.6)$$

We will estimate the expressions in the above inequality. For this purpose, let $k \geq 1$ be an arbitrary integer and $B_g \leq k \leq B_{g+1}-1$ for some $g \geq 0$. Let $x \in [0,1)$ be an arbitrary and fixed real and let q be the integer such that $0 \leq q \leq B_g - 1$ and $\frac{q}{B_g} \leq x < \frac{q+1}{B_g}$. Then, by using Lemma 3.1 (ii) we obtain that

$$|{}_BJ_k(x)| = \left|\int_0^x {}_B\gamma_k(t)\,dt\right| = \left|\int_0^{\frac{q}{B_g}} {}_B\gamma_k(t)\,dt + \int_{\frac{q}{B_g}}^x {}_B\gamma_k(t)\,dt\right| = \left|\int_{\frac{q}{B_g}}^x {}_B\gamma_k(t)\,dt\right|$$

$$\leq \int_{\frac{q}{B_g}}^x |{}_B\gamma_k(t)|\,dt = \int_{\frac{q}{B_g}}^x dt = x - \frac{q}{B_g} \leq \frac{1}{B_g},$$

i.e., the inequality holds

$$|_B J_k(x)| \leq \frac{1}{B_g}, \quad x \in [0,1). \tag{4.7}$$

For arbitrary integers $b \geq 2$ and $\beta \in \{1, \dots, b-1\}$ we have that $|\Delta(b;\beta)| = \frac{b}{2} \cdot \frac{1}{\sin \pi \frac{\beta}{b}}$. We will use the facts that for $t \in \left[0, \frac{\pi}{2}\right]$ and $t \in \left[\frac{\pi}{2}, \pi\right]$, respectively the inequalities $\sin t \geq \frac{2}{\pi} t$ and $\sin t \geq 2 - \frac{2}{\pi} t$ hold. In this way, we obtain that $\sin \pi \frac{\beta}{b} \geq \frac{2}{b}$ and hence,

$$|\Delta(b;\beta)| \leq \frac{b^2}{4}. \tag{4.8}$$

For an arbitrary real t the equality $|e^{2\pi \mathbf{i} t} - 1| = 2|\sin \pi t|$ holds. Hence, for each $1 \leq j \leq u$ we have that

$$\left| \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)} - 1} \right| \leq \frac{1}{\sin \pi \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)}.$$

The inequalities

$$\frac{1}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \leq \frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \leq 1 - \frac{1}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}$$

give us that the lower bound

$$\sin \pi \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right) \geq \frac{2}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}$$

holds. In this way, we obtain that

$$\left| \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)} - 1} \right| \leq \frac{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}{2}. \tag{4.9}$$

From (4.6)–(4.9) we obtain that

$$\left|_{\mathcal{B}_s} J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\right|$$
$$\leq \prod_{j=1}^{u} \frac{1}{B_{g_j}^{(\alpha_j)}} + \prod_{j=1}^{u} \frac{1}{\left[B_{g_j}^{(\alpha_j)}\right]^2} \cdot \frac{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}{2} \cdot \frac{\left[b_{g_j}^{(\alpha_j)}\right]^2}{4} \leq \left[1 + \left(\frac{C^2}{8}\right)^u\right] \prod_{j=1}^{u} \frac{1}{B_{g_j}^{(\alpha_j)}}.$$

In this way, inequality (4.5) is finally proved.

From (4.4) and (4.5) we obtain that

$$\begin{aligned}
&A(M;s;\xi_N)\\
&\le \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{1}{\left[B_{g_1}^{(\alpha_1)}\right]^2}\sum_{g_2=0}^{\infty}\frac{1}{\left[B_{g_2}^{(\alpha_2)}\right]^2}\cdots\sum_{g_u=0}^{\infty}\frac{1}{\left[B_{g_u}^{(\alpha_u)}\right]^2}\\
&\qquad\times\prod_{j=1}^{u}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\ \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1} 1\\
&= \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{b_{g_1}^{(\alpha_1)}-1}{B_{g_1}^{(\alpha_1)}}\sum_{g_2=0}^{\infty}\frac{b_{g_2}^{(\alpha_2)}-1}{B_{g_2}^{(\alpha_2)}}\cdots\sum_{g_u=0}^{\infty}\frac{b_{g_u}^{(\alpha_u)}-1}{B_{g_u}^{(\alpha_u)}}\\
&\le \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 (C-1)^u \sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{1}{B_{g_1}^{(\alpha_1)}}\sum_{g_2=0}^{\infty}\frac{1}{B_{g_2}^{(\alpha_2)}}\cdots\sum_{g_u=0}^{\infty}\frac{1}{B_{g_u}^{(\alpha_u)}}.
\end{aligned} \tag{4.10}$$

We will use the following estimations

$$\sum_{g_1=M}^{\infty}\frac{1}{B_{g_1}^{(\alpha_1)}}\le\sum_{g=M}^{\infty}\frac{1}{b^g}=\frac{b}{b-1}\cdot\frac{1}{b^M}\ \text{and for } 2\le j\le u\ \sum_{g_j=0}^{\infty}\frac{1}{B_{g_j}^{(\alpha_j)}}\le\sum_{g=0}^{\infty}\frac{1}{b^g}=\frac{b}{b-1}.$$

