ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 94, 2000

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 94, 2000

EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR THE EQUATION $f(t, x, x', x'') = 0$ WITH FULLY NONLINEAR BOUNDARY CONDITIONS

PETIO KELEVEDJIEV, NEDYU POPIVANOV

Existence results for two-point boundary value problems are established, the equation is not solved with respect to the high derivative, and the boundary conditions are nonlinear and full. The proofs are based on a variant of a basic theorem of Granas, Guenther and Lee. The a priori bounds needed for its application are obtained by the barrier strips technique.

Keywords: boundary value problems, nonlinear boundary conditions, existence, barrier strips

Mathematics Subject Classification 2000: 34B15

1. INTRODUCTION

In this paper we study the solvability of boundary value problems (BVPs) of the form

$$\begin{cases} f(t,x,x',x'') &=& 0, \quad t \in [0,1], \\ W_i(x) &=& V_i(x), \quad i = 1, 2. \end{cases} \tag{1.1}$$

Here the scalar function $f(t, x, p, q)$ is continuous and has continuous first derivatives only on suitable subsets of $[0,1] \times R^3$,

$$V_1(x) = \varphi\left(x(0), x'(0), x(1), x'(1)\right), \quad V_2(x) = \psi\left(x(0), x'(0), x(1), x'(1)\right),$$

$\phi, \psi : R^4 \to R$ are continuous, and $(W_1(x), W_2(x))$ are of the type

$$(M_1)\left(x(0), x'(1)\right), \quad (M_2)\left(x'(0), x(1)\right) \quad \text{or} \quad (D)\left(x(0), x(1)\right).$$

Further, we will write as (M_1), (M_2) and (D) the BVP (1.1) in the cases (M_1), (M_2) and (D), respectively.

The solvability of BVPs for the equation $x'' = f(t, x, x')$ with various nonlinear boundary conditions is studied in [1–8], for example, under various conditions on $f(t, x, p)$.

The results [9–14], see also [15], guarantee the existence of $C^2[0,1]$-solutions to BVPs for the equation $x'' = f(t, x, x', x'') - y(t)$. Moreover, the solutions satisfy mixed boundary conditions (M_1) or (M_2) in [9], periodic ones in [10], Neumann ones in [9, 11], Dirichlet or periodic ones in [9, 12, 13], and either Dirichlet, Neumann, Sturm-Liouvile, periodic or antiperiodic ones in [14]; in the last work uniqueness results are also obtained. Moreover, the growth of $f(t, x, p, q)$ is linear with respect to x, p and q in [10–12], semilinear in [13], quadratic with respect to p and linear with respect to q in [14]. In addition f satisfies various further conditions. The results [16] guarantee an existence and an uniqueness of $C^2[0,1]$-solution to the BVP for the equation $x'' = f(t, x, x', x'')$ with boundary conditions of the form

$$\mathbf{a}_{i1}x(0) + \mathbf{a}_{i2}x'(0) + \mathbf{a}_{i3}x(1) + \mathbf{a}_{i4}x'(1) = 0, i = 1, 2.$$

In [16] $f(t, x, p, q)$ satisfies a growth condition, which is a Nagumo one with respect to p and a linear one with respect to q, and some further conditions. The approach [10–14, 16] relies on the topological transversality [8] or similar arguments. The existence results [17] guarantee $W^{2,\infty}[0,1]$-solutions or $C^2[0,1]$-solutions to the Dirichlet BVP for the equation (1.1). The function $f(t, x, p, q)$ is defined on $[0,1] \times R^n \times R^n \times Y$, where Y is a non-empty closed connected and locally connected subset of R^n. Growth conditions on f are not used. The approach [17] follows that introduced in [18] with regard to the Cauchy problem. The results [19] guarantee an existence of $C^2[0,1]$-solutions to the BVP for the equation $x'' + g(t, x, x', x'') = y(t)$ with either Dirichlet, Neumann or mixed boundary conditions. The authors use conditions of Lipschitz type on g and barrier strips [20].

In this paper we also do not use assumptions on the growth of f. Using again the barrier strips technique [20], see also [19] for similar conditions, we obtain some uniformely a priori bounds for x', x and x'' (in this order) for the eventual solutions $x(t) \in C^2[0,1]$ to the family of BVPs

$$\begin{cases} Kx'' &= \lambda(Kx'' + f(t, x, x', x'')), \quad t \in [0,1], \\ W_i(x) &= \lambda V_i(x), \quad i = 1, 2, \end{cases} \qquad (1.1)_\lambda$$

where $\lambda \in [0,1]$, and K is a suitable positive constant; further, we will write as $(M_1)_\lambda$, $(M_2)_\lambda$ and $(D)_\lambda$ the family $(1.1)_\lambda$ in the cases (M_1), (M_2) and (D), respectively. Then the solvability of the problems considered follows by a basic existence result (Theorem 4.1) proved by an application of the topological transversality theorem [8].

