ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

TWO PROPERTIES OF THE PARTIAL THETA FUNCTION

VLADIMIR PETROV KOSTOV

For the partial theta function $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, q, $z \in \mathbb{C}$, $|q| < 1$, we prove that its zero locus is connected. This set is smooth at every point $(q^\flat, z^\flat)$ such that $z^\flat$ is a simple or double zero of $\theta(q^\flat, .)$. For $q \in (0,1)$, $q \to 1^-$ and $a \geq e^\pi$, there are $o(1/(1-q))$ and $(\ln(a/e^\pi))/(1-q) + o(1/(1-q))$ real zeros of $\theta(q,.)$ in the intervals $[-e^\pi, 0)$ and $[-a, -e^{-\pi}]$ respectively (and none in $[0,\infty)$). For $q \in (-1,0)$, $q \to -1^+$ and $a \geq e^{\pi/2}$, there are $o(1/(1+q))$ real zeros of $\theta(q,.)$ in the interval $[-e^{\pi/2}, e^{\pi/2}]$ and $(\ln(a/e^{\pi/2})/2)/(1+q) + o(1/(1+q))$ in each of the intervals $[-a, -e^{\pi/2}]$ and $[e^{\pi/2}, a]$.

Keywords: partial theta function, separation in modulus, limit density of the real zeros

2020 Mathematics Subject Classification: 26A06

1. Introduction

1.1. Definition of the partial theta function

For $q \in \mathbb{D}_1$, $z \in \mathbb{C}$, where $\mathbb{D}_r$ stands for the open disk of radius r centered at $0 \in \mathbb{C}$, one defines the *partial theta function* by the formula

$$\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j. \tag{1.1}$$

This terminology is explained by the resemblance of formula (1.1) with the one defining the *Jacobi theta function* $\Theta(q,z) := \sum_{j=-\infty}^{\infty} q^{j^2} z^j$; the word “partial” refers to the summation in the case of θ taking place only over the nonnegative values of j. One has $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$. We consider q as a parameter and z as a variable. For each q fixed, $\theta(q,.)$ is an entire function.

DOI: 10.60063/GSU.FMI.112.77-91

The function θ has been studied as Ramanujan-type series in [21]. Its applications in statistical physics and combinatorics are explained in [20]. Other fields, where θ is used, are the theory of (mock) modular forms (see [4]) and asymptotic analysis (see [2]). Asymptotics, modularity and other properties of partial and false theta functions are considered in [5] with regard to conformal field theory and representation theory, and in [3] when asymptotic expansions of regularized characters and quantum dimensions of the $(1,p)$-singlet algebra modules are studied.

A recent impetus for the interest in θ (in the case when the parameter q is real) was given by the theory of *section-hyperbolic polynomials*, i.e., real univariate polynomials of degree ≥ 2 with all roots real negative and such that, when their highest-degree monomial is deleted, this gives again a polynomial having only real negative roots. The classical results of Hardy, Petrovitch and Hutchinson in this direction (see [6,7,19]) have been continued in [8,17,18]. Various analytic properties of θ are studied in [9–11,13–15]. See more about θ in [1].

1.2. The zero locus and the spectrum of the partial theta function

In the present paper we consider the zero locus of θ, i.e., the set $S := \{(q,z) \in \mathbb{D}_1 \times \mathbb{C}, \theta(q,z) = 0\}$. In Section 2 we prove the following theorem.

Theorem 1.1. *In the space $\mathbb{D}_1 \times \mathbb{C}$, the zero locus S is an irreducible hence connected analytic curve. It is smooth at every point $(q^\flat, z^\flat)$ such that $z^\flat$ is a simple or double zero of $\theta(q^\flat, .)$.*

Remarks 1.2. (1) B. Z. Shapiro has introduced the notion of *spectrum* of θ as the set of values of q for which $\theta(q,.)$ has a multiple zero, see [17]. Suppose that q is real, i.e., $q \in (-1,0) \cup (0,1)$ (the case $q = 0$ is of little interest since $\theta(0,z) \equiv 1$). If $q \in (0,1)$, then $\theta(q,.)$ has infinitely-many real zeros and they are all negative. There are also infinitely-many spectral numbers $0 < \tilde{q}_1 < \tilde{q}_2 < \cdots < \tilde{q}_k < \cdots < 1$, $\lim_{k\to\infty} \tilde{q}_k = 1^-$, see [9].

(2) For $q \in (0, \tilde{q}_1)$ (where $\tilde{q}_1 = 0.3092\ldots$), all zeros of $\theta(q,.)$ are real, negative and distinct: $\cdots < \xi_2 < \xi_1 < 0$; one has $\theta(q,x) > 0$ for $x \in (\xi_{2j+1}, \xi_{2j})$ and $\theta(q,x) < 0$ for $x \in (\xi_{2j}, \xi_{2j-1})$. For $q \in (\tilde{q}_k, \tilde{q}_{k+1}) \subset (0,1)$, $k \in \mathbb{N}$, $\tilde{q}_0 := 0$, the function $\theta(q,.)$ has exactly k pairs of complex conjugate zeros (counted with multiplicity). When $q \in (0,1)$ increases and passes through the spectral value $\tilde{q}_k$, the two zeros ξ_{2k-1} and ξ_{2k} coalesce and then form a complex conjugate pair, see [9]. The index j of the zero ξ_j is meaningful as long as ξ_j is real, i.e., for $q \in (0, \tilde{q}_{[(j+1)/2]}]$, where $[.]$ stands for "the integer part of".

(3) Asymptotic expansions of the numbers $\tilde{q}_k$ are obtained in [10] and [14]. The formula of [14] reads:

$$\tilde{q}_k = 1 - \pi/2k + (\ln k)/8k^2 + O(1/k^2), \quad \tilde{y}_k = -e^\pi e^{-(\ln k)/4k + O(1/k)} \tag{1.2}$$

where $e^\pi = 23.1407\ldots$ and $\tilde{y}_k < 0$ is the double zero of $\theta(\tilde{q}_k, .)$. It is the rightmost of its real zeros and $\theta(\tilde{q}_k, .)$ has a minimum at $\tilde{y}_k$.

(4) For $k \in \mathbb{N}^*$, one has $\theta(q, -q^{-k}) \in (0, q^k)$, see [9, Proposition 9]. For $q > 0$ small enough, one has $\operatorname{sgn}(\theta(q, -q^{-k-1/2})) = (-1)^k$ and $|\theta(q, -q^{-k-1/2})| > 1$, see [9, Proposition 12].

Remarks 1.3. (1) If $q \in (-1, 0)$ is sufficiently small, then $\theta(q, .)$ has infinitely-many real negative and infinitely-many real positive zeros: $\cdots < \xi_4 < \xi_2 < 0 < \xi_1 < \xi_3 < \cdots$; one has $\theta(q, x) < 0$ for $x \in (\xi_{4j+4}, \xi_{4j+2})$ and $x \in (\xi_{4j+1}, \xi_{4j+3})$, and $\theta(q, x) > 0$ for $x \in (\xi_{4j+2}, \xi_{4j})$, $x \in (\xi_{4j+3}, \xi_{4j+5})$ and $x \in (\xi_2, \xi_1)$. For $q \in (-1, 0)$, there are also infinitely-many spectral numbers, see [13]. We denote them by $\bar{q}_k$, where $-1 < \bar{q}_k < 0$.