Then, from inequality (4.10) we obtain that

$$\begin{aligned}
A(M;s;\xi_N) &\le \frac{1}{b^M}\sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\left[(C-1)\frac{b}{b-1}\right]^u\sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S} 1\\
&= \left[\sum_{u=0}^{s} C_s^u\cdot\left[(C-1)\frac{b}{b-1}\right]^u\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\right]\frac{1}{b^M}.
\end{aligned} \tag{4.11}$$

Let us assume that $C=2$. Then, for $1\le u\le s$ we have that $\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\le \frac{9}{4}$. In the case, when $C\ge 3$ we have that $\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\le\left[1+\left(\frac{C^2}{8}\right)^s\right]^2$. In this way, from inequality (4.11) we obtain that

$$\begin{aligned}
&A(M;s;\xi_N)\\
&\quad\le K(C;s)\left[\sum_{u=0}^{s}C_s^u\left[(C-1)\frac{b}{b-1}\right]^u\right]\frac{1}{b^M}=K(C;s)\left[1+(C-1)\frac{b}{b-1}\right]^s\frac{1}{b^M}
\end{aligned}$$

with the defined in the condition of the theorem constant $K(C;s)$. Equality (4.3) and the above estimation finally prove the statement of the theorem. □

Theorem 4.4 (The Integral Weyl's criterion)**.** *Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. The sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the following conditions hold:*

(i) *The limit equality* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) = 0$ *holds;*

(ii) *The limit equality*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n) = 0$$

holds for each choice of the parameters $1 \leq u \leq s$, $\{\alpha_1,\dots,\alpha_u\} \subseteq S$, $(g_1,\dots,g_u) \in \mathbb{N}_0^s$, for $1 \leq j \leq u$ $k_{g_j}^{(\alpha_j)} \in \{1,\dots,b_{g_j}^{(\alpha_j)}-1\}$ and $k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)} \leq k_{\alpha_j} \leq (k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)} - 1$.

Proof. To prove the theorem we will use the so-called integral criterion of Weyl [22]. Thus, the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if for each Riemann integrable over $[0,1]^s$ function, the limit equality holds

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x})\,d\mathbf{x}.$$

In this way, we must prove that the equality $\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})\,d\mathbf{x} = 0$ holds, and for each choice of the parameters $1 \leq u \leq s$, $\{\alpha_1,\dots,\alpha_u\} \subseteq S$, $(g_1,\dots,g_u) \in \mathbb{N}_0^u$, $1 \leq j \leq u$, $k_{g_j}^{(\alpha_j)} \in \{1,\dots,b_{g_j}^{(\alpha_j)}-1\}$ and $k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)} \leq k_{\alpha_j} \leq (k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)} - 1$, the equality $\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\,d\mathbf{x} = 0$ holds.

By using the definition of the modified integral ${}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})$, we directly obtain that

$$\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})\,d\mathbf{x} = 0.$$

Now, let the above parameters be fixed. By using the definition of the modified integral ${}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})$ and the statement of Lemma 3.3 we consecutively obtain that

$$\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\,d\mathbf{x}$$

$$= (-1)^u \prod_{j=1}^{u} \int_0^1 {}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j} \prod_{j=1}^{s-u} \int_0^1 (1 - x_{\beta_j})\,dx_{\beta_j}$$

$$- \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \times \frac{1}{2^{s-u}}$$

$$= \prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\ldots b_{g_j}^{(\alpha_j)}}\right)}-1}\cdot\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right]\times\frac{1}{2^{s-u}}$$
$$-\prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\ldots b_{g_j}^{(\alpha_j)}}\right)}-1}\cdot\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right]\times\frac{1}{2^{s-u}}=0.$$

In this way, the proof of the theorem is finally completed. □

5. Conclusion

The application of the functions of some classes of complete orthonormal function systems to the theory of the uniformly distributed sequences is realized as two main approaches.

The first approach is their direct applications. The main results in this direction are the exponential Weyl's criterion, which is a necessary and sufficient condition that a sequence to be uniformly distributed and the Erdős-Turán-Koksma's inequality, which gives an upper bound of the extreme discrepancy. Both results are presented in the terms of the trigonometric sum of these functions. Also, an important result here, is to define the notion of diaphony, which is based on using some classes of complete orthonormal function systems.

The second approach is to introduce the notions of modified integrals from these functions and to show some their applications for an investigation of the uniformly distributed sequences. Grozdanov and Stoilova [8] realized this approach with respect to the functions of the system $\mathrm{Vil}_{\mathcal{B}_s}$ of the Vilenkin functions.

As it was shown, as an appropriate tool for investigation of sequences and nets constructed in Cantor systems, the functions of some complete orthonormal systems constructed over these number systems, are used.

In this article, we realized our main purpose – to use the function system $\Gamma_{\mathcal{B}_s}$ and to show some of its applications to the quantitative and qualitative theory of the uniformly distributed sequences. The LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion were presented in the terms of these integrals. We consider that the obtained results are related to a wide aspect of the quantitative and the qualitative theory of the uniformly distributed sequences.

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