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ON THE CONNECTION BETWEEN FIXED POINT THEOREMS ON METRIC SPACES WITH GRAPHS AND $\mathbb{P}$ SETS

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The Banach contraction principle is one of the most famous and applied results in recent mathematical history. Due to its utility, plenty of generalizations have been established. One of them considers a contraction principle on metric spaces with graphs, while another confines the contraction principle to pairs of elements inside a $\mathbb{P}$ set, a generalization of partial orders. In this work we examine the similarities of both approaches, establishing the connection between theorems of metric spaces with graphs and metric spaces with $\mathbb{P}$ sets and restating results from one approach to the other and vice versa.

Keywords: fixed point, metric space endowed with a graph, $\mathbb{P}$ sets

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1. Introduction

The Banach Contraction Principle [1] has long served as a cornerstone in fixed point theory, guaranteeing the existence and uniqueness of fixed points for certain classes of self-maps defined on complete metric spaces. Over time, considerable effort has been invested in extending and generalizing this principle to encompass broader settings. Two notable extensions focus on restricting the contraction condition to certain elements of the metric space. One of these involves the study of mappings defined on metric spaces endowed with a directed graph, where the notion of a G-contraction is used to yield refined fixed point results. The other is concerned with $\mathbb{P}$ sets, a binary relation structure allowing the contraction condition to be imposed in a more flexible and generalized manner.

Studies of fixed points in graph-based frameworks demonstrate that adding a directed graph G to a metric space (X, ρ) enriches the classical theory. This approach

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leads to fixed point theorems that unify and extend numerous known results obtained for mappings defined on partially ordered sets, cone metric spaces, or ordered normed spaces [7,9,11,14]. Such graph-based results have broadened the applicability of the Banach Contraction Principle, enabling one to treat nonlinear integral equations, coupled fixed points, and iterative approximation in function spaces within a unified setting.

Simultaneously, the introduction of $\mathbb{P}$ sets has provided a versatile tool to handle fixed point problems where the contraction condition may vary depending on the points under consideration. These structures are inspired by results in partially ordered metric spaces, first introduced in [17] and later popularized in [2,15]. By abstracting the partial order as a $\mathbb{P}$ set, researchers have proven natural formulation of contractive iterates, generalizing classical assumptions and linking them to iterative processes often seen in nonlinear analysis [3–5,12,13]. This setting encapsulates a wide spectrum of known contraction-type maps, from single-valued mappings on metric spaces to more sophisticated structures that underlie iterative approximation schemes. A deep observation in [12] makes a connection between fixed points and coupled fixed points, utilizing $\mathbb{P}$ sets.

Although these two directions (graph-based fixed point theory and $\mathbb{P}$-based approaches) originate from different motivations and employ distinct technical tools, they share a profound conceptual similarity. Both paradigms embed the classical contraction condition into a richer structural environment, capitalizing on additional relational properties to produce more general fixed point results. The key question this paper addresses is how these two seemingly different approaches relate to each other. We aim to demonstrate that many results obtained in the context of G-contractions have direct analogues within the $\mathbb{P}$ set framework, and vice versa.

By examining the conditions and conclusions of fixed point theorems in both settings, we establish a correspondence between the assumptions on the graph G and those on the $\mathbb{P}$ sets. This correspondence allows us to transfer results, insights, and techniques from one realm to the other, thereby yielding a unified perspective on fixed point theory that transcends the particularities of the chosen framework. Such a unification not only streamlines the existing theory but also opens new avenues for research, enabling known fixed point principles to be translated and applied in broader contexts.

In what follows, we present a reformulation of several theorems, translating from G-contractions to $\mathbb{P}$ sets and the other way around, highlighting their equivalences in terms of existence, uniqueness, and ordered structural properties of fixed points. The results herein show that the interplay between metric completeness, contractive behavior, and additional relational structures, be it a directed graph or a binary relation, gives rise to a more comprehensive and robust theory of fixed points.

2. Preliminaries

In what follows, we will use the notation $\mathbb{N}$ for the natural numbers ($\mathbb{N} = 1, 2, \dots$), $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rational numbers, $\mathbb{R}$ for the reals, (X, ρ) for a metric space with a metric ρ, G for a graph a metric space is endowed with, f for a mapping from X to X when discussing metric spaces endowed with a graph, T for a mapping from X to X when discussing metric spaces with a $\mathbb{P}$ set and F for mappings from $X \times X$ to X for both metric spaces endowed with a graph and those with a $\mathbb{P}$ set.