We will say that $(\mathbf{A}_1)$ holds for the constants F and L if:

$(\mathbf{A}_1)$ $L \geq F$ and there are functions $F_i^-(t), L_i^+(t) \in C[0,1]$, $i = 1,2$, such that

$$L_1^+(1) \geq L, F \geq F_1^-(1),$$

$$L_1^+(t) \text{ is nonincreasing and } F_1^-(t) \text{ is nondecreasing on } [0,1],$$

$$L_2^+(t) > L_1^+(t) \quad \text{and} \quad F_1^-(t) > F_2^-(t) \quad \text{for} \quad t \in [0,1],$$

and there is a constant $K > 0$ for which

$$f(t,x,p,q) \geq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (-\infty,0), t \in [0,1] \text{ and } L_1^+(t) \leq p \leq L_2^+(t)\Big\}$,

$$f(t,x,p,q) \leq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (0,\infty), t \in [0,1] \text{ and } F_2^-(t) \leq p \leq F_1^-(t)\Big\}$.

We will say that $(\mathbf{A}_2)$ holds for the constants F and L if:

$(\mathbf{A}_2)$ $L \geq F$ and there are functions $F_i^+(t), L_i^-(t) \in C[0,1]$, $i = 1,2$, such that

$$L_1^-(0) \geq L, F \geq F_1^+(0),$$

$$L_1^-(t) \text{ is nondecreasing and } F_1^+(t) \text{ is nonincreasing on } [0,1],$$

$$L_2^-(t) > L_1^-(t) \quad \text{and} \quad F_1^+(t) > F_2^+(t) \quad \text{for} \quad t \in [0,1],$$

and there is a constant $K > 0$ for which

$$f(t,x,p,q) \leq -Kq$$

on $\Big\{(t,x,p,q) : x \in R,\ q \in (0,\infty),\ t \in [0,1] \text{ and } L_1^-(t) \leq p \leq L_2^-(t)\Big\}$,

$$f(t,x,p,q) \geq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (-\infty,0), t \in [0,1] \text{ and } F_2^+(t) \leq p \leq F_1^+(t)\Big\}$.

Remark. The constant K from $(\mathbf{A}_1)$ and the constant K from $(\mathbf{A}_2)$ could be different.

Lemma 2.1. *Let the condition $(\mathbf{A}_1)$ hold for some F and L and $x(t) \in C^2[0,1]$ be a solution to $(1.1)_\lambda$ (with the constant K from $(\mathbf{A}_1)$). Suppose there is an interval $T_1 \subseteq [0,1]$ such that*

$$L_1^+(t) \leq x'(t) \leq L_2^+(t) \quad \text{for} \quad t \in T_1. \tag{2.1}$$

Then $x''(t) \geq 0$ for $t \in T_1$. If there is an interval $T_2 \subseteq [0,1]$ such that

$$F_2^-(t) \leq x'(t) \leq F_1^-(t) \quad \text{for} \quad t \in T_2,$$

then $x''(t) \le 0$ *for* $t \in T_2$.

Proof. We will show only that (2.1) yields $x''(t) \ge 0$ for $t \in T_1$. The assertion is true for $\lambda = 0$. Now let $\lambda \in (0,1]$. Assume there is $t_0 \in T_1$ such that $x''(t_0) < 0$. Then

$$0 > Kx''(t_0) = \lambda [Kx''(t_0) + f(t_0, x(t_0), x'(t_0), x''(t_0))] \ge 0.$$

The contradiction obtained yields the assertion. □

Lemma 2.2. *Let the condition* $(\mathbf{A}_2)$ *hold for some* F *and* L *and* $x(t) \in C^2[0,1]$ *be a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_2)$*). Suppose there is an interval* $T_1 \subseteq [0,1]$ *such that*

$$L_1^-(t) \le x'(t) \le L_2^-(t) \quad \text{for} \quad t \in T_1.$$

Then $x''(t) \le 0$ *for* $t \in T_1$. *If there is an interval* $T_2 \subseteq [0,1]$ *such that* $F_2^+(t) \le x'(t) \le F_1^+(t)$ *for* $t \in T_2$, *then* $x''(t) \ge 0$ *for* $t \in T_2$.

Proof. The proof is the same as for Lemma 2.1 except for a few inessential changes in the details. □

Denote

$$\max u(t) := \max_{[0,1]} u(t), \quad \min u(t) := \min_{[0,1]} u(t), \quad \text{and} \quad \|u\|_0 := \max |u(t)|.$$

Let M, $Q \in R^+$ be some constants, and $L(t), F(t) \in C[0,1]$ be some functions such that $L(t) \ge F(t)$ on $[0,1]$. Let the functions $G_i^-(t), G_i^+(t)$, $H_i^-(t), H_i^+(t) \in C[0,1], i = 1,2$, be such that for

$$C = \max\{\|F\|_0, \|L\|_0\} \tag{2.2}$$

we have

$$\begin{cases} G_1^+(t) \ge 2C, G_1^-(t) \ge 2C \text{ for } t \in [0,1], \\ H_1^+(t) \le -2C, H_1^-(t) \le -2C \text{ for } t \in [0,1], \\ G_1^+(t) \text{ and } H_1^+(t) \text{ are nonincreasing on } [0,1], \\ G_1^-(t) \text{ and } H_1^-(t) \text{ are nondecreasing on } [0,1], \\ G_2^+(t) > G_1^+(t), G_2^-(t) > G_1^-(t), \text{ for } t \in [0,1], \\ H_1^+(t) > H_2^+(t), H_1^-(t) > H_2^-(t) \text{ for } t \in [0,1]. \end{cases} \tag{2.3}$$