(2) For $s \geq 1$, one has $-1 < \bar{q}_{2s+1} < \bar{q}_{2s-1} < 0$, see [13, Lemma 4.11]. For k sufficiently large, one has $\bar{q}_{k+1} < \bar{q}_k$, see [13, Lemmas 4.10, 4.11 and 4.17]. The inequality $\bar{q}_{k+1} < \bar{q}_k$ being proved only for k sufficiently large we admit the possibility finitely-many equalities of the form $\bar{q}_i = \bar{q}_j$ to hold true, where at least one of the numbers i and j is even.

(3) When $q \in (-1, 0)$ decreases and passes through a spectral value $\bar{q}_k$, then for $k = 2s-1$ (resp. $k = 2s$), $s \in \mathbb{N}^*$, the zeros ξ_{4s-2} and ξ_{4s} (resp. ξ_{4s-1} and ξ_{4s+1}) coalesce. Thus for $q \in (\bar{q}_{k+1}, \bar{q}_k) \subset (-1, 0)$ and k sufficiently large, the function $\theta(q, .)$ has exactly k pairs of complex conjugate zeros (counted with multiplicity). The zero ξ_1 remains real and simple for any $q \in (-1, 0)$.

(4) Asymptotic expansions of the numbers $\bar{q}_k$ are found in [13]:

$$\bar{q}_k = 1 - (\pi/8k) + o(1/k), \quad |\bar{y}_k| = e^{\pi/2} + o(1), \tag{1.3}$$

where $\bar{y}_k$ is the double zero of $\theta(\bar{q}_k, .)$ and $e^{\pi/2} = 4.81477382\ldots$. For k odd (respectively, k even) $\theta(\bar{q}_k, .)$ has a local minimum (respectively, maximum) at $\bar{y}_k$, and $\bar{y}_k$ is the rightmost of the real negative zeros of $\theta(\bar{q}_k, .)$ (respectively, for k sufficiently large, $\bar{y}_k$ is the second from the left of the real positive zeros of $\theta(\bar{q}_k, .)$).

Remarks 1.4. (1) All coefficients in the series (1.1) are real. Hence a priori spectral numbers are either real or they form complex conjugate pairs. It is proved in [15] that there exists at least one such pair which equals $0.4353184958\cdots \pm i0.1230440086\ldots$. Numerical results suggest that one should expect there to be infinitely-many such pairs.

(2) In any set of the form $\mathbb{D}_r \setminus \{0\}$, $r \in (0, 1)$, the number of spectral values of θ is finite (because the spectrum is locally a codimension 1 analytic subset in $\mathbb{D}_1 \setminus \{0\}$). For any spectral number q, the function $\theta(q, .)$ has finitely-many multiple zeros, see [11]. The number $\tilde{q}_1 = 0.3092\ldots$ is the only spectral number of θ in the disk $\overline{\mathbb{D}_{0.31}}$.

(3) For all spectral numbers $\tilde{q}_j \in (0, 1)$ and, for k sufficiently large, for all spectral numbers $\bar{q}_k \in (-1, 0)$, it is true that the function $\theta(\tilde{q}_j, .)$, resp. $\theta(\bar{q}_k, .)$, has exactly one double real zero while all its other real zeros are simple (see [9] and [13]). It would be interesting to prove (or disprove) that for any spectral value (real or complex) the partial theta function has just one double zero all its other zeros being simple. If true, this would mean in particular (see Theorem 1.1) that S is globally smooth and connected. If false, it would be of interest to describe the eventual singularities of S.

1.3. The limit distribution of the real zeros

In the present subsection we consider the case $q \in \mathbb{R}$, i.e., $q \in (-1,0) \cup (0,1)$.

Notation 1.5. For $q \in (-1,0)\cup(0,1)$, given a finite interval $J \subset \mathbb{R}$, we denote by $Z_J(q)$ the number of zeros of $\theta(q,.)$ (counted with multiplicity) belonging to J. For $q \in (0,1)$ and $a \geq e^{\pi}$, we set $\ell_a(q) := Z_{[-a,-e^{\pi}]}(q)$. For $q \in (-1,0)$ and $a \geq e^{\pi/2}$, we set $n_a(q) := Z_{[-a,-e^{\pi/2}]}(q)$ and $p_a(q) := Z_{[e^{\pi/2},a]}(q)$.

Theorem 1.6. (1) *For $q \in (0,1)$, one has $Z_{[-e^{\pi},0)}(q) = o(1/(1-q))$.*

(2) *The set of zeros of $\theta(\tilde{q}_k,.)$ (over all $k \in \mathbb{N}^*$) is everywhere dense in $(-\infty,-e^{\pi}]$. One has $\lim_{q\to 1^-} \ell_a(q)(1-q) = \ln(a/e^{\pi})$.*

(3) *For $q \in (-1,0)$, one has $Z_{[-e^{\pi/2},0)}(q) = o(1/(1+q))$.*

(4) *The set of zeros of $\theta(\bar{q}_{2s-1},.)$ (over all $s \in \mathbb{N}^*$) is everywhere dense in $(-\infty,-e^{\pi/2}]$. One has $\lim_{q\to -1^+} n_a(q)(1+q) = \ln(a/e^{\pi/2})/2$.*

(5) *For $q \in (-1,0)$, one has $Z_{(0,e^{\pi/2}]}(q) = o(1/(1+q))$.*

(6) *The set of zeros of $\theta(\bar{q}_{2s},.)$ (over all $s \in \mathbb{N}^*$) is everywhere dense in $[e^{\pi/2},\infty)$. One has $\lim_{q\to -1^+} p_a(q)(1+q) = \ln(a/e^{\pi/2})/2$.*

The theorem is proved in Section 3.

Remark 1.7. The quantity $1/a = \lim_{\varepsilon\to 0^+}((\ln((a+\varepsilon)/e^{\pi}) - \ln(a/e^{\pi})/\varepsilon)$ can be interpreted as limit density of the real zeros of $\theta(q,.)$ as $q \to 1^-$ at the point $-a \leq -e^{\pi}$. Similarly for the quantity $1/(2a)$ at $\pm a$, $a \geq e^{\pi/2}$, as $q \to -1^+$. For the rest of the real line the limit density is 0. Indeed, for $q \in (0,1)$, there are no nonnegative zeros of $\theta(q,.)$; for $0 < a < e^{\pi}$, see part (1) of Theorem 1.6. For $q \in (-1,0)$, see parts (3) and (5) of Theorem 1.6.

2. Proof of Theorem 1.1

2.1. Smoothness

We prove the smoothness first. If $z^\flat$ is a simple zero of $\theta(q^\flat,.)$, then $(\partial\theta/\partial z)(q^\flat,z^\flat)\neq 0$ hence

$$\mathrm{Grad}(\theta)(q^\flat,z^\flat) \neq 0 \tag{2.1}$$

and S is smooth at $(q^\flat,z^\flat)$. The function θ satisfies the following differential equation (see (1.1)):

$$2q(\partial\theta/\partial q) = z(\partial^2/\partial z^2)(z\theta). \tag{2.2}$$

The right-hand side equals $2z(\partial\theta/\partial z)+z^2(\partial^2\theta/\partial z^2)$. If $z^\flat$ is a double zero of $\theta(q^\flat,.)$, then

$$\theta(q^\flat,z^\flat) = (\partial\theta/\partial z)(q^\flat,z^\flat) = 0 \neq (\partial^2\theta/\partial z^2)(q^\flat,z^\flat).$$

One has neither $q^\flat = 0$, because $\theta(0,.) \equiv 1 \neq 0$, nor $z^\flat = 0$, because $\theta(q,0) \equiv 1$. Hence

$$(\partial\theta/\partial q)(q^\flat, z^\flat) = ((z^\flat)^2/2q^\flat)(\partial^2\theta/\partial z^2)(q^\flat, z^\flat) \neq 0.$$

Therefore one has (2.1), so S is smooth at $(q^\flat, z^\flat)$.