2.1. Metric spaces endowed with a graph

In this subsection we follow the exposition in [7].

Definition 2.1 ([12])**.** Let (X, ρ) be a metric space. Two sequences $x_n, y_n \in X$, $n \in \mathbb{N}$ are said to be Cauchy equivalent if $\lim_{n\to\infty} \rho(x_n, y_n) = 0$.

Whenever we discuss a metric space (X, ρ) endowed with a graph, it will be understood that G is a weighted directed graph with a set of vertices $V(G) = X$ and an edge set $E(G) \subseteq X \times X$, where the weights of the edges will be calculated as the distance between their endpoints. We will also require some general notions from graph theory.

By G^{-1} we will denote the conversion of G, i.e., $V(G^{-1}) = V(G)$ and

$$E(G^{-1}) = \{(x, y) \in X \times X \colon (y, x) \in E(G)\}.$$

Further, $\widetilde{G}$ will be the undirected graph obtained from G, that is, $V(\widetilde{G}) = V(G)$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. A subgraph of G is called a graph (V', E') such that $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for each edge $(x, y) \in E'$, it holds that $x, y \in V'$.

If x and y are vertices of G, then a path of length n, $n \in \mathbb{N} \cup \{0\}$ is a sequence of vertices $\{x_i\}_{i=0}^n$ such that

$$x_0 = x,\ x_n = y,\ (x_{i-1}, x_i) \in E(G) \text{ for } i = 1, 2, \ldots, n.$$

A graph is said to be connected if there is a path between any two vertices. Given that $\widetilde{G}$ is connected, G is weakly connected. If the edge set $E(G)$ of a graph G is symmetric, then the component of G containing a vertex x is defined as the subgraph G_x that includes all vertices and edges that lie on a path starting from x. By $[x]_G$ we will denote the equivalence class induced by the relation R defined on $V(G)$ as

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

It follows that $V(G_x) = [x]_G$. Let us also point out that $\widetilde{G}$ clearly has a symmetric edge set.

Definition 2.2 ([7])**.** Let (X, ρ) be a metric space endowed with a graph G. We say that a mapping $f \colon X \to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,

$$\text{for all } x, y \in X((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$

$$((x, y) \in E(G) \Rightarrow d(fx, fy) \leq \alpha\rho(x, y)).$$

Let us recollect that the self-map f is said to be a Picard operator (PO) if for every $x \in X$, we have that $\lim_{n\to\infty} f^n(x) = x^*$, where $x^* \in X$ is the unique fixed point of the operator. A weaker notion is that of a weak Picard operator, that is, for the mapping f it holds that $\lim_{n\to\infty} f^n(x)$ is convergent for all $x \in X$ to a fixed point of f that may not be unique.

For the following theorem to hold, the assumption that $(x, x) \in E(G)$ for all $x \in X$ is made.

Theorem 2.3 ([7])**.** *Let (X, ρ) be a complete metric space and G be a directed graph on X. Assume that (X, ρ, G) satisfies the following property:*

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k.

Let $f\colon X \to X$ be a G-contraction and define

$$X_f := \{x \in X \colon (x, f(x)) \in E(G)\}.$$

Then the following hold:

(1) $\operatorname{card}(\operatorname{Fix}(f)) = \operatorname{card}\big(\{[x]_{\widetilde{G}} \colon x \in X_f\}\big)$.

(2) $\operatorname{Fix}(f) \neq \varnothing \iff X_f \neq \varnothing$.

(3) *f has a unique fixed point if and only if there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*

(4) *For every $x \in X_f$, the restriction $f|_{[x]_{\widetilde{G}}}$ is PO (Picard operator).*

(5) *If $X_f \neq \varnothing$ and G is weakly connected, then f is PO.*

(6) *If $X' = \bigcup_{x\in X_f} [x]_{\widetilde{G}}$, then $f|_{X'}$ is WPO (Weak Picard operator).*

(7) *If $f \subseteq E(G)$, then f is WPO.*

2.2. Metric spaces endowed with a $\mathbb{P}$ set

In order to state a part of the results, proven in the context of metric spaces with a $\mathbb{P}$ set, the following definitions are oftentimes used, following the exposition in [12, 13].