Replace

$$Y_1 := \Bigg\{ (t,x,p,q) \ : \ |x| \le M + \varepsilon, \ t \in [0,1], \ p \in [F(t) - \varepsilon, \ L(t) + \varepsilon] \quad \text{and}$$

$$q \in \Big[\min\Big\{ \min H_2^+(t), \min H_2^-(t) \Big\} - \varepsilon, \ \max\Big\{ \max G_2^+(t), \max G_2^-(t) \Big\} + \varepsilon \Big] \Bigg\},$$

where

$$\begin{cases} \varepsilon > 0 \text{ is small enough and such that } H_1^+(t) > H_2^+(t) + \varepsilon, \\ H_1^-(t) > H_2^-(t) + \varepsilon, \ G_1^+(t) > G_2^+(t) + \varepsilon, \ G_1^-(t) > G_2^-(t) + \varepsilon, \end{cases} \tag{2.4}$$

$$Y_2 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [\min\{H_2^+(t), H_2^-(t)\},\ \max\{G_2^+(t), G_2^-(t)\}] \Big\}$$

$$Y_3 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [H_2^+(t), H_1^+(t)] \cup [G_1^+(t), G_2^+(t)] \Big\}$$

$$Y_4 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [H_2^-(t), H_1^-(t)] \cup [G_1^-(t), G_2^-(t)] \Big\}$$

$$Y_5 := \Big\{ (\lambda,t,x,p) : \lambda \in [0,1],\ x \in [-Q,Q],\ t \in [0,1],\ p \in [F(t),L(t)] \Big\}.$$

We will say that **(B)** holds for the functions $L(t), F(t) \in C[0,1]$ and the constant $M \in R^+$ if:

(B) There are functions $G_i^-(t)$, $G_i^+(t)$, $H_i^-(t)$, $H_i^+(t) \in C[0,1]$, $(i = 1,2)$, which satisfy (2.3) and are such that

$$\begin{cases} f(t,x,p,q) \text{ and } f_q(t,x,p,q) \text{ are continuous on } Y_1 \\ \qquad \text{and} \quad f_q(t,x,p,q) < 0 \quad \text{on } Y_1, \end{cases} \tag{2.5}$$

$$f_t(t,x,p,q),\ f_x(t,x,p,q) \text{ and } f_q(t,x,p,q) \text{ are continuous on } Y_2,$$

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \geq 0 \text{ on } Y_3,$$

and

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \leq 0 \text{ on } Y_4.$$

We will say that **(C)** holds for the functions $L(t), F(t) \in C[0,1]$ and for the constants $Q \in R^+$, Q_1, Q_2 if:

(C) $\qquad F(\lambda,t,x,p,Q_1)F(\lambda,t,x,p,Q_2) \leq 0$ for $(\lambda,t,x,p) \in Y_5$,

where $F(\lambda,t,x,p,q) = (\lambda-1)Kq + \lambda f(t,x,p,q)$, and K is the constant from $(1.1)_\lambda$.

3. TOPOLOGICAL PRELIMINARIES

For the sake of completeness, we give the topological transversality theorem which will be used later; moreover, we follow [8].

Let X be a metric space, and Y be a convex subset of a Banach space E. The continuous map $F : X \to Y$ is called compact if $F(X)$ is a compact subset of Y. The continuous map $F : X \to Y$ is completely continuous if it maps bounded subsets in X into compact subsets of Y.

Theorem 3.1 (Shauder's fixed point theorem). *Let Y be a convex subset of E, and $F : Y \to Y$ be a compact map. Then there exists a point $x_0 \in Y$ such that $F(x_0) = x_0$.*

We say that the homotopy $\{H_\lambda : X \to Y\}$, $0 \le \lambda \le 1$, is compact if the map $H(x, \lambda) : X \times [0,1] \to Y$ given by $H(x,\lambda) \equiv H_\lambda(x)$ for $(x,\lambda) \in X \times [0,1]$ is compact.

Let $U \subset Y$ be open in Y, ∂U be the boundary of$U$ in Y, and $\overline{U} = \partial U \cup U$. The compact map $F : \overline{U} \to Y$ is called admissible if it is fixed point free on ∂U. We denote the set of all such maps by $\mathbf{L}_{\partial U}(\overline{U}, Y)$.

Definition 3.1. The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if there is a fixed point free compact map $G : \overline{U} \to Y$ such that $G|\partial U = F|\partial U$. The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$, which is not inessential, is called essential.