2.2. Separation in modulus

For fixed $q \in \mathbb{D}_1 \setminus \{0\}$, we denote by $\mathcal{C}_k$, $k \in \mathbb{N}^*$, the circumference in the z-space $|z| = |q|^{-k-1/2}$. When q is close to 0, one can enumerate the zeros of θ, because there exists exactly one zero such that $\xi_k \sim -q^{-k}$ (see [9, Proposition 10]). For $0 < |q| \leq c_0 := 0.2078750206\ldots$, one has

$$|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2}, \tag{2.3}$$

see [15, Lemma 1]. In this sense we say that for $q \in \mathbb{D}_{c_0} \setminus \{0\}$, the zeros of θ are separated in modulus (that is, their moduli are separated by the circumferences $\mathcal{C}_k$). We say that, for given q, *strong separation* of the zeros of θ takes place for $k \geq k_0$, if for any $k \geq k_0$, there exists exactly one zero ξ_k of θ satisfying conditions (2.3).

Set $\alpha_0 := \sqrt{3}/2\pi = 0.2756644477\ldots$. The following result can be found in [15].

Theorem 2.1. *For $n \geq 5$ and for $|q| \leq 1 - 1/(\alpha_0 n)$, strong separation of the zeros of θ takes place for $k \geq n$.*

Theorem 2.1 has several important corollaries:

i) For each path $\gamma \subset \mathbb{D}_1 \setminus \{0\}$ in the q-space which avoids the spectral numbers of θ, one can define by continuity the zeros of θ as functions of q as q varies along γ. One can find $k \in \mathbb{N}$ such that $\gamma \subset \mathbb{D}_{1-1/(\alpha_0 k)}$. For $n \geq k$, the zero ξ_n is an analytic function in $q \in \mathbb{D}_{1-1/(\alpha_0 k)}$. Thus the indices of the zeros ξ_n are meaningful for $n \geq k$ and $q \in \mathbb{D}_{1-1/(\alpha_0 k)}$.

ii) Denote by Γ the spectrum of θ. If $\gamma \subset D := \mathbb{D}_{1-1/(\alpha_0 k)} \setminus \{\Gamma \cup \{0\}\}$ is a loop, then the zeros of θ lying inside $\mathcal{C}_k$ might undergo a *monodromy* as q varies along γ, i.e., a permutation which depends on the class of homotopy equivalence of γ in D. Therefore it might not be possible to correctly define the indices of these zeros for $q \in D$.

iii) For no $q_* \in \mathbb{D}_1 \setminus \{0\}$ does a zero of θ go to infinity as $q \to q_*$. That is, zeros are not born and do not disappear at infinity.

iv) For $(0,1) \ni q = \tilde{q}_j \in \Gamma$, the function $\theta(q,.)$ has one double zero and infinitely-many simple zeros, see part (3) of Remarks 1.4 and part (3) of Remarks 1.2. The double zero is a Morse critical point for θ. Suppose that γ is a small loop in $\mathbb{D}_1 \setminus \{0\}$ circumventing $\tilde{q}_j$. Then the two zeros ξ_{2j-1} and ξ_{2j} of $\theta(q,.)$ which coalesce for $q = \tilde{q}_j$ are exchanged as q varies along γ. For $(-1,0) \ni q = \bar{q}_k \in \Gamma$, $k = 2s-1$ or $2s$, where $s \geq 1$ is sufficiently large, the same remark applies to the zeros ξ_{4s-2} and ξ_{4s} or ξ_{4s-1} and

ξ_{4s+1}, see part (3) of Remarks 1.3. For the remaining values of k, if, say, p spectral values $\bar{q}_i$ coincide, then the function $\theta(\bar{q}_i, .)$ has p double real zeros (its other real zeros are simple) and the monodromy defined by the class of homotopy equivalence of γ exchanges the zeros in p non-intersecting couples of zeros (which are close to the double zeros of $\theta(\bar{q}_i, .)$).

v) Theorem 2.1 implies that the monodromy around $0 \in \mathbb{D}_1$ is trivial.

2.3. Connectedness of S

When $q \in \mathbb{D}_{c_0} \setminus \{0\}$, the zeros ξ_j can be considered as analytic functions in q. We discuss the possible monodromies which they can undergo when the parameter q runs along certain loops in $\mathbb{D}_1 \setminus \{0\}$. First of all we recall that for $q \in (0, \tilde{q}_j)$, the zeros $0 > \xi_{2j-1} > \xi_{2j} > \xi_{2j+1} > \cdots$ are simple, real negative and continuously depending on q, see part (2) of Remarks 1.2; for $q = \tilde{q}_j$, the zeros ξ_{2j-1} and ξ_{2j} coalesce.

Suppose that $a \in (0, c_0)$ and that $\mathcal{C}^\sharp \subset \mathbb{D}_1 \setminus \{0\}$ is a small circumference of radius ε centered at the spectral number $\tilde{q}_j$, see parts (1) and (2) of Remarks 1.2; no spectral number other than $\tilde{q}_j$ belongs to the circumference $\mathcal{C}^\sharp$ or to its interior. Define $\gamma_j \subset \mathbb{D}_1 \setminus \{0\}$ as the path consisting of the segment $\sigma_+ := [a, \tilde{q}_j - \varepsilon] \subset \mathbb{R}$, the circumference $\mathcal{C}^\sharp$ (which is run, say, counterclockwise) and the segment $\sigma_- := [\tilde{q}_j - \varepsilon, a]$. Hence if one considers the analytic continuation of the function ξ_{2j-1} (resp. ξ_{2j}) along the loop γ_j, the result will be the function ξ_{2j} (resp. ξ_{2j-1}), see *iv)* in Subsection 2.2. We denote this symbolically by $\gamma_j : \xi_{2j-1} \leftrightarrow \xi_{2j}$. If we need to indicate only the image of ξ_{2j-1} we might write $\gamma_j : \xi_{2j-1} \mapsto \xi_{2j}$.

Remark 2.2. For $j > 1$, the two segments $\sigma_\pm$ of the path γ_j pass through the spectral numbers $\tilde{q}_1, \ldots, \tilde{q}_{j-1}$. If one insists the path γ_j to bypass all spectral numbers $\tilde{q}_j$, then one should modify γ_j. Namely, parts of the two segments $\sigma_\pm$ which are segments of the form $\sigma_s := [\tilde{q}_s - \varepsilon', \tilde{q}_s + \varepsilon']$, $0 < \varepsilon' \ll \varepsilon$, $1 \leq s \leq j-1$, should be replaced by small half-circumferences with diameters σ_s which bypass the spectral numbers $\tilde{q}_s$ from above or below.

Suppose that $q \in (-1, 0)$. We will make use of Remarks 1.3. We construct a path δ_j consisting of a segment $\tau_- := [-a, \bar{q}_j + \varepsilon] \subset \mathbb{R}$, $-c_0 < -a < 0$ (c_0 is defined at the beginning of Subsection 2.2), a circumference $\mathcal{C}^\triangle \subset \mathbb{D}_1 \setminus \{0\}$ of radius ε centered at $\bar{q}_j$ (and run, say, counterclockwise) and the segment $\tau_+ := [\bar{q}_j + \varepsilon, -a]$. If $\bar{q}_{j_1} \neq \bar{q}_j$, then the spectral number $\bar{q}_{j_1}$ does not belong to $\mathcal{C}^\triangle$ or to its interior.