Definition 2.4 ([13])**.** Let X be a non-empty set, $\mathbb{P} \subseteq X \times X$ and $T\colon X \to X$ be a map. We say that $\mathbb{P}$ is T-closed if whenever $(x, y) \in \mathbb{P}$, it follows that $(Tx, Ty) \in \mathbb{P}$.

Next, we will present some examples of T-closed $\mathbb{P}$ sets.

Example 2.5 ([12])**.** Let $(X, \rho, \preccurlyeq)$ be a partially ordered metric space. Let the mapping $T\colon X \to X$ be an increasing function, i.e., $Tx \preccurlyeq Ty$, provided that $x \preccurlyeq y$. Then the set $\mathbb{P} = \{(x, y) \in X \times X \colon x \preccurlyeq y\}$ is T-closed.

Example 2.6 ([12])**.** Let $(X, \rho, \preccurlyeq)$ be a partially ordered metric space. For the mapping $T\colon X \to X$ let Tx be comparable with Ty, i.e., $Tx \asymp Ty$. Then the set $\mathbb{P} = \{(x, y) \in X \times X\colon x \asymp y\}$ is T-closed.

Example 2.7. Let us consider $\mathbb{R}$ with the usual metric. For $Tx = x^2$, we have that the set $\mathbb{P} = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon x, y \in \mathbb{Q}\}$ is T-closed.

Definition 2.8. ([13]) Let (X, ρ) be a metric space and $\mathbb{P} \subseteq X \times X$. The triple $(X, \rho, \mathbb{P})$ is said to be:

(a) i-$\mathbb{P}$-regular if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, such that for all n $(x_n, x_{n+1}) \in \mathbb{P}$, there holds $(x_n, x) \in \mathbb{P}$ for all n.

(b) d-$\mathbb{P}$-regular if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, such that for all n $(x_{n+1}, x_n) \in \mathbb{P}$, there holds $(x, x_n) \in \mathbb{P}$ for all n.

Definition 2.9 ([10])**.** Let X and Y be topological spaces. The graph of a map $T\colon X \to Y$ is the set $\{(x, y) \in X \times Y\colon y = T(x)\}$. It is said that T has a closed graph if its graph is a closed subset of $X \times Y$ endowed with the product topology.

We will also use the following notation – $T \subset \mathbb{P}$ if from $x \in X$ it follows that $(x, Tx) \in \mathbb{P}$.

Definition 2.10. ([5]) We say $\mathbb{P}$ has the transitive property on a set X if for any $x, y, z \in X$, whenever $(x, y) \in \mathbb{P}$ and $(y, z) \in \mathbb{P}$, it follows that $(x, z) \in \mathbb{P}$.

The next theorem is a generalization of the results for mappings with a contractive iterate at a point, first considered in [16] and later developed in [5, 6, 8].

Theorem 2.11 ([5])**.** *Let (X, ρ) be a complete metric space, $\mathbb{P} \subset X \times X$, $T\colon X \to X$ be a map and there hold*

(i) *$\mathbb{P}$ is T-closed and has the transitive property;*

(ii) *T either has a closed graph or the triple $(X, \rho, \mathbb{P})$ is* i-$\mathbb{P}$*-regular;*

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathbb{P}$;*

(iv) *there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \le \alpha\rho(x, y).$$

Then

(a) $\mathrm{Fix}(f) \neq \varnothing$ *and for any arbitrarily chosen $x_0 \in X$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_n = T^n x_0$ converges to an element $x^* \in \mathrm{Fix}(T)$;*

(b) *For any $x \in X$ and x_0 so that $(x_0, Tx_0) \in \mathbb{P}$, satisfying $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent and hence u_n converges to $x^* \in \mathrm{Fix}(f)$, where $x^* = \lim_{n\to\infty} T^n x_0$;*