Theorem 3.2. *Let $p \in U$ be arbitrary and $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then F is essential.*

Proof. Let $G : \overline{U} \to Y$ be a compact map such that $G|\partial U = F|\partial U$. Define the map $H : Y \to Y$ by

$$H(x) = p \quad \text{for} \quad x \in Y \setminus \overline{U},$$

$$H(x) = G(x) \quad \text{for} \quad x \in \overline{U}.$$

Clearly, $H : Y \to Y$ is a compact map. By Shauder's theorem H has a fixed point $x_0 \in Y$, i. e. $H(x_0) = x_0$. By definition of H we have $x_0 \in U$. Thus, $G(x_0) = x_0$ since H equals G on U. So every compact map from $\overline{U}$ into Y, which agrees with F on ∂U, has a fixed point. That is, F is essential. □

Definition 3.2. The maps $F, G \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ are called homotopic ($F \sim G$) if there is a compact homotopy $H_\lambda : \overline{U} \to Y$ such that H_λ is admissible for each $\lambda \in [0,1]$ and $G = H_0$, $F = H_1$.

Lemma 3.1. *The map $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if and only if it is homotopic to a fixed point free map.*

Proof. Let F be inessential and $G : \overline{U} \to Y$ be a compact fixed point free map such that $G|\partial U = F|\partial U$. Then the homotopy $H_\lambda : \overline{U} \to Y$, defined by

$$H_\lambda(x) = \lambda F(x) + (1-\lambda)G(x), \quad \lambda \in [0,1],$$

is compact, admissible and such that $G = H_0$, $F = H_1$.

Now let $H_0 : \overline{U} \to Y$ be a compact fixed point free map, and $H_\lambda : \overline{U} \to Y$ be an admissible homotopy joining H_0 and F. To show that $H_\lambda, \lambda \in [0,1]$, is an inessential map, consider the map $H : \overline{U} \times [0,1] \to Y$ such that $H(x,\lambda) \equiv H_\lambda(x)$ for each $x \in \overline{U}$ and $\lambda \in [0,1]$ and define the set $B \subset \overline{U}$ by

$$B = \{x \in \overline{U} : H_\lambda(x) \equiv H(x,\lambda) = x \quad \text{for some} \quad \lambda \in [0,1]\}.$$

If B is empty, then $H_1 = F$ has no fixed point which means that F is inessential. So we may assume that B is non-empty. In addition, B is closed and such that $B \cap \partial U = \varnothing$ since $H_\lambda, \lambda \in [0,1]$, is an admissible map. Now consider the Urysohn function $\theta : \overline{U} \to [0,1]$ with

$$\theta(x) = 1 \text{ for } x \in \partial U \quad \text{and} \quad \theta(x) = 0 \text{ for } x \in B$$

and define the homotopy $H^*_\lambda : \overline{U} \to Y, \lambda \in [0,1]$, by

$$H^*_\lambda = H(x, \theta(x)\lambda) \quad \text{for} \quad (x,\lambda) \in \overline{U} \times [0,1].$$

It is easy to see that $H^*_\lambda : \overline{U} \to Y$ is inessential. In particular, $H_1 = F$ is inessential, too. The proof is completed. □

As a consequence of Lemma 3.1 we have:

Theorem 3.3 (Topological transversality theorem). *Let $F, G \in \mathrm{L}_{\partial U}(\overline{U}, Y)$ be homotopics maps. Then one of these maps is essential if and only if the other one is.*

Theorem 3.3 is used in the following equivalent form:

Theorem 3.4 (Topological transversality theorem). *Let Y be a convex subset of a Banach space E, and $U \subset Y$ be open. Suppose:*

(i) *$F, G : \overline{U} \to Y$ are compact maps;*
(ii) *$G \in \mathrm{L}_{\partial U}(\overline{U}, Y)$ is essential;*
(iii) *$H_\lambda(x)$, $\lambda \in [0,1]$, is a compact homotopy joining F and G, i. e. $H_0(x) = G(x)$, $H_1(x) = F(x)$;*
(iv) *$H_\lambda(x)$, $\lambda \in [0,1]$, is a fixed point free on ∂U.*

Then H_λ, $\lambda \in [0,1]$, has at least one fixed point $x_0 \in U$, and, in particular, there is an $x_0 \in U$ such that $x_0 = F(x_0)$.

4. A BASIC EXISTENCE RESULT, ANCILLARY RESULTS

The following theorem is a modification of [8, Chapter II, Theorem 6.1].

Theorem 4.1. *Let $\varphi, \psi : R^4 \to R$ be continuous. Assume there are constants Q, Q_1, Q_2 (independent of λ) and functions $L(t), F(t) \in C[0,1]$ (independent of λ) such that:*

(i) $|x(t)| < Q$, $F(t) < x'(t) < L(t)$, $Q_1 < x''(t) < Q_2$, $t \in [0,1]$, *for each solution $x(t) \in C^2[0,1]$ to $(1.1)_\lambda$ (with fixed $K > 0$) and for $\lambda \in [0,1]$;*

(ii) *$f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous, and $f_q(t,x,p,q) < 0$ on*

$$\Big\{(t,x,p,q) : x \in [-Q,Q],\ q \in [Q_1,Q_2],\ t \in [0,1] \text{ and } p \in [F(t),L(t)]\Big\}.$$

(iii) *$F(\lambda,t,x,p,Q_1)F(\lambda,t,x,p,Q_2) \le 0$ for $(\lambda,t,x,p) \in \Lambda =$*

$$\Big\{(\lambda,t,x,p) : \lambda \in [0,1],\ x \in [-Q,Q],\ t \in [0,1] \text{ and } p \in [F(t),L(t)]\Big\};$$

Then the BVP (1.1) has at least one $C^2[0,1]$-solution.

