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A SHORT NOTE ON A WEIGHTED OPIAL-TYPE INEQUALITY

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This paper establishes new weighted Opial-type inequalities by employing recent advances in weighted Hardy inequalities. Our approach builds upon results of Nikolov and Uluchev and develops inequalities for both single and two functions.

Keywords: Opial's inequality, Opial-type inequalities, weighted Hardy inequality

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1. Introduction and preliminaries

Opial's inequality, introduced in 1960, has played a central role in the theory of integral inequalities. The original result relates a function to its derivative in a sharp inequality with significant applications in analysis. Following Opial's work, many authors provided new proofs, refinements, and discrete analogues.

In this section we recall some classical results related to Opial's inequality, which provide the background and motivation for the results established later in this paper.

We begin with the original inequality due to Zdzisław Opial and then recall some of its later versions.

Theorem 1.1 ([9]). *If $f \in C^1[0,h]$ satisfies $f(0)=f(h)=0$ and $f(x)>0$ for $x \in (0,h)$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{4}\int_0^h \left[f'(x)\right]^2 dx, \tag{1.1}$$

where the constant factor $h/4$ is best possible.

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This inequality has found much attention among many mathematicians, and many different proofs, variants, extensions, and discrete analogues of Opial's theorem have been developed. We mention here only a few [2–4, 6, 8–10, 12]. For a detailed bibliography on the topic we refer to [1, 11].

Soon after the publication of Opial's paper, C. Olech [8], in a short note, showed that the condition $f(x) > 0$ is not necessary; however, without this assumption the constant on the right-hand side of the inequality becomes double. The result reads as follows.

Theorem 1.2 ([8]). *If f is absolutely continuous on $[0, h]$ with $f(0) = f(h) = 0$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h \left[f'(x)\right]^2 dx. \tag{1.2}$$

The sign of equality holds if and only if $f(x) = cx$, where c is a constant.

In 1965, C. Mallows [6], presented what is arguably the simplest and most elegant proof of Opial's inequality. Moreover, Mallows' version of Opial inequality does not require any condition on the function f at the right end of the interval $[0, h]$.

Theorem 1.3 ([6]). *If f is absolutely continuous on $[0, h]$ with $f(0) = 0$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h \left[f'(x)\right]^2 dx.$$

The sign of equality holds if and only if $f(x) = cx$, where c is a constant.

This is in fact the most well-known version of Opial inequality.

In 1962, Beesack [3], provided conditions under which a generalized weighted version of Opial inequality

$$\left(\int_a^b |g(x)|^q\,|g'(x)|^s v(x)\,dx\right)^{\frac{1}{q+s}} \le c\left(\int_a^b |g'(x)|^p u(x)\,dx\right)^{\frac{1}{p}}$$

holds for all $g(x)$, such that $g(a) = 0$, $p > q > 0$, $0 < s < p$, and $u(x)$, $v(x)$ are weight functions.

A key contribution was made by Sinnamon [12], who demonstrated that Hardy-type inequalities naturally yield Opial-type inequalities. Moreover, Sinnamon proposed a two-function generalization of Opial's inequality, where the inequality is formulated in terms of products of two functions and their derivatives. This result complemented Pachpatte's earlier contributions [10] and illustrated once again the link between Hardy- and Opial-type inequalities.

In 1997, exploiting Sinnamon's approach, Bloom [4] established some Opial-type inequalities involving generalized Hardy operators.

Research on Opial-type inequalities continues to develop rapidly, with ongoing work on refinements, generalizations, and discrete analogues, as well as important recent applications to fractional calculus [1].

The above short historical overview outlines some of the main steps in the evolution of Opial-type inequalities, without attempting to be complete. Given the extensive literature on the subject, we restrict ourselves to citing mainly works that are directly relevant to our approach.

In the next section we turn to recent advances in weighted Hardy inequalities.

2. Recent results on weighted Hardy inequalities

Recent work by Nikolov and Uluchev [7] studies a weighted Hardy-type inequality in the particular finite-dimensional space

$$\mathcal{H}_{n,\alpha} := \left\{ f\colon \int_0^x f(t)\,dt = x^{-\alpha/2} e^{-x/2}\, p(x),\ p\in\mathcal{P}_n,\ p(0)=0 \right\}, \quad \alpha<1,$$

where $\mathcal{P}_n$ is the set of algebraic polynomials of degree at most n with real coefficients. More precisely, the authors examine the sharp constant $C_{n,\alpha}$ in a Hardy inequality with weight $w(x)=x^\alpha$:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^2 x^\alpha\,dx \le C_{n,\alpha}\int_0^\infty f^2(x)\,x^\alpha\,dx, \quad f\in\mathcal{H}_{n,\alpha}. \tag{2.1}$$

Lower and upper bounds for $C_{n,\alpha}$ of correct order in space dimension n are proved. The particular case $\alpha=0$ was considered by Dimitrov, Gadjev, Nikolov and Uluchev in [5].