(c) *If* $y^* \in \mathrm{Fix}(T)$ *and either* $(x_0, y^*) \in \mathbb{P}$ *or* $(y^*, x_0) \in \mathbb{P}$ *or there is* $z \in X$ *so that either* $(x_0, z), (y^*, z) \in \mathbb{P}$ *or* $(z, x_0), (z, y^*) \in \mathbb{P}$*, then* $y^* = x^*$*;*

(d) *If in addition we suppose that for every* $x, y \in X$ *such that neither* $(x, y) \in \mathbb{P}$ *nor* $(y, x) \in \mathbb{P}$ *there is* $z \in X$ *so that* $(x, z), (y, z) \in \mathbb{P}$ *or* $(z, x), (z, y) \in \mathbb{P}$*, then* $\mathrm{Fix}(T) = \{x^*\}$*.*

Proposition 2.12 ([5])**.** *Given the conditions of Theorem 2.11, we have that any two sequences* $\{T^n u_0\}_{n=0}^{\infty}$ *and* $\{T^n v_0\}_{n=0}^{\infty}$ *are Cauchy equivalent, given that* $(u_0, v_0) \in \mathbb{P}$ *or* $(v_0, u_0) \in \mathbb{P}$*.*

3. Main result

3.1. From graphs to $\mathbb{P}$ sets

Let us point out that since the proof of Theorem 2.3 depends on $V(G) = X$ and for all $x \in X$ to hold that $(x, x) \in E(G)$, to restate the result in terms of $\mathbb{P}$ sets, we would require that for every $\mathbb{P}$ set of this subsection it holds that if $x \in X$, then $(x, x) \in \mathbb{P}$.

Lemma 3.1. *Let* (X, ρ) *be a metric space and* $f\colon X \to X$ *be a self-map. Then there exists a directed graph* G *with* $E(G) \subseteq X \times X$ *such that* f *is a Banach* G*-contraction if and only if there exists* $\mathbb{P} \subseteq X \times X$ *such that* $\mathbb{P}$ *is* f*-closed and* $\alpha \in (0, 1)$ *such that*

$$\rho(fx, fy) \leq \alpha\rho(x, y)$$

for all $(x, y) \in \mathbb{P}$*.*

Proof. Let G be such a directed graph with $E(G) \subseteq X \times X$ such that f is a Banach G-contraction. Let $\mathbb{P} = E(G)$. Then

$$(x, y) \in \mathbb{P} \Rightarrow (fx, fy) \in \mathbb{P},$$

or $\mathbb{P}$ is f-closed. Also, from f being a G-contraction, we get that there exists $\alpha \in (0, 1)$ such that

$$\rho(fx, fy) \leq \alpha\rho(x, y)$$

for all $(x, y) \in \mathbb{P}$.

Now let there exist $\mathbb{P} \subseteq X \times X$ such that $\mathbb{P}$ is f-closed and there exists $\alpha \in (0, 1)$ so that $\rho(fx, fy) \leq \alpha\rho(x, y)$ for all $(x, y) \in \mathbb{P}$. Let us construct a graph G such that $(x, y) \in E(G)$ if and only if $(x, y) \in \mathbb{P}$. Then clearly f is a G-contraction. □

We introduce the following notation:

$$\mathbb{P}^{-1} = \{(x, y) \in X \times X\colon (y, x) \in \mathbb{P}\},$$
$$\widetilde{\mathbb{P}} = \mathbb{P} \cup \mathbb{P}^{-1},$$
$$\widetilde{\mathbb{P}}(x) := \{y \in X\colon \text{there exists } z_i \in X \text{ such that } (x, z_1), (z_j, z_{j+1}), (z_n, y) \in \tilde{\mathbb{P}},$$
$$i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n-1,\ n \in \mathbb{N}\}.$$

It is clear that these sets are analogues of the converse, the undirected graph and $V(G_x)$ notions in graphs.

Lemma 3.2. *Let (X,ρ) be a metric space, G be a directed graph with $E(G) \subseteq X \times X$ and $\mathbb{P} = E(G)$. Then $[x]_{\widetilde{G}} = \widetilde{\mathbb{P}}(x)$.*

Proof. Let us point out that $E(\widetilde{G}) = \widetilde{\mathbb{P}}$. Then $y \in X$ being in $[x]_{\widetilde{G}}$ means that there exists a path in $\widetilde{G}$ from x to y, or there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. This is equivalent to the existence of $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in \tilde{\mathbb{P}}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$, which means that $y \in \widetilde{\mathbb{P}}(x)$. Therefore, $[x]_{\widetilde{G}} = \widetilde{\mathbb{P}}(x)$. □

Let us first state Theorem 2.3 in terms of $\mathbb{P}$ sets.