Proof. From (ii) and (iii) it follows that there is an unique function $G(\lambda,t,x,p)$ continuous on Λ and such that

$$q = G(\lambda,t,x,p) \text{ for } (\lambda,t,x,p) \in \Lambda$$

is equivalent to the equation $F(\lambda,t,x,p,q) = 0$ on $\Lambda \times [Q_1,Q_2]$. Thus, the family $(1.1)_\lambda$ is equivalent to the family of BVPs

$$\begin{cases} x'' &= G(\lambda,t,x,x'), \quad t \in [0,1], \\ W_i(x) &= \lambda V_i(x), \quad i = 1,2, \end{cases} \tag{4.1}$$

$\lambda \in [0,1]$. Note that $F \equiv -Kq$ for $\lambda = 0$ and it yields

$$G(0,t,x,p) = 0 \text{for} (t,x,p) \in \Omega, \tag{4.2}$$

where $\Omega = \Big\{(t,x,p) : x \in [-Q,Q], t \in [0,1], \text{ and } p \in [F(t),L(t)]\Big\}$.

Define the map

$$L_1 : C^2[0,1] \to C[0,1] \times R^2 \quad \text{by} \quad L_1 x = \Big(x'',\, W_1(x),\, W_2(x)\Big)$$

and the maps

$$G_\lambda : C^1[0,1] \to C[0,1] \times R^2 \quad \text{by}$$

$$G_\lambda(x) = \Big(G(\lambda,t,x,x'),\, \lambda V_1(x),\, \lambda V_2(x)\Big) \quad \text{for} \quad \lambda \in [0,1].$$

It is easy to see that L_1 is a continuous, linear, one-to-one map of $C^2[0,1]$ onto $C[0,1] \times R^2$. So L_1 has a continuous inverse L_1^{-1}. Finally, define $j : C^2[0,1] \to C^1[0,1]$ by $jx = x$, which is a completely continuous embeding.

Now define the set

$$U = \Big\{x \in C^2[0,1] : \text{for } t \in [0,1],\ |x(t)| < M,\ F(t) < x'(t) < L(t),\ Q_1 < x''(t) < Q_2\Big\}$$

and consider the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \text{ defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_1^{-1} \circ G_\lambda \circ j(x).$$

This homotopy is compact since $j(\overline{U})$ is a compact subset of $C^1[0,1]$, and G_λ, $\lambda \in [0,1]$, and L_1^{-1} are continuous on $j(\overline{U})$ and $G_\lambda\big(j(\overline{U})\big)$, respectively. In addition, the equation

$$L_1^{-1} \circ G_\lambda \circ j(x) = x \quad \text{yields} \quad L_1(x) = G_\lambda(x),$$

which is the BVPs (4.1). Then it follows from **(i)** that $H_\lambda(x)$ is a fixed point free on ∂U, i. e. $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Finally, $H_0 = 0$ by (4.2).

So H_0 is an essential map by [8, Chapter I, Theorem 2.2]. Now we are in a position to apply Theorem 3.4. It implies that $H_1 = L_1^{-1} \circ G_1 \circ j$ is essential, too, which means that the original problem (1.1) has a solution in $C^2[0,1]$. □

The next results prepare the application of Theorem 4.1. They guarantee the a priori bounds from (i) of Theorem 4.1.

Lemma 4.1.A. *Let* $(\mathbf{A}_1)$ *hold for some constants* M_1 *and* M_2 *(i. e.* $(\mathbf{A}_1)$ *holds for* $F(t) \equiv M_1$ *and* $L(t) \equiv M_2$, $t \in [0,1]$*). Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_1)$*) such that* $M_1 \le x'(1) \le M_2$. *Then*

$$F_1^-(t) \le x'(t) \le L_1^+(t) \quad \textit{for} \quad t \in [0,1].$$

Proof. Suppose the set

$$S_0 = \{t \in [0,1] : L_1^+(t) < x'(t) \le L_2^+(t)\}$$

or

$$S_1 = \{t \in [0,1] : F_2^-(t) \le x'(t) < F_1^-(t)\}$$

is not empty. The continuity of $x'(t)$ and the inequalities $F_1^-(1) \le x'(1) \le L_1^+(1)$ imply that there are closed intervals

$$[t_0, t_0'] \subseteq S_0 \quad \text{or} \quad [t_1, t_1'] \subseteq S_1$$

such that

$$x'(t_0) > x'(t_0') \quad \text{or} \quad x'(t_1) < x'(t_1'). \tag{4.3}$$

On the other hand, by Lemma 2.1, we have

$$x''(t) \ge 0 \text{fort} \in [t_0, t_0'] \quad \text{or} \quad x''(t) \le 0 \text{ for } t \in [t_1, t_1'].$$

Consequently,

$$x'(t_0) \le x'(t_0') \quad \text{or} \quad x'(t_1) \ge x'(t_1').$$

The contradiction to (4.3) shows that S_0 and S_1 are empty, which yields the lemma. □