Suppose that $j = 2s-1$ (resp. $j = 2s$). If one considers the analytic continuation of the functions ξ_{4s-2} and ξ_{4s} (resp. of ξ_{4s-1} and ξ_{4s+1}) along the loop δ_j, the result will be that the functions ξ_{4s-2} and ξ_{4s} (resp. ξ_{4s-1} and ξ_{4s+1}) exchange their values, see iv) in Subsection 2.2. We denote this symbolically by $\delta_{2s-1} : \xi_{4s-2} \leftrightarrow \xi_{4s}$ or $\delta_{2s} : \xi_{4s-1} \leftrightarrow \xi_{4s+1}$.

Remark 2.3. Similarly to what was done with the path γ_j, see Remark 2.2, one can modify the path δ_j so that it should pass through no spectral value of θ. We do not claim, however, that an equality of the form $\bar{q}_{j_1} = \bar{q}_{j_2}$, $j_1 \neq j_2$, does not take place (this is not proved in [13]; see part (2) of Remarks 1.3). Nevertheless, even if such an equality holds true, then it does not affect our reasoning, because when q runs along $\mathcal{C}^{\triangle}$ close to the coinciding spectral numbers $\bar{q}_{j_1}$ and $\bar{q}_{j_2}$, the exchange of zeros ξ_i which occurs concerns two couples of zeros with no zero in common.

By combining the monodromies defined by the paths γ_j and δ_j one can obtain any monodromy $\xi_k \mapsto \xi_m$. Indeed, denote by η_+ a half-circumference centered at 0, of radius a, belonging to the upper half-plane (hence the segment $[-a, a]$ is its diameter) and run counterclockwise, by η_- the same half-circumference run clockwise, by $\gamma_j\gamma_\ell$ the concatenation of the paths γ_j and γ_ℓ (defined for one and the same value of a, γ_j is followed by γ_ℓ) and similarly for the loops (all with base point a) $\gamma_j\eta_+\delta_s\eta_-$, $\eta_+\delta_s\eta_-\gamma_j$, etc. Thus for $s \geq 1$, one obtains the monodromies

$$\begin{aligned}
\gamma_{2s-1} &: \xi_{4s-3} \leftrightarrow \xi_{4s-2}, \\
\gamma_{2s} &: \xi_{4s-1} \leftrightarrow \xi_{4s}, \\
\delta_{2s-1} &: \xi_{4s-2} \leftrightarrow \xi_{4s}, \\
\delta_{2s} &: \xi_{4s-1} \leftrightarrow \xi_{4s+1}, \\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_- &: \xi_{4s-3} \mapsto \xi_{4s}, \\
\eta_+\delta_{2s-1}\eta_-\gamma_{2s-1} &: \xi_{4s} \mapsto \xi_{4s-3}, \\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_-\gamma_{2s} &: \xi_{4s-3} \mapsto \xi_{4s-1}, \\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_-\gamma_{2s}\eta_+\delta_{2s}\eta_- &: \xi_{4s-3} \mapsto \xi_{4s+1}, \text{ etc.}
\end{aligned}$$

This means that, for suitably chosen loops, the root ξ_{4s-3} can be mapped by the corresponding monodromies into any of the roots ξ_{4s-2}, ξ_{4s-1}, ξ_{4s} or ξ_{4s+1}. After this one can repeat the reasoning with $\xi_{4s+1} = \xi_{4(s+1)-3}$ (i.e., one can shift the value of s by 1) and so on.

Thus the subset S^0 of S on which all zeros of θ are simple is connected. The set $S \setminus S^0$ belongs to the topological closure of S (because the zeros of θ depend continuously on q), so S is connected. The theorem is proved.

3. Proof of Theorem 1.6

Part (1). In the proof of parts (1) and (2) of the theorem, when considering the values of q from an interval of the form $(\tilde{q}_k, \tilde{q}_{k+1})$, we take into account the first of formulae (1.2), so as q tends to 1^- (hence k tends to ∞) one has $1 - q = O(1/k)$. We prove first the following lemma.

Lemma 3.1.

(1) *For every $r \in (0, 1)$, there exists $K_r \in \mathbb{N}$ such that for every $q \in (0, r]$, one has $Z_{[-e^{\pi},0)}(q) \leq K_r$.*

(2) *When the zeros ξ_{2s-1} and ξ_{2s} are real (see part (2) of Remarks 1.2), they belong to the interval $(-q^{-2s}, -q^{-2s+1})$.*

(3) *For $q \in [\tilde{q}_k, \tilde{q}_{k+1})$, one has $Z_{[-e^\pi,0)}(q) = o(k)$.*

Proof. Part (1). It follows from part (4) of Remarks 1.2 that for $q > 0$ small enough, all zeros ξ_j of $\theta(q,.)$ are real and the zeros ξ_{2s-1} and ξ_{2s} belong to the interval $(-q^{-2s}, -q^{-2s+1})$, so they are smaller than $-r^{-2s+1}$. And in the same way, for any $q \in (0,1)$, the zeros ξ_{2s-1} and ξ_{2s}, when they are real, belong to the interval $(-q^{-2s}, -q^{-2s+1})$ (which proves part (2)).

When q increases and becomes equal to $\tilde{q}_s$, the zeros ξ_{2s-1} and ξ_{2s} coalesce. For $q > \tilde{q}_s$, they form a complex conjugate pair, see part (2) of Remarks 1.2. For $q \in (0, r]$ and $2s - 1 > \pi/\ln(1/r)$, i.e., $q^{-2s+1} \geq r^{-2s+1} > e^\pi$ hence $-q^{-2s+1} < -e^\pi$, the zero ξ_j, $j \geq 2s-1$, is either smaller than $-e^\pi$ or it has given birth (together with ξ_{j-1} or ξ_{j+1} depending on the parity of j) to a complex conjugate pair. Therefore $Z_{[-e^\pi,0)}(q) \leq [\pi/\ln(1/r)] + 1$ and one can set $K_r := [\pi/\ln(1/r)] + 1$.

Part (3). Suppose first that $q = \tilde{q}_k$. The interval $I := [-e^\pi, \tilde{y}_k]$ contains all real zeros of $\theta(\tilde{q}_k,.)$ belonging to the interval $J := [-e^\pi, 0)$. The rightmost of these zeros which is in I is the double zero $\tilde{y}_k$ which is the result of the confluence of ξ_{2k-1} and ξ_{2k}, see parts (2) and (3) of Remarks 1.2. Denote by s_0 the smallest of the numbers s for which $-(\tilde{q}_k)^{-2s+1} < -e^\pi$. Hence there are not more than

$$t_0 := 2(s_0 - 1) - 2(k-1) + 1 = 2(s_0 - k) + 1$$

real zeros of $\theta(\tilde{q}_k,.)$ in I (counted with multiplicity), see the proof of part (1) of the present lemma. One has $\tilde{q}_k = 1 - \pi/2k + o(1/k)$, see (1.2). Therefore

$$-(\tilde{q}_k)^{-2s_0+1} < -e^\pi \;\Leftrightarrow\; (-2s_0+1)\ln(\tilde{q}_k) > \pi \;\Leftrightarrow\; 2s_0 - 1 > \pi/(\ln(1/\tilde{q}_k)) = 2k + o(k).$$

On the other hand, it follows from the definition of s_0 that $2s_0 - 3 \leq \pi/(\ln(1/\tilde{q}_k)) = 2k + o(k)$. Thus $s_0 = k + o(k)$ and $t_0 = o(k)$. Suppose now that $q \in (\tilde{q}_k, \tilde{q}_{k+1})$. Hence when one counts the real zeros of $\theta(q,.)$ in J, one should take into account that:

1) The double root $\tilde{y}_k$ gives birth to a complex conjugate pair of zeros, i.e., two real zeros are lost; for $q \in (\tilde{q}_k, \tilde{q}_{k+1})$, these are the only real zeros that are lost, see part (2) of Remarks 1.2;

2) Denote by $s_*(q)$ the smallest of the numbers s for which one has $-q^{-2s+1} < -e^\pi$ (hence $s_*(\tilde{q}_k) = s_0$). For fixed s, the number $-q^{-2s+1}$ increases with q, so $s_*(q)$ also increases, i.e., new real zeros might enter the interval J from the left.