Theorem 3.3. *Let (X,ρ) be a complete metric space, $\mathbb{P} \subseteq X \times X$ and the triple $(X, \rho, \mathbb{P})$ have the following property:*

$$\begin{gathered}\text{For any sequence } \{x_n\}_{n\in\mathbb{N}} \subseteq X, \text{ if } \lim_{n\to\infty} x_n = x \text{ and } (x_n, x_{n+1}) \in \mathbb{P} \text{ for all} \\ n, \text{ then there exists a subsequence } \{x_{n_k}\}_{k\in\mathbb{N}} \text{ such that } (x_{n_k}, x) \in \mathbb{P} \text{ for all } k.\end{gathered} \tag{3.1}$$

Let $f\colon X \to X$, $\mathbb{P}$ be f-closed, $W = \{x \in X\colon (x, fx) \in \mathbb{P}\}$ and there exist $\alpha \in (0,1)$ such that

$$\rho(fx, fy) \le \alpha\rho(x, y)$$

for all $(x,y) \in \mathbb{P}$. Then:

1. $\operatorname{card}(\operatorname{Fix}(f)) = \operatorname{card}\left(\left\{\widetilde{\mathbb{P}}(x)\colon x \in W\right\}\right)$.

2. $\operatorname{Fix}(f) \neq \varnothing \iff W \neq \varnothing$.

3. *f has a unique fixed point if and only if there exists $x_0 \in W$ such that $W \subseteq \widetilde{\mathbb{P}}(x_0)$.*

4. *For every $x \in W$, the restriction $f|_{\widetilde{\mathbb{P}}(x)}$ is PO.*

5. *If $W \neq \varnothing$ and there exists $x \in X$ such that $X \subseteq \widetilde{\mathbb{P}}(x)$, then f is PO.*

6. *If $X' = \bigcup_{x\in W} \widetilde{\mathbb{P}}(x)$, then $f|_{X'}$ is WPO.*

7. *If $f \subseteq \mathbb{P}$, then f is WPO.*

In order to present a proof, we will use the following proposition.

Proposition 3.4. *Theorem 2.3 holds if and only if Theorem 3.3 holds.*

Proof. From Lemmas 3.1 and 3.2, we see that the statements of both theorems are equivalent. The only thing we need to show is that G being weakly connected is equivalent to $X \subseteq \widetilde{\mathbb{P}}(x)$ for some $x \in X$. Indeed, let G be weakly connected, i.e., for all $x, y \in X$ there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. But that is equivalent to the statement that for all $x, y \in X$ there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in \widetilde{\mathbb{P}}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. This is true if and only if from $y \in X$ it follows that $y \in \widetilde{\mathbb{P}}(x)$, or $X \subseteq \widetilde{\mathbb{P}}(x)$. Thus, the proposition is proven. □

As a consequence of Proposition 3.4 and Theorem 2.3 having been proven, it follows that Theorem 3.3 holds as well.

Example 3.5. Let us consider $\mathbb{R}$ with the usual metric $\rho(x, y) = |x - y|$. Let $\mathbb{E} = \{x \in \mathbb{R} \colon |x| = 2^n, n \in \mathbb{Z}\} \cup \{0\}$ and let us have the map

$$Tx = \begin{cases} \frac{x}{2}, & x \in \mathbb{E}, \\ 3x + \sqrt{3}, & x \notin \mathbb{E} \end{cases}.$$

Let $\mathbb{P} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon x, y \in \mathbb{E} \text{ or } x = y\}$. Then clearly condition (3.1) holds, $\mathbb{P}$ is T-closed and the contractive condition is fulfilled with $\alpha = \frac{1}{2}$. Therefore, we can apply Theorem 3.3. Clearly, $\widetilde{\mathbb{P}} = \mathbb{P}$. If $x \in \mathbb{E}$, then $\widetilde{\mathbb{P}}(x) = \mathbb{E}$, whereas if $x \notin \mathbb{E}$, then $\widetilde{\mathbb{P}}(x) = \{x\}$. Let us consider W. If $x \in \mathbb{E}$, it holds that $Tx \in \mathbb{E}$ and $(x, Tx) \in \mathbb{P}$. If however, $x \notin \mathbb{E}$, then we have that $\left(-\frac{\sqrt{3}}{2}, T\left(-\frac{\sqrt{3}}{2}\right)\right) \in \mathbb{P}$. Therefore, $W = \mathbb{E} \cup \left\{-\frac{\sqrt{3}}{2}\right\}$.