Lemma 4.1.B. *Let* $(\mathbf{A}_1)$ *hold for some constants* M_1 *and* M_2 *and* $(\mathbf{B})$ *hold for* $L(t) \equiv L_1^+(t), F(t) \equiv F_1^-(t), t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant (2.2),* N *is some constant, and the functions* $L_1^+(t)$ *and* $F_1^-(t)$ *are from the condition* $(\mathbf{A}_1)$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* K *from* $\mathbf{A}_1$*)) such that*

$$|x(0)| \le N, t \in [0,1], \quad \textit{and} \quad M_1 \le x'(1) \le M_2.$$

Then

$$|x(t)| \le M \quad \textit{for} \quad t \in [0,1] \tag{4.4}$$

and

$$\min\Big\{H_1^+(1),\, H_1^-(0)\Big\} \le x''(t) \le \max\Big\{G_1^+(0), G_1^-(1)\Big\},\ t \in [0,1]. \tag{4.5}$$

Proof. In fact $C = \max\{\|L_1^+\|_0, \|F_1^-\|_0\}$. By the mean value theorem there is $d \in (0,1)$ such that $x''(d) = x'(1) - x'(0)$. Lemma 4.1.A implies

$$F_1^-(t) \le x'(t) \le L_1^+(t) \quad \text{for} \quad t \in [0,1],$$

i. e. $|x'(t)| \le C$ for $t \in [0,1]$. So

$$x''(d) \le 2C \le G_1^+(t) \quad \text{for} \quad t \in [0,d]. \tag{4.6}$$

On the other hand, for each $t \in (0,d]$ there is $c \in (0,t)$ such that

$$x(t) - x(0) = x'(c)t,$$

which yields

$$|x(t)| \le M \text{ for } t \in [0,d].$$

Now suppose the set

$$S = \{t \in [0,d] : G_1^+(t) < x''(t) \le G_2^+(t)\}$$

is not empty. The continuity of $x''(t)$ and (4.6) imply that there is a closed interval

$$[t_0, t_0'] \subseteq S \quad \text{such that} \quad x''(t_0) > x''(t_0'). \tag{4.7}$$

Since for $t \in [t_0, t_0']$

$$-M \le x(t) \le M, \quad F_1^-(t) \le x'(t) \le L_1^+(t), \quad G_1^+(t) < x''(t) \le G_2^+(t),$$

we have

$$f_q(t, x(t), x'(t), x''(t)) < 0, \ t \in [t_0, t_0'], \tag{4.8}$$

and

$$f_t(t, x(t), x'(t), x''(t)) + f_x(t, x(t), x'(t), x''(t))x' + f_p(t, x(t), x'(t), x''(t))x'' \ge 0$$

for $t \in [t_0, t_0']$. From the differential equation $(1.1)_\lambda$ for $t \in [t_0, t_0']$ we obtain

$$\begin{cases} [K(1-\lambda) - \lambda f_q(t, x(t), x'(t), q_h)]\,[x''(t+h) - x''(t)] \\ = h f_t(P_{1h}) + f_x(P_{2h})[x(t+h) - x(t)] + f_p(P_{3h})[x'(t+h) - x'(t)] \\ \to f_t(P) + f_x(P)x'(t) + f_p(P)x''(t), \end{cases} \tag{4.9}$$

where $(t, x(t), x'(t), x''(t))$ and the points P_{1h}, P_{2h} and P_{3h} tend to P. Because of (4.8) it follows from (4.9) that $x'''(t)$ exists and

$$x''' = \lambda\,(f_t + f_x x' + f_p x'')\,/\,[K(1-\lambda) - \lambda f_q]\,, \tag{4.10}$$

which yields

$$x'''(t) \ge 0 \quad \text{for} \quad t \in [t_0, t_0'].$$

Then

$$x''(t_0) \le x''(t_0'),$$

a contradiction to (4.7). Consequently,

$$x''(t) \le G_1^+(t) \quad \text{for} \quad t \in [0,d].$$

The inequality

$$H_1^-(t) \le x''(t), \quad t \in [0,d],$$

may be obtained in a similar way.

Similarly, the inequalities

$$|x(t)| \leq M \quad \text{and} \quad H_1^+(t) \leq x''(t) \leq G_1^-(t), \quad t \in [d,1],$$

may be established. □

Lemma 4.2.A. *Let* $(\mathbf{A}_2)$ *hold for some constants* M_3 *and* M_4. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_2)$*) such that* $M_3 \leq x'(0) \leq M_4$. *Then*

$$F_1^+(t) \leq x'(t) \leq L_1^-(t) \quad \textit{for} \quad t \in [0,1].$$

Proof. The lemma can be obtained by using Lemma 2.2 and following the proof of Lemma 4.2.A. □

Lemma 4.2.B. *Let* $(\mathbf{A}_2)$ *hold for some constants* M_3 *and* M_4, *and* $(\mathbf{B})$ *hold for* $L(t) \equiv L_1^-(t), F(t) \equiv F_1^+(t)$, $t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant* (2.2), N *is some constant, and the functions* $L_1^-(t)$ *and* $F_1^+(t)$ *are from the condition* $(\mathbf{A}_2)$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* K *from* $(\mathbf{A}_2)$*) such that*

$$|x(1)| \leq N, \ t \in [0,1], \quad \textit{and} \quad M_3 \leq x'(0) \leq M_4.$$