Thus for $q \in (\tilde{q}_k, \tilde{q}_{k+1})$, one has $Z_J(q) \leq t_1 + 2$, where t_1 is the quantity t_0 defined for $k+1$ instead of k, hence $Z_J(q) = o(k)$. Indeed, the numbers $-q^{-2s+1}$ increase with q. We cannot claim that if for $s = s_*(q) - 1$, one has $-q^{-2s+1} \geq -e^\pi$, then the zeros ξ_{2s-1} and ξ_{2s} are larger or smaller than $-e^\pi$; this is why 2 is added to t_1. □

The proof of part (1) of Theorem 1.6 results from part (3) of Lemma 3.1. Indeed, one has $k = O(1/(1 - \tilde{q}_k))$, see (1.2). □

Part (2). The function θ satisfies the following functional equation

$$\theta(q,x) = 1 + qx\theta(q,qx). \tag{3.1}$$

For $q = \tilde{q}_k \in \Gamma$, we denote by $\cdots < x_2 < x_1 < x_0 < 0$ the numbers $x_0 = \tilde{y}_k$, $x_s = x_{s-1}/\tilde{q}_k$, $s \in \mathbb{N}$ (i.e., $x_s = \tilde{y}_k/(\tilde{q}_k)^s$). Hence $\theta(\tilde{q}_k, x_0) = 0$, $\theta(\tilde{q}_k, x_1) = 1 + x_0\theta(\tilde{q}_k, x_0) = 1$ (see (3.1)), and for $s > 1$,

(i) if $\theta(\tilde{q}_k, x_s) < 0$, then $\theta(\tilde{q}_k, x_{s+1}) = 1 + x_s\theta(\tilde{q}_k, x_s) > 1$;

(ii) if $\theta(\tilde{q}_k, x_s) \geq 1$ (this is the case for $s = 1$), then for k sufficiently large, one has $x_s < -e^\pi/2$ (see (1.2)), $\tilde{q}_k \in (0.3, 1)$ (see parts (1) and (2) of Remarks 1.2) hence $\tilde{q}_k x_s < -0.3 \times e^\pi/2 < -3$ and

$$\theta(\tilde{q}_k, x_{s+1}) = 1 + \tilde{q}_k x_s \theta(\tilde{q}_k, x_s) < 1 - 3 = -2 < 0.$$

Thus for k sufficiently large, we have $\theta(\tilde{q}_k, x_s) < 0$ for $s \geq 2$ even and $\theta(\tilde{q}_k, x_s) > 0$ for $s \geq 3$ odd. Hence each interval (x_{s+1}, x_s) contains a zero of θ. For a fixed interval $[-a, -e^\pi]$, consider the intervals (x_{s+1}, x_s) which are its subintervals. As $k \to \infty$ (hence $\tilde{q}_k \to 1^-$) the lengths of these intervals tend uniformly to 0. Indeed, the largest of them is the last one and its length is $\leq (a - aq) = (1-q)a$. Therefore for any $a > e^\pi$, the set of zeros of $\theta(\tilde{q}_k, .)$ (over all k sufficiently large) is everywhere dense in the interval $[-a, -e^\pi]$. This proves the first claim of part (2) of the theorem. To prove the second one we first consider the case $q = \tilde{q}_k \in \Gamma$. We define the quantities u_0, $u_1 \in \mathbb{N}$ by the conditions

$$|\tilde{y}_k|/q^{u_0} = |x_{u_0}| \leq e^\pi < |x_{u_0+1}| = |\tilde{y}_k|/q^{u_0+1},$$

and

$$|\tilde{y}_k|/q^{u_1} = |x_{u_1}| \leq a < |x_{u_1+1}| = |\tilde{y}_k|/q^{u_1+1}.$$

Hence (remember that $\ln q < 0$)

$$(u_0+1)\ln q < \ln(|\tilde{y}_k|/e^\pi) \leq u_0 \ln q \quad \text{and} \quad (u_1+1)\ln q < \ln(|\tilde{y}_k|/a) \leq u_1 \ln q$$

which, taking into account that as $q \to 1^-$, implies $\ln q = \ln(1 + (q-1)) = (q-1) + o(q-1)$, one has

$$u_0(q-1) = \ln(|\tilde{y}_k|/e^\pi) + o(q-1) \quad \text{and} \quad u_1(q-1) = \ln(|\tilde{y}_k|/a) + o(q-1).$$

It is clear that $\ell_a(q) = u_1 - u_0 + O(1)$. Thus

$$\ell_a(q)(1-q) = (u_1 - u_0)(1-q) + O(1)(1-q) = \ln(a/e^\pi) + O(1-q).$$

Now suppose that $q \in (\tilde{q}_k, \tilde{q}_{k+1})$. Our reasoning is similar to the one in the proof of Lemma 3.1. The double zero $\tilde{y}_k$ gives birth to a complex conjugate pair, so two real zeros are lost. If for $q = q_* \in (\tilde{q}_k, \tilde{q}_{k+1})$, the interval $(-q_*^{-2s}, -q_*^{-2s+1})$ is a subset of the interval $[-a, 0)$, then the same is true for $q = \tilde{q}_{k+1}$. Thus

$$\ell_a(q_*) \leq Z_{[-a,0)}(\tilde{q}_{k+1}) + 2. \tag{3.2}$$

One adds 2 in order to take into account the two zeros of $\theta(\tilde{q}_{k+1},.)$ of the not more than one interval $(-q_*^{-2s},-q_*^{-2s+1})$ which belongs partially, but not completely, to $[-a,0)$. The number 2 of lost zeros and the number 2 in (3.2) are $o(1/(1-q_*))$. According to part (1) of the theorem

$$Z_{[-a,0)}(\tilde{q}_{k+1})=\ell_a(\tilde{q}_{k+1})+o(1/(1-q_*)),$$

and for $q=\tilde{q}_{k+1}$, it was shown that $(1-\tilde{q}_{k+1})\ell_a(\tilde{q}_{k+1})=\ln(a/e^{\pi})+o(1)$, so $\ell_a(q_*)=\ell_a(\tilde{q}_{k+1})+o(1/(1-q_*))$ and $(1-q_*)\ell_a(q_*)=\ln(a/e^{\pi})+o(1)$ which proves part (2) of the theorem. □

Part (3). We need the following lemma.