The following two-sided estimate for the sharp constant $C_{n,\alpha}$, $n\ge 3$ of (2.1) was proved in [7, Theorem 1.1]:

$$C_{n,\alpha} < \left(\frac{2}{1-\alpha}\right)^2 \left(1 - \frac{4\sqrt{6}}{(1-\alpha)^2\left(\log\lfloor\frac{n+1}{2}\rfloor + 4\right)^2 + 4\sqrt{6}}\right) =: \overline{C}_{n,\alpha}, \tag{2.2}$$

$$C_{n,\alpha} > \left(\frac{2}{1-\alpha}\right)^2 \left(1 - \frac{16}{(1-\alpha)^2\left(\log\lfloor\frac{n+1}{2}\rfloor + \frac{8}{3}\right)^2 + 16}\right).$$

While the estimates in [5] for the lower and the upper bound of $C_{n,0}$ are of different order, in [7] the asymptotic rate of the lower estimate and the upper estimate is the same as $n\to\infty$ for any $\alpha<1$, namely $O((\log n)^{-2})$.

3. A weighted Opial-type inequality

We now describe the framework in which a weighted Opial-type inequality will be derived. We begin with introducing some notations.

In what follows, $\|\cdot\|_{2,\alpha}$ denotes the weighted $L_2[0,\infty)$-norm with weight x^α,

$$\|f\|_{2,\alpha} := \left(\int_0^\infty f^2(x) x^\alpha\,dx\right)^{1/2}.$$

We will consider the class

$$\mathcal{G}_{n,\alpha} := \{g \colon g(x) = x^{-\alpha/2} e^{-x/2} p(x),\ p \in \mathcal{P}_n,\ p(0) = 0\}, \quad \alpha < 1,$$

and the class used in [7]

$$\mathcal{H}_{n,\alpha} = \left\{ f \colon \int_0^x f(t)\, dt \in \mathcal{G}_{n,\alpha} \right\}.$$

Introducing the classes $\mathcal{G}_{n,\alpha}$ and $\mathcal{H}_{n,\alpha}$ allows us to translate the results on weighted Hardy-type estimates from [7] into Opial-type inequalities. The link between the two classes will be essential in the subsequent derivation.

Proposition 3.1. *If $g(x) \in \mathcal{G}_{n,\alpha}$, then $g'(x) \in \mathcal{H}_{n,\alpha}$.*

Proof. Indeed, any function $g(x) \in \mathcal{G}_{n,\alpha}$ has the form $x^{-\alpha/2} e^{-x/2} p(x)$, $p \in \mathcal{P}_n$ and $g(0) = 0$. Then

$$g(x) = g(x) - g(0) = \int_0^x g'(t)\, dt$$

is of the form $x^{-\alpha/2} e^{-x/2} p(x)$, where $p \in \mathcal{P}_n$, i.e., $g' \in \mathcal{H}_{n,\alpha}$. □

Theorem 3.2. *Let $g(x) \in \mathcal{G}_{n,\alpha}$, $q, s \geq 0$, $q + s = 2$ and $\alpha < 1$. Then the following weighted Opial-type inequality holds true*

$$\int_0^\infty |g(x)|^q\, |g'(x)|^s\, x^{\alpha - q}\, dx \leq (C_{n,\alpha})^{q/2} \int_0^\infty \left[g'(x)\right]^2 x^\alpha\, dx.$$

Proof. Since $g \in \mathcal{G}_{n,\alpha}$, we have that $g'(x) \in \mathcal{H}_{n,\alpha}$. Then using (2.1) for $g'(x) \in \mathcal{H}_{n,\alpha}$, we get that the following inequality holds true

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x g'(t)\, dt \right)^2 x^\alpha\, dx \right)^{1/2} \leq (C_{n,\alpha})^{1/2} \left(\int_0^\infty \left[g'(x)\right]^2 x^\alpha\, dx \right)^{1/2}.$$

Or, in other words, we get

$$\|g\|_{2,\alpha-2} \leq (C_{n,\alpha})^{1/2}\, \|g'\|_{2,\alpha}. \tag{3.1}$$

Denote

$$I := \int_0^\infty |g(x)|^q |g'(x)|^s x^{\alpha - q}\, dx = \int_0^\infty \left(|g(x)|^2 x^{\alpha-2}\right)^{q/2} \left(|g'(x)|^2 x^\alpha\right)^{s/2} dx.$$