Then (4.4) *and* (4.5) *hold with current notations.*

Proof. It is not too different from the proof of Lemma 4.1.B. □

Lemma 4.3.A. *Let* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$, *where* $M_i \in R$, $i = \overline{1,4}$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* $K = \min\{K_1, K_2\}$, *where* K_i *is the "value" of the constant* K *from* $(\mathbf{A}_i)$, $i = 1,2$*) such that*

$$M_1 \leq x(0) \leq M_2 \quad \textit{and} \quad M_3 \leq x(1) \leq M_4.$$

Then for $t \in [0,1]$

$$\min\left\{F_1^-(0), F_1^+(1)\right\} \leq x'(t) \leq \max\left\{L_1^+(0), L_1^-(1)\right\}.$$

Proof. There is $d \in (0,1)$ such that

$$x'(d) = x(1) - x(0) = \lambda[\psi(x(0), x'(0), x(1), x'(1)) - \varphi(x(0), x'(0), x(1), x'(1))],$$

from where it follows

$$\min\{0, M_3 - M_2\} \leq \lambda(M_3 - M_2) \leq x'(d)$$

and

$$x'(d) \leq \lambda(M_4 - M_1) \leq \max\{0, M_4 - M_1\}.$$

For $t \in [0,d]$ we have

$$F_1^-(t) \leq x'(t) \leq L_1^+(t), \quad \text{by Lemma 4.1.A},$$

and for $t \in [d, 1]$ we have

$$F_1^+(t) \le x'(t) \le L_1^-(t), \quad \text{by Lemma 4.2.A},$$

and the assertion follows. □

Lemma 4.3.B. *Let* $(\mathbf{A}_1)$ *and* $\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$, *where* $M_i \in R$, $i = \overline{1,4}$, *and* $(\mathbf{B})$ *hold for*

$$L(t) \equiv \max\left\{L_1^+(0), L_1^-(1)\right\}, \qquad F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\},$$

and $M = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\}$, *where* C *is the constant* (2.2). *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ (*with* $K = \min\{K_1, K_2\}$, *where* K_i *is the "value" of the constant* K *from* $(\mathbf{A}_i)$, $i = 1,2$) *such that*

$$M_1 \le x(0) \le M_2 \quad \text{and} \quad M_3 \le x(1) \le M_4.$$

Then (4.4) *and* (4.5) *hold with a current notations.*

Proof. It is not too different from the proof of Lemma 4.1.B. □

5. EXISTENCE RESULTS

Theorem 5.1. *Let* $\varphi, \psi : R^4 \to R$ *be continuous. Suppose there are constants* M_i, $i = 1,2$, *and* N *such that:*

(i) $M_1 \le \psi(s_1, s_2, s_3, s_4) \le M_2$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_1)$ *holds for* M_1 *and* M_2;

(iii) $|\varphi(s_1, s_2, s_3, s_4)| \le N$ *for* $(s_1, s_2, s_3, s_4) \in R \times \left[F_1^-(0), L_1^+(0)\right] \times R \times [M_1, M_2]$;

(iv) $(\mathbf{B})$ *holds for* $L(t) \equiv L_1^+(t)$, $F(t) \equiv F_1^-(t)$, $t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant* (2.2);

(v) $(\mathbf{C})$ *holds for* $L(t) \equiv L_1^+(t) + \varepsilon$, $F(t) \equiv F_1^-(t) - \varepsilon$, $t \in [0,1]$, *for*

$$Q = C+N+\varepsilon,\ Q_1 = \min\left\{H_1^+(1), H_1^-(0)\right\} - \varepsilon,\ Q_2 = \max\left\{G_1^+(0), G_1^-(1)\right\} + \varepsilon,$$

where C *is the constant* (2.2), *and* ε *satisfies* (2.4).

Then the mixed BVP (M_1) *has a* $C^2[0,1]$*-solution.*

Proof. Let $x(t) \in C^2[0,1]$ be a solution to $(\mathrm{M}_1)_\lambda$. Then

$$F_1^-(t) - \varepsilon < x'(t) < L_1^+(t) + \varepsilon \quad \text{for} \quad t \in [0,1], \text{by Lemma 4.1.A},$$

and Lemma 4.1.B yields the bounds

$$|x(t)| < Q \quad \text{for} \quad t \in [0,1],$$

$$Q_1 < x''(t) < Q_2 \quad \text{for} \quad t \in [0,1].$$

Then the condition (i) of Theorem 4.1 holds. From (2.5) it follows that the condition (ii) of Theorem 4.1 holds. Finally, (v) implies that the condition (iii) of Theorem

4.1 holds. So we can apply Theorem 4.1 to conclude that the problem (M_1) has a solution in $C^2[0,1]$. □

Theorem 5.2. *Let $\varphi, \psi : R^4 \to R$ be continuous. Suppose there are constant M_i, $i = 3, 4$, and N such that:*

(i) $M_3 \le \varphi(s_1, s_2, s_3, s_4) \le M_4$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_2)$ *holds for* M_3 *and* M_4;