From conclusions (1) and (2), we know that $\operatorname{card}(\operatorname{Fix}(f)) = 2$. Conclusions (3), (5) and (7) do not hold, whereas (4) and (6) do hold. If $x \in \mathbb{E}$, then $\lim_{n \to \infty} T^n x = 0$ and if $x = -\frac{\sqrt{3}}{2}$, then $T^n\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$.

If we would prefer to express this example utilizing a graph, we could construct a graph G such that $V(G) = \mathbb{R}$ and $E(G) = \mathbb{P}$. Then the results from Theorem 2.3 can be applied. Due to Proposition 3.4, we will arrive at the same conclusions as we did using Theorem 3.3.

3.2. From $\mathbb{P}$ sets to graphs

Definition 3.6. We say that a mapping $f \colon X \to X$ is G-contraction with a contractive iterate at a point if f preserves edges of G, i.e,

$$\text{for all } x, y \in X((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way: there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in E(G)$ it holds

$$\rho(f^{n(x)}(x), f^{n(x)}(y)) \leq \alpha \rho(x, y).$$

This definition is a clear analogue of Definition 2.2.

In order to produce a simpler proof of the graph version of 2.11, we will first generalize the result by replacing the i-$\mathbb{P}$-regularity with a weaker assumption via the next Lemma.

Lemma 3.7 ([7]). *Let (X, d) be a complete metric space, G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. Let $E(G)$ be transitive. Then for the triple (X, d, G) the following properties are equivalent:*

1. *for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ so that $(x_{n_k}, x) \in E(G)$ for all k;*
2. *the i-$E(G)$-regular property.*

In view of this, we can restate Theorem 2.11 in the following way.

Theorem 3.8. *Let (X, ρ) be a complete metric space, $\mathbb{P} \subset X \times X$, $T: X \to X$ be a map and there hold*

(i) *$\mathbb{P}$ is T-closed and has the transitive property;*

(ii) *T either has a closed graph or the triple $(X, \rho, \mathbb{P})$ has the following property:*

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k;

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathbb{P}$;*

(iv) *there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \leq \alpha\rho(x, y).$$

Then

(a) *$\mathrm{Fix}(f) \neq \varnothing$ and for any arbitrary chosen $x_0 \in X$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_n = T^n x_0$ converges to an element $x^* \in \mathrm{Fix}(T)$;*

(b) *For any $x \in X$ and x_0 so that $(x_0, Tx_0) \in \mathbb{P}$, satisfying $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent and hence u_n converges to $x^* \in \mathrm{Fix}(f)$, where $x^* = \lim_{n\to\infty} T^n x_0$;*

(c) *If $y^* \in \mathrm{Fix}(T)$ and either $(x_0, y^*) \in \mathbb{P}$ or $(y^*, x_0) \in \mathbb{P}$ or there is $z \in X$ so that either $(x_0, z), (y^*, z) \in \mathbb{P}$ or $(z, x_0), (z, y^*) \in \mathbb{P}$, then $y^* = x^*$;*

(d) *If in addition we suppose that for every $x, y \in X$ such that neither $(x, y) \in \mathbb{P}$ nor $(y, x) \in \mathbb{P}$ there is $z \in X$ so that $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (z, y) \in \mathbb{P}$, then $\mathrm{Fix}(T) = \{x^*\}$.*

Before we move on to the proof, let us prove the following lemmas.