Lemma 3.2. *Suppose that $q\in(-1,0)$ and set $\rho:=|q|$. Then:*

(1) *For $\rho>0$ small enough, one has*

$$\begin{array}{lll}\xi_{4s}\in(-\rho^{-4s-1},-\rho^{-4s+1}), & & \xi_{4s+2}\in(-\rho^{-4s-3},-\rho^{-4s-1}),\\ \xi_{4s-1}\in(\rho^{-4s+2},\rho^{-4s}), & \text{and} & \xi_{4s+1}\in(\rho^{-4s},\rho^{-4s-2}).\end{array}\tag{3.3}$$

Moreover, the mentioned zeros ξ_j are the only zeros of $\theta(q,.)$ in the indicated intervals.

(2) *For $\rho\in(0,1)$, one has $\theta(q,-q^{-2k})=\theta(q,-\rho^{-2k})\in(0,\rho^{2k}+\rho^{4k+1})$.*

(3) *For $q\in[\bar{q}_{2s-1},0)$, the zeros ξ_{4s-2} and ξ_{4s} belong to the interval $I^{\bullet}:=(-\rho^{-4s},-\rho^{-4s+2})$.*

Proof. Part (1). We consider the following four series:

$$\theta^{\diamond}:=\theta(-\rho,-\rho^{-4s+1})=\sum_{j=0}^{\infty}d_j,\quad d_j:=(-1)^{j(j+3)/2}\rho^{-(4s-1)j+j(j+1)/2},$$

$$\theta^{\nabla}:=\theta(-\rho,-\rho^{-4s-1})=\sum_{j=0}^{\infty}h_j,\quad h_j:=(-1)^{j(j+3)/2}\rho^{-(4s+1)j+j(j+1)/2},$$

$$\theta^{\heartsuit}:=\theta(-\rho,\rho^{-4s})=\sum_{j=0}^{\infty}r_j,\qquad r_j:=(-1)^{j(j+1)/2}\rho^{-4sj+j(j+1)/2},\quad\text{and}$$

$$\theta^{\star}:=\theta(-\rho,\rho^{-4s+2})=\sum_{j=0}^{\infty}\lambda_j,\qquad \lambda_j:=(-1)^{j(j+1)/2}\rho^{-(4s-2)j+j(j+1)/2}.$$

For the first series, its terms of largest modulus are d_{4s-1} and d_{4s-2}; one has $d_{4s-1}=d_{4s-2}=-\rho^{-8s^2+6s-1}$. The moduli of the terms decrease rapidly as $j>4s-1$ increases or as $j<4s-2$ decreases. In this series the sign $(-1)^{j(j+3)/2}$ is positive for $j=4\nu$ and $j=4\nu+1$ and negative for $j=4\nu+2$ and $j=4\nu+3$. Hence for ρ small enough, the sign of $\theta^{\diamond}$ is the same as the one of $d_{4s-1}+d_{4s-2}$, i.e., one has $\theta^{\diamond}<0$.

For the other three series the largest modulus terms are respectively $h_{4s} = h_{4s+1} = \rho^{-8s^2-2s} > 0$, $r_{4s-1} = r_{4s} = \rho^{-8s^2+2s} > 0$ and $\lambda_{4s-3} = \lambda_{4s-2} = -\rho^{-8s^2+10s-3} < 0$, so in the same way $\theta^{\nabla} > 0$, $\theta^{\heartsuit} > 0$ and $\theta^{\star} < 0$. Hence there is at least one zero of θ in the interval $(-\rho^{-4s-1}, -\rho^{-4s+1})$. In fact, there is exactly one zero, and this is ξ_{4s}. Indeed, for ρ small enough this is true, because one has $\xi_m \sim -q^{-m}$, see [12] (the zeros ξ_{4s-1} and ξ_{4s+1} are positive, so only ξ_{4s} belongs to $(-\rho^{-4s-1}, -\rho^{-4s+1})$). For any $\rho \in (0,1)$, this follows from the fact that as ρ increases, new complex conjugate pairs are born, but the inverse does not take place, see part (2) of Remarks 1.2. In the same way one proves the rest of part (1) of the lemma.

Part (2). One checks directly that

$$\theta(q, -\rho^{-2k}) = \sum_{j=0}^{\infty} q^{j(j+1)/2}(-\rho^{-2k})^j = \sum_{j=0}^{\infty} (-1)^{j(j+3)/2} \rho^{-2kj+j(j+1)/2}$$
$$= \sum_{j=4k}^{\infty} (-1)^{j(j+3)/2} \rho^{-2kj+j(j+1)/2}.$$

The last of these equalities follows from the fact that the first $4k$ terms of the series cancel (the first with the $(4k)$th, the second with the $(4k-1)$st, etc.). The signs of the terms of the last of these series are $+,+,-,-,+,+,-,-,\cdots$ and the exponents $-2kj + j(j+1)/2$ are increasing for $j \geq 4k$. Hence the series is the sum of two Leibniz series with positive first terms, so its sum is positive and not larger than the sum of the first terms of these two series. The latter sum is $\rho^{2k} + \rho^{4k+1}$ which proves part (2).

Part (3). For ρ sufficiently small, the zeros ξ_{4s-2} and ξ_{4s} belong to $I^{\bullet}$. Indeed, by part (2) of the present lemma, at the endpoints of $I^{\bullet}$ the function $\theta(q,.)$ is positive while it is negative at $-\rho^{-4s+1}$ (we showed already that $\theta^{\diamond} < 0$). As $\theta(q,.)$ is positive at the endpoints for any $q \in (-1,0)$, the zeros ξ_{4s-2} and ξ_{4s} belong to $I^{\bullet}$ exactly for $q \in [\bar{q}_{2s-1}, 0)$, see part (3) of Remarks 1.3. This proves part (3) of Lemma 3.2. □

Suppose first that $q = \bar{q}_{2\nu-1}$, $\nu \in \mathbb{N}$. The rightmost of the negative zeros of $\theta(\bar{q}_{2\nu-1},.)$ is the double zero $\bar{y}_{2\nu-1} = \xi_{4\nu-2} = \xi_{4\nu}$, see part (3) of Remarks 1.3. Denote by $s^{\dagger} = s^{\dagger}(\bar{q}_{2s-1})$ the largest of the numbers $s \in \mathbb{N}$ for which one has $-(\bar{q}_{2\nu-1})^{-4s} \geq -e^{\pi/2}$. Hence the zero $\xi_{4s^{\dagger}}$ is in the interval $[-e^{\pi/2}, 0)$ and the zero $\xi_{4(s^{\dagger}+1)}$ is to its left, i.e., outside it. Thus the number $\tilde{N}(\bar{q}_{2\nu-1}) := Z_{[-e^{\pi/2},0)}(\bar{q}_{2\nu-1})$ (the zeros in $[-e^{\pi/2}, 0)$ have only even indices i, see Remarks 1.3) is

$$\tilde{N}(\bar{q}_{2\nu-1}) = (4s^{\dagger} - 4\nu + 2)/2 + u = 2(s^{\dagger} - \nu) + 1 + u,$$

where $u \leq 1$ (the presence of the number u reflects the fact that we do not say whether the zero $\xi_{4s^{\dagger}+2}$ belongs or not to the interval $[-e^{\pi/2}, 0)$). The conditions

$$-(\bar{q}_{2\nu-1})^{-4(s^{\dagger}+1)} < -e^{\pi/2} \leq -(\bar{q}_{2\nu-1})^{-4(s^{\dagger})}$$

are equivalent to $-4(s^\dagger+1)\ln|\bar{q}_{2\nu-1}|>\pi/2\geq -4s^\dagger\ln|\bar{q}_{2\nu-1}|$ or to

$$\begin{cases} 4(s^\dagger+1)>(\pi/2)/(\ln(1/|\bar{q}_{2\nu-1}|))=(\pi/2)/(\ln(1+\pi/(8(2\nu-1))+o(1/\nu))) \\ \qquad\qquad =4\nu+O(1), \\ \qquad 4s^\dagger\leq(\pi/2)/(\ln(1/|\bar{q}_{2\nu-1}|)), \end{cases}$$

see the first of formulae (1.3). Thus

$$s^\dagger=\nu+O(1) \quad \text{and} \quad \tilde{N}(\bar{q}_{2\nu-1})=O(1). \tag{3.4}$$

One can also write $\tilde{N}(\bar{q}_{2\nu-1})=o(\nu)=o(1/(1+\bar{q}_{2\nu-1}))$. Hence $\tilde{N}(\bar{q}_{2\nu+1})=o(\nu)$.