(iii) $|\psi(s_1, s_2, s_3, s_4)| \le N$ *for* $(s_1, s_2, s_3, s_4) \in R \times [M_3, M_4] \times R \times [F_1^+(1), L_1^-(1)]$;

(iv) $(\mathbf{B})$ *holds for* $L(t) \equiv L_1^-(t)$, $F(t) \equiv F_1^+(t)$, $t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant* (2.2);

(v) $(\mathbf{C})$ *holds for* $L(t) \equiv L_1^-(t) + \varepsilon$, $F(t) \equiv F_1^+(t) - \varepsilon$, $t \in [0,1]$, *for*

$$Q = C+N+\varepsilon,\ Q_1 = \min\left\{H_1^+(1), H_1^-(0)\right\} - \varepsilon,\ Q_2 = \max\left\{G_1^+(0), G_1^-(1)\right\} + \varepsilon,$$

where C *is the constant* (2.2), *and* ε *satisfies* (2.4).

Then the mixed BVP (M_2) *has a* $C^2[0,1]$*-solution.*

Proof. It is not too different from the proof of Theorem 5.1. Consider $(M_2)_\lambda$. Now Lemma 4.2.A guarantees the a priori bound for x', and Lemma 4.2.B guarantees the a priori bounds for x and x''. □

Theorem 5.3. *Let $\varphi, \psi : R^4 \to R$ be continuous. Suppose there are constants M_i, $i = \overline{1,4}$, such that:*

(i) $M_1 \le \varphi(s_1, s_2, s_3, s_4) \le M_2$ *and* $M_3 \le \psi(s_1, s_2, s_3, s_4) \le M_4$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$;

(iii) $(\mathbf{B})$ *holds for* $L(t) \equiv \max\left\{L_1^+(0),\, L_1^-(1)\right\}$, $F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\}$ *and* $M = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\}$, *where* C *is the constant* (2.2);

(iv) $(\mathbf{C})$ *holds for the functions*

$$L(t) \equiv \max\left\{L_1^+(0),\, L_1^-(1)\right\} + \varepsilon, \quad F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\} - \varepsilon,$$

$$Q = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\} + \varepsilon, \quad Q_1 = \min\left\{H_1^+(1), H_1^-(0)\right\} - \varepsilon,$$

$$Q_2 = \max\left\{G_1^+(0), G_1^-(1)\right\} + \varepsilon,$$

where C *is the constant* (2.2), *and* ε *satisfies* (2.4).

Then the Dirichlet BVP (D) *has a* $C^2[0,1]$*-solution.*

Proof. It is not too different from the proof of Theorem 5.1. Now consider the family $(D)_\lambda$. The a priori bound for x' follows by Lemma 4.3.A, and the a priori bounds for x and x'' follow by Lemma 4.3.B. □

Example 6.1. Consider the boundary value problem

$$-(2-t)x'' - tx''^3 + \sin(x' - 0.2) = 0, \quad t \in [0,1],$$

$$x(0) = 0, \quad x'(1) = 0.15.$$

For $L = F = 0.15$ $(\mathbf{A}_1)$ holds. Moreover, we can choose

$$L_1^+(t) = 0.25,\ L_2^+(t) = 0.3,\ F_1^-(t) = 0.1,\ F_2^-(t) = 0.05,\ t \in [0,1],$$

and K is sufficiently small; to say $K = 10^{-10}$. It is easy to see that $f_q = -(2-t)q - 3tq^2 < 0$ for $t \in [0,1]$ and each q. This fact allows us to conclude that $(\mathbf{B})$ holds for $L(t) = 0.25$, $F(t) = 0.1$, $t \in [0,1]$, and $M = 0.25$. Moreover, we can choose

$$G_1^+(t) = 0.9, G_2^+(t) = 1,\ G_1^-(t) = 2, G_2^-(t) = 3,$$

$$H_1^-(t) = -0.9,\ H_2^-(t) = -1,\ H_1^+(t) = -2,\ H_2^+(t) = -3,\ t \in [0,1].$$

Finally, from

$$2.01(\lambda - 1)K + \lambda\left[-2.01(2-t) - (2.01)^3 t + \sin(p - 0.2)\right] \le 0$$

and

$$-2.01(\lambda - 1)K + \lambda\left[-(-2.01)(2-t) - (-2.01)^3 t + \sin(p - 0.2)\right] \ge 0$$

for $\lambda, t \in [0,1]$ and each p we conclude that $(\mathbf{C})$ holds for $Q_1 = -2.01$ and $Q_2 = 2.01$. Thus the problem considered has a $C^2[0,1]$-solution by Theorem 5.1.

Example 6.2. Consider the boundary value problem

$$x'^2 - 4 - 50(2-t)x'' - tx''^5 = 0, \quad t \in [0,1],$$

$$x(0) = [x^2(0) + x'^2(0) + x^2(1) + x'^2(1) + 1]^{-1}, \quad x(1) = \sin^2 x'(1).$$

For $L = 1$ and $F = -1$ $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ hold. Moreover, we can choose

$$L_1^+(t) = 2.1,\ L_2^+(t) = 2.2, F_1^-(t) = -1.1,\ F_2^-(t) = -1.