Applying Hölder's inequality and (3.1), we obtain

$$\begin{aligned} I &\leq \left(\int_0^\infty |g(x)|^2 x^{\alpha-2}\, dx \right)^{q/2} \left(\int_0^\infty |g'(x)|^2 x^\alpha\, dx \right)^{s/2} \\ &\leq \|g\|_{2,\alpha-2}^q\, \|g'\|_{2,\alpha}^s \\ &\leq (C_{n,\alpha})^{q/2}\, \|g'\|_{2,\alpha}^q\, \|g'\|_{2,\alpha}^s \\ &= (C_{n,\alpha})^{q/2}\, \|g'\|_{2,\alpha}^2, \end{aligned}$$

which concludes the proof. □

To obtain an explicit, though non-sharp, version of the inequality in Theorem 3.2, we replace the sharp constant $C_{n,\alpha}$ by its computable upper bound $\overline{C}_{n,\alpha}$ from [7, Theorem 1.1], which yields the following result.

Corollary 3.3. *Let $g \in \mathcal{G}_{n,\alpha}$ with $\alpha < 1$, and let $q, s \geq 0$ satisfy $q + s = 2$. For $n \geq 3$,*

$$\int_0^\infty |g(x)|^q \, |g'(x)|^s \, x^{\alpha - q} \, dx \leq \left(\overline{C}_{n,\alpha}\right)^{q/2} \int_0^\infty |g'(x)|^2 \, x^\alpha \, dx, \tag{3.2}$$

where $\overline{C}_{n,\alpha}$, given in (2.2) is an explicit upper bound for the sharp Hardy constant $C_{n,\alpha}$.

Moreover, the constant satisfies the asymptotic relation

$$\left(\overline{C}_{n,\alpha}\right)^{q/2} = \left(\frac{2}{1-\alpha}\right)^q \left(1 + O\big((\log n)^{-2}\big)\right), \quad n \to \infty,$$

so that $\left(\overline{C}_{n,\alpha}\right)^{q/2}$ preserves the asymptotic rate of convergence of $(C_{n,\alpha})^{q/2}$.

Remark 3.4. The inequality (3.2) is non-sharp, since $C_{n,\alpha} < \overline{C}_{n,\alpha}$, and equality can occur only in the trivial case $g \equiv 0$.

4. An Opial-type inequality for two functions

Having derived the weighted Opial-type inequality for a single function, we next obtain the corresponding result for two functions.

Theorem 4.1. *Let $g(x) \in \mathcal{G}_{n,\alpha}$, $h(x) \in \mathcal{G}_{m,\beta}$, $m, n \in \mathbb{N}$, $\alpha, \beta < 1$, $q + s = 2$, $q = q_1 + q_2$, $s = s_1 + s_2$ and*

$$v(x) := x^{(\alpha-2)q_1/2} \, x^{\alpha s_1/2} \, x^{(\beta-2)q_2/2} \, x^{\beta s_2/2} = x^{\frac{\alpha(q_1+s_1)+\beta(q_2+s_2)}{2} - q}.$$

Then

$$\int_0^\infty |g(x)|^{q_1} \, |g'(x)|^{s_1} \, |h(x)|^{q_2} \, |h'(x)|^{s_2} \, v(x) \, dx \leq \left(C_{n,\alpha}\right)^{q_1/2} \left(C_{m,\beta}\right)^{q_2/2} \|g'\|_{2,\alpha}^{q_1+s_1} \, \|h'\|_{2,\beta}^{q_2+s_2}.$$

Proof. Denote

$$J := \int_0^\infty |g(x)|^{q_1} \, |g'(x)|^{s_1} \, |h(x)|^{q_2} \, |h'(x)|^{s_2} \, v(x) \, dx. \tag{4.1}$$

Since $g'(x) \in \mathcal{H}_{n,\alpha}$ and $h'(x) \in \mathcal{H}_{m,\beta}$, then from (3.1) it follows that

$$\|g\|_{2,\alpha-2} \leq \left(C_{n,\alpha}\right)^{1/2} \|g'\|_{2,\alpha}, \quad \|h\|_{2,\beta-2} \leq \left(C_{m,\beta}\right)^{1/2} \|h'\|_{2,\beta}.$$