Lemma 3.9. *The conditions of Theorem 2.11 and Theorem 3.8 are equivalent.*

Proof. Let us construct a graph G such that $V(G) = X$ and $E(G) = \mathbb{P}$. Then, from $\mathbb{P}$ having the transitive property, it is clear that $E(G)$ also has the transitive property. Therefore, the result quickly follows from Lemma 3.7. □

Lemma 3.10. *Given the conditions of Theorem 3.8, we have that any two sequences $\{T^n u_0\}_{n=0}^{\infty}$ and $\{T^n v_0\}_{n=0}^{\infty}$ are Cauchy equivalent, given that $(u_0, v_0) \in \mathbb{P}$ or $(v_0, u_0) \in \mathbb{P}$.*

Proof. This is a simple consequence of Lemma 3.9. □

Now we will prove Theorem 3.8.

Proof. The conditions of Theorems 2.11 and 3.8 are the same by Lemma 3.9. □

Lemma 3.11. *Let (X, d) be a metric space and $f\colon X \to X$ be a self-map. Then there exists a directed graph G with $E(G) \subseteq X \times X$ such that f is a G-contraction with a contractive iterate at a point if and only if there exists $\mathbb{P} \subseteq X \times X$ such that $\mathbb{P}$ is f-closed and there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \leq \alpha\rho(x, y).$$

Proof. The proof is analogous to the proof of Lemma 3.1. □

We will now restate Theorem 3.8 in the context of a metric space with a graph. Let us note that in the following result, we do not require that $(x, x) \in E(G)$ for any $x \in X$.

Theorem 3.12. *Let (X, ρ) be a complete metric space and G be a directed graph on X with edge set $E(G)$ and $T\colon X \to X$ be a G contraction with a contractive iterate at a point. Let $W = \{x \in X\colon (x, Tx) \in E(G)\}$ and assume that*

(I) *$E(G)$ has the transitive property;*

(II) *T has a closed graph or $(X, \rho, E(G))$ satisfies the following property:*

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k;

(III) *$W \neq \varnothing$.*

Then

(A) *$\mathrm{Fix}(f) \neq \varnothing$. If $X' = \bigcup_{x\in W} [x]_{\widetilde{G}}$, then the restriction $T|_{X'}$ is a WPO;*

(B) *For any $x \in W$, $T|_{[x]_{\widetilde{G}}}$ is a PO;*

(C) *If G is weakly connected, then T is a PO.*

Proposition 3.13. *Theorem 3.8 holds if and only if Theorem 3.12 holds.*

Proof. Without loss of generality, let $E(G) = \mathbb{P}$. From Lemmas 3.11 and 3.9, we see that the statements I and II of Theorem 3.12 are equivalent to conditions i, ii and iv of Theorem 3.8. Condition III is clearly equivalent to condition iii of Theorem 3.8.

By definition, $x \in X'$ if and only if $(x, Tx) \in E(G) = \mathbb{P}$. Thus, for any $x \in X$ such that $(x, Tx) \in \mathbb{P}$ the sequence $T^n x$ converges to an element $x^* \in \operatorname{Fix}(T)$ if and only if $T|_{X'}$ is a WPO. Thus Theorem 3.8 a is equivalent to Theorem 3.12 A.

There exists an $x \in X$ such that $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, which is equivalent to $x \in \widetilde{\mathbb{P}}(x_0)$. Therefore by Lemma 3.2, $(x_0, Tx_0) \in \mathbb{P}$ and $x \in \widetilde{\mathbb{P}}(x_0)$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent if and only if for any $x \in V$, $T_{[x]_{\widetilde{G}}}$ is a PO. Thus Theorem 3.8 b and c are equivalent to Theorem 3.12 B.

Let us examine the last condition. The graph G being weakly connected means that for all $x, y \in X$ there exist $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. This is equivalent to the existence of $z \in X$ such that $(x, z), (z, y) \in E(\widetilde{G})$. Since $E(\widetilde{G}) = \widetilde{\mathbb{P}}$, the previous statement is true if and only if for all $x, y \in X$ there exists $z \in X$ such that $(x, z), (z, y) \in \widetilde{\mathbb{P}}$, which is equivalent to $(x, z), (z, y) \in \mathbb{P}$, $(z, x), (z, y) \in \mathbb{P}$, $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (y, z) \in \mathbb{P}$. If $(x, z), (z, y) \in \mathbb{P}$, then $(x, y) \in \mathbb{P}$. If $(z, x), (y, z) \in \mathbb{P}$, then $(y, x) \in \mathbb{P}$. Thus Theorem 3.8 b and d are equivalent to Theorem 3.12 C.