Now suppose that $q\in(\bar{q}_{2\nu+1},\bar{q}_{2\nu-1})$. When counting the zeros ξ_i in the interval $[-e^{\pi/2},0)$ one takes into account that the double zero $\xi_{4\nu-2}=\xi_{4\nu}$ is lost (it gives birth to a complex conjugate pair). The numbers $-\rho^{-4s}$ (which are left endpoints of intervals $I^\bullet$) increase, so new zeros ξ_i might enter the interval $[-e^{\pi/2},0)$ from the left. The number of such intervals $I^\bullet$ which belong entirely to $[-e^{\pi/2},0)$ is not greater than their number for $q=\bar{q}_{2\nu+1}$. There is at most one interval $I^\bullet$ which belongs only partially to $[-e^{\pi/2},0)$, so ignoring it means not counting at most 2 zeros $\xi_i\in[-e^{\pi/2},0)$. Therefore $\tilde{N}(q)=\tilde{N}(\bar{q}_{2\nu+1})+O(1)=o(\nu)=o(1/(1+q))$. Part (3) of Theorem 1.6 is proved. □

Part (4). Consider an interval of the form $[-a,-e^{\pi/2}]$ and its subinterval $(-a^*,-a^\triangle)$, $e^{\pi/2}<a^\triangle<a^*<a$. For $\nu\in\mathbb{N}$ sufficiently large, the double zero $\bar{y}_{2\nu-1}=\xi_{4\nu-2}=\xi_{4\nu}$ of $\theta(\bar{q}_{2\nu-1},.)$ is to the right of $-a^\triangle$ (see the second of formulae (1.3)) and there exists an interval of the form $I^\bullet$ (see Lemma 3.2) such that $I^\bullet\subset(-a^*,-a^\triangle)$. Indeed, the length of $I^\bullet$ equals $\rho^{-4s}(1-\rho^2)$. For each s sufficiently large, one can choose $\rho\in(0,1)$ such that

$$-\rho^{-4s}\in(-a^*,(-a^*-a^\triangle)/2). \tag{3.5}$$

If one chooses a larger s, then one can achieve condition 3.5 by choosing ρ closer to 1. This means that, as ρ^{-4s} remains bounded, the length of $I^\bullet$ tends to 0 and one can attain both conditions (3.5) and $-\rho^{-4s+2}\in(-a^*,-a^\triangle)$. Thus $\xi_{4s-2},\xi_{4s}\in(-a^*,-a^\triangle)$, see part (3) of Lemma 3.2. This proves the first claim of part (4) of Theorem 1.6.

To prove the second claim, for $q^*\in(-1,0)$, we denote by $s^\sharp(q^*)$ the value of $s\in\mathbb{N}$ corresponding to the leftmost of the numbers $-(q^*)^{-4s}$ belonging to the interval $[-a,0)$. In the proof of part (4) of Theorem 1.6 we set $\rho:=|q^*|$, so $-(q^*)^{-4s}=-\rho^{-4s}$. Hence

$$\lim_{\rho\to1^-}(-\rho^{-4s^\sharp(q^*)})=-a, \quad -\rho^{-4s^\sharp(q^*)}>-a \quad \text{and} \quad -\rho^{-4(s^\sharp(q^*)+1)}<-a.$$

From the latter two inequalities, having in mind that $\ln(1/\rho)=(1-\rho)+o(1-\rho)$, one gets

$$s^\sharp(q^*)\sim(\ln a)/(4(1-\rho)). \tag{3.6}$$

Now we partition the zeros of $\theta(q^*,.)$ with negative real parts in several sets (we remind that there are no zeros of $\theta(q^*,.)$ on the imaginary axis for any $q^*\in(-1,0)$, see [16]):

1) The set S_∞ of zeros ξ_j belonging to the intervals $I^\bullet$ with $s \geq s^\sharp(q^*)+2$. These zeros (when considered as depending continuously on $q \in [q^*, 0)$) are real and do not belong to the interval $[-a, 0)$ for any $q \in [q^*, 0)$.

2) The set S_0 of the two zeros of the interval $I^\bullet$ with $s = s^\sharp(q^*)+1$.

3) The set S_R of the other real negative zeros of $\theta(q^*, .)$. We subdivide this set into $S_R([-a, -e^{\pi/2}])$ and $S_R((-e^{\pi/2}, 0))$ of zeros belonging to the respective intervals.

4) The set S_I of the complex conjugate pairs of zeros of $\theta(q^*, .)$ which have negative real parts. For $q^* \in (\bar{q}_{2\nu+1}, \bar{q}_{2\nu-1})$, their number is ν. For $q^* < 0$ close to zero, the zeros of the set S_I are real and belong to intervals $I^\bullet$, and as q^* decreases, they form complex conjugate pairs, see Remarks 1.3.

By abuse of notation we denote by the same symbols sets (e.g. S_I, S_R, etc.) and the number of zeros of θ which they contain. We remind that the numbers $n_a(q^*)$ and $s^\dagger(q^*)$ are defined in Notation 1.5 and in the proof of part (3) of the present theorem respectively; the number $s^\dagger(q^*)$ satisfies the first of conditions (3.4). Hence for $q^* \in (\bar{q}_{2\nu+1}, \bar{q}_{2\nu-1})$, one has

$$n_a(q^*) = S_R([-a, -e^{\pi/2}]) + A, \tag{3.7}$$

where $A = 0$, 1 or 2 is the number of zeros of the set S_0 which belong to the interval $[-a, -e^{\pi/2}]$. On the other hand,

$$S_R([-a, -e^{\pi/2}]) = 2s^\sharp(q^*) - S_R((-e^{\pi/2}, 0)) - S_I. \tag{3.8}$$

Recall that $S_I = 2\nu$. By the first of equations (3.4) one has $\nu = s^\dagger(q^*) + O(1)$, and by part (3) of the present theorem one has $S_R((-e^{\pi/2}, 0)) = o(\nu)$. That's why equations (3.7) and (3.8) imply

$$n_a(q^*) = 2s^\sharp(q^*) - 2s^\dagger(q^*) + o(\nu). \tag{3.9}$$

The factor 2 corresponds to the fact that there are two zeros ξ_i in the interval $I^\bullet$. One can apply formula (3.6) with $a = e^{\pi/2}$ to obtain $s^\dagger(q^*) \sim \ln(e^{\pi/2})/(4(1-\rho))$ and from (3.9) one concludes that $n_a(q^*) = (\ln(a/e^{\pi/2}))/(2(1-\rho)) + o(1/(1-\rho))$ from which part (4) of the theorem follows. □