Since $\frac{q_1}{2} + \frac{q_2}{2} + \frac{s_1}{2} + \frac{s_2}{2} = \frac{q+s}{2} = 1$, the Hölder inequality applied to (4.1) yields

$$J \le \left(\int_0^\infty |g(x)|^{q_1 \frac{2}{q_1}} x^{\frac{\alpha-2}{2} q_1 \frac{2}{q_1}} \, dx \right)^{q_1/2} \left(\int_0^\infty |g'(x)|^2 x^\alpha \, dx \right)^{s_1/2}$$
$$\times \left(\int_0^\infty |h(x)|^{q_2 \frac{2}{q_2}} x^{\frac{\beta-2}{2} q_2 \frac{2}{q_2}} \, dx \right)^{q_2/2} \left(\int_0^\infty |h'(x)|^2 x^\beta \, dx \right)^{s_2/2}$$
$$= \left(\|g\|_{2,\alpha-2} \right)^{q_1} \|g'\|_{2,\alpha}^{s_1} \left(\|h\|_{2,\beta-2} \right)^{q_2} \|h'\|_{2,\beta}^{s_2}$$
$$\le \left(C_{n,\alpha} \right)^{q_1/2} \|g'\|_{2,\alpha}^{q_1+s_1} \left(C_{m,\beta} \right)^{q_2/2} \|h'\|_{2,\beta}^{q_2+s_2}. \qquad \square$$

In the particular case $\alpha = \beta$, we have $v(x) = x^{\alpha-q}$ and the weighted Opial's-type inequality has a much simpler form.

Corollary 4.2. *Let $g(x) \in \mathcal{G}_{n,\alpha}$, $h(x) \in \mathcal{G}_{m,\alpha}$, $m, n \in \mathbb{N}$, $\alpha < 1$, $q + s = 2$, $q = q_1 + q_2$, $s = s_1 + s_2$. Then the following weighted Opial's type inequality holds true*

$$\int_0^\infty |g(x)|^{q_1} |g'(x)|^{s_1} |h(x)|^{q_2} |h'(x)|^{s_2} x^{\alpha-q} \, dx$$
$$\le \left(C_{n,\alpha} \right)^{q_1/2} \left(C_{m,\alpha} \right)^{q_2/2} \|g'\|_{2,\alpha}^{q_1+s_1} \|h'\|_{2,\alpha}^{q_2+s_2}.$$

To obtain an explicit, though non-sharp, form of the inequality in Theorem 4.1, we replace the sharp constants $C_{n,\alpha}$ and $C_{m,\beta}$ by their computable upper bounds $\overline{C}_{n,\alpha}$ and $\overline{C}_{m,\beta}$ given in (2.2), which leads to the following result.

Corollary 4.3. *Let $g(x) \in G_{n,\alpha}$ and $h(x) \in G_{m,\beta}$ with $\alpha, \beta < 1$ and $m, n \in \mathbb{N}$. Let $q + s = 2$, $q = q_1 + q_2$, $s = s_1 + s_2$, and define*

$$v(x) = x^{\frac{\alpha(q_1+s_1)+\beta(q_2+s_2)}{2} - q}.$$

Then, for $n, m \ge 3$,

$$\int_0^\infty |g(x)|^{q_1} |g'(x)|^{s_1} |h(x)|^{q_2} |h'(x)|^{s_2} v(x) \, dx$$
$$\le \left(\overline{C}_{n,\alpha} \right)^{q_1/2} \left(\overline{C}_{m,\beta} \right)^{q_2/2} \|g'\|_{2,\alpha}^{\frac{q_1+s_1}{2}} \|h'\|_{2,\beta}^{\frac{q_2+s_2}{2}}. \qquad (4.2)$$

Moreover, the constant satisfies the asymptotic relation

$$\left(\overline{C}_{n,\alpha} \right)^{q_1/2} \left(\overline{C}_{m,\beta} \right)^{q_2/2} = \left(\frac{2}{1-\alpha} \right)^{q_1} \left(\frac{2}{1-\beta} \right)^{q_2} \left(1 + O\big((\log n)^{-2} + (\log m)^{-2} \big) \right),$$

$n, m \to \infty$, so that the explicit bound preserves the asymptotic rate of convergence of $C_{n,\alpha}^{q_1/2} C_{m,\beta}^{q_2/2}$, $n, m \to \infty$.

Remark 4.4. The inequality in Corollary 4.2 is non-sharp, since $C_{n,\alpha} < \overline{C}_{n,\alpha}$ and $C_{m,\beta} < \overline{C}_{m,\beta}$, and equality can occur only in the trivial case $g \equiv h \equiv 0$.

The presented results demonstrate how recent advances in weighted Hardy inequalities can be employed to establish new weighted Opial-type inequalities with explicit constant estimates and clear asymptotic behavior.

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