The proposition is proven. □

As a consequence of Proposition 3.13 and Theorem 3.8 having been proven, it follows that Theorem 3.12 holds as well.

3.3. Coupled fixed points

We will conclude by establishing a similar connection between metric spaces endowed with a graph and metric spaces with $\mathbb{P}$ sets by considering a result concerning coupled fixed points. Let us recall that for a mapping $F\colon X \times X \to X$ a coupled fixed point $(x, y) \in X \times X$ is called a point such that $(F(x, y), F(y, x)) = (x, y)$. The analogues to $\mathbb{P}$ sets in this context are often called $\mathbb{M}$ sets.

Definition 3.14 ([12])**.** Let (X, d) be a metric space and $F\colon X \times X \to X$ be an operator. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-closed if for all $x, y, u, v \in X$ the following implication holds:

$$(x, y, u, v) \in \mathbb{M} \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}.$$

Theorem 3.15 ([12])**.** *Let (X, d) be a complete metric space, $\mathbb{M} \subseteq X^4$ and $F\colon X \times X \to X$ be an operator with a closed graph. Suppose*:

1. *$\mathbb{M}$ is F-closed*;

2. *there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$*;

3. *there is* $k \in [0,1]$ *such that if* $(x,y) \in X \times X$ *and* $(x,y,F(x,y),F(y,x)) \in \mathbb{M}$, *then*

$$d\left(F(x,y),F^2(x,y)\right)+d\left(F(y,x),F^2(y,x)\right)\leq k\left(d\left(x,F(x,y)\right)+d\left(y,F(y,x)\right)\right).$$

Then F *has at least one coupled fixed point* $(x^*,y^*) \in X \times X$ *and the sequences* $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ *and* $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ *converge to* x^* *and* y^* *respectively.*

For the restatement of this result, we will require the following definition.

Definition 3.16. Let (X,d) be a metric space and G be a directed graph on $X \times X$ with edge set $E(G)$. We say that a mapping $F\colon X \times X \to X$ is a coupled G-contraction if F preserves edges of G, i.e.,

$$\begin{gathered}\text{for all } x,y \in X(((x,y),(u,v)) \in E(G) \text{ it follows that}\\ \left(\left(F(x,y),F(y,x)\right),\left(F(u,v),F(v,u)\right)\right) \in E(G)),\end{gathered} \tag{3.2}$$

and F decreases weights of edges of G in the following way: there exists $k \in [0,1)$ so that for any $(x,y) \in X$ such that

$$((x,y),(F(x,y),F(y,x))) \in E(G),$$

it is true that

$$d\left(F(x,y),F^2(x,y)\right)+d\left(F(y,x),F^2(y,x)\right) \leq k\left(d\left(x,F(x,y)\right)+d\left(y,F(y,x)\right)\right). \tag{3.3}$$

The equivalent theorem in the context of metric spaces with a graph is:

Theorem 3.17. *Let* (X,d) *be a complete metric space,* G *be a directed graph on* $X \times X$ *with edge set* $E(G)$ *and* $F\colon X \times X \to X$ *be a coupled* G *contraction with a closed graph. Let*

$$W = \{(x,y) \in X \times X\colon\ ((x,y),(F(x,y),F(y,x))) \in E(G)\}$$

and assume that $W \neq \varnothing$. *Then* F *has at least one coupled fixed point* (x^*,y^*) *and the sequences* $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ *and* $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ *converge to* x^* *and* y^* *respectively.*

Proposition 3.18. *Theorem 3.15 holds if and only if Theorem 3.17 holds.*

Proof. The operator F being a coupled G contraction from Theorem 3.17 is equivalent to conditions (1) and (3) of Theorem 3.15. Furthermore, $W \neq \varnothing$ from Theorem 3.17 is equivalent to (2) of Theorem 3.15. Both theorems require that F has a closed graph. The conditions of Theorem 3.15 and Theorem 3.17 are equivalent. Therefore, Theorem 3.15 holds if and only if Theorem 3.17 holds. □

Since Theorem 3.15 is proven, by Proposition 3.18 we can conclude that Theorem 3.17 holds as well.

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