Parts (5) *and* (6). We begin by proving the first claim of part (6); in this part of the proof we write q instead of $\bar{q}_{2s}$. For any $\varepsilon > 0$, there exists $s^\nabla \in \mathbb{N}$ such that for $s \geq s^\nabla$, one has $\bar{y}_{2s} \in (e^{\pi/2} - \varepsilon,\ e^{\pi/2} + \varepsilon)$, see formulae (1.3). We assume that $\varepsilon < 1/2$, so $\bar{y}_{2s} > 3$. For $s \geq s^\nabla$, we set $x_j := \bar{y}_{2s}/q^j$, $j \in \mathbb{N}$. One has $\theta(q, x_0) = 0$, $x_{2m} > 0$, $x_{2m+1} < 0$ and $|x_j| > 3$. Therefore

$$\begin{aligned}
&\theta(q, x_1) = 1 + x_0\theta(q, x_0) = 1 > 0,\\
&\theta(q, x_2) = 1 + x_1\theta(q, x_1) = 1 + x_1(1 + x_0\theta(q, x_0)) < 1 - 3 = -2 < 0,\\
&\theta(q, x_3) = 1 + x_2\theta(q, x_2) < -2 < 0 \quad \text{and}\\
&\theta(q, x_4) = 1 + x_3\theta(q, x_3) > 2 > 0.
\end{aligned}$$

In the same way one shows that $\theta(q, x_{4m+2}) < -2 < 0$ and $\theta(q, x_{4m+4}) > 2 > 0$. Hence at least one zero of $\theta(q, .)$ belongs to the interval (x_{4m+2}, x_{4m+4}). The longest of these intervals for which $x_{4m+2} \in [e^{\pi/2} + \varepsilon, a]$ is the last one, i.e., the one with largest value of m. Its length is $\leq a((1/q^2) - 1)$ which quantity tends to 0 as $q \to -1^+$ (i.e., as $s \to \infty$). Hence the zeros of $\theta(q, .)$ are everywhere dense in the interval $[e^{\pi/2} + \varepsilon, a]$, and as $\varepsilon > 0$ is arbitrary, they are everywhere dense in $[e^{\pi/2}, a]$. This proves the first claim of part (6).

To prove part (5) we observe that for $x \in (\xi_{4s+4}, \xi_{4s+2})$, one has $\theta(q, x) \leq 0$ and according to (3.1), $\theta(q, qx) = \theta(q, x)/(qx) - 1/(qx) < 0$ (because $qx > 0$). Hence $(q\xi_{4s+2}, q\xi_{4s+4}) \subset (\xi_{4s+1}, \xi_{4s+3})$, see [13, Fig. 3] (in [13] the latter inclusion is proved only for $q \in [-0.108, 0)$; for any $q \in (-1, 0)$, provided that the zeros ξ_{4s+1}, ξ_{4s+2}, ξ_{4s+3} and ξ_{4s+4} are real, it follows by continuity). Thus

$$Z_{(0,e^{\pi/2}]}(q) = Z_{[-e^{\pi/2}/|q|,0)}(q) + B = Z_{[-e^{\pi/2},0)}(q) + Z_{[-e^{\pi/2}/|q|,-e^{\pi/2})}(q) + B,$$

where $B = -1$, 0 or 1 indicates that the count might not concern the leftmost zero in $[-e^{\pi/2}/|q|, 0)$ and/or the rightmost zero in $(0, e^{\pi/2}]$. By parts (3) and (4) of the present theorem each of the summands $Z_{[-e^{\pi/2},0)}(q)$ and $Z_{[-e^{\pi/2}/|q|,-e^{\pi/2})}(q)$ is $o(1/(1 + q))$ which proves part (5). In the same way one proves the second claim of part (6) as well:

$$Z_{[e^{\pi/2},a]}(q) = Z_{[-a/|q|,-e^{\pi/2}/|q|]}(q) + B = Z_{[-a/|q|,-e^{\pi/2}]}(q) - Z_{(-e^{\pi/2}/|q|,-e^{\pi/2}]}(q) + B,$$

where $Z_{(-e^{\pi/2}/|q|,-e^{\pi/2}]}(q) = o(1/(1 + q))$ and

$$Z_{[-a/|q|,-e^{\pi/2}]}(q) = (\ln((a/|q|)/e^{\pi/2})/2)/(1+q) = (\ln(a/e^{\pi/2})/2)/(1+q) + o(1/(1+q)).$$

The theorem is proved. □

References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part II, Springer, New York, 2009.

[2] B. C. Berndt and B. Kim, Asymptotic expansions of certain partial theta functions, Proc. Amer. Math. Soc. 139(11) (2011) 3779–3788.

[3] K. Bringmann, A. Folsom and A. Milas, Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters, J. Math. Phys. 58(1) (2017) 011702, 19 pp.

[4] K. Bringmann, A. Folsom and R. C. Rhoades, Partial theta functions and mock modular forms as q-hypergeometric series, Ramanujan J. 29(1–3) (2012) 295–310.

[5] T. Creutzig, A. Milas and S. Wood, On regularised quantum dimensions of the singlet vertex operator algebra and false theta functions, Int. Math. Res. Not. 5 (2017) 1390–1432.

[6] G. H. Hardy, On the zeros of a class of integral functions, Messenger of Mathematics 34 (1904) 97–101.

[7] J. I. Hutchinson, On a remarkable class of entire functions, Trans. Amer. Math. Soc. 25 (1923) 325–332.

[8] O. M. Katkova, T. Lobova and A. M. Vishnyakova, On power series having sections with only real zeros, Comput. Methods Funct. Theory 3(2) (2003) 425–441.
[9] V. P. Kostov, On the zeros of a partial theta function, Bull. Sci. Math. 137(8) (2013) 1018–1030.
[10] V. P. Kostov, Asymptotics of the spectrum of partial theta function, Revista Mat. Complut. 27(2) (2014) 677–684.
[11] V. P. Kostov, A property of a partial theta function, C. R. Acad. Bulg. Sci. 67(10) (2014) 1319–1326.
[12] V. P. Kostov, Asymptotic expansions of zeros of a partial theta function, C. R. Acad. Bulg. Sci. 68(4) (2015) 419–426.
[13] V. P. Kostov, On a partial theta function and its spectrum, Proc. Royal Soc. Edinb. A 146(3) (2016) 609–623.
[14] V. P. Kostov, On the double zeros of a partial theta function, Bull. Sci. Math. 140(4) (2016) 98–111.
[15] V. P. Kostov, A separation in modulus property of the zeros of a partial theta function, Analysis Mathematica 44(4) (2018) 501–519.
[16] V. P. Kostov, On the zero set of the partial theta function, Serdica Math. J. 45 (2019) 225–258.
[17] V. P. Kostov and B. Shapiro, Hardy-Petrovitch-Hutchinson's problem and partial theta function, Duke Math. J. 162(5) (2013) 825–861.
[18] I. V. Ostrovskii, On zero distribution of sections and tails of power series, Israel Math. Conf. Proceedings 15 (2001) 297–310.
[19] M. Petrovitch, Une classe remarquable de séries entières, Atti del IV Congresso Internationale dei Matematici, Roma, Ser. 1, 2 (1908) 36–43.
[20] A. Sokal, The leading root of the partial theta function, Adv. Math. 229(5) (2012) 2603–2621.
[21] S. O. Warnaar, Partial theta functions. I. Beyond the lost notebook, Proc. London Math. Soc. 87(2) (2003) 363–395.

Received on February 3, 2025
Revised on June 16, 2025
Accepted on June 26, 2025

Vladimir Petrov Kostov

Université Côte d'Azur
CNRS, LJAD
FRANCE

E-mail: `vladimir.kostov@unice.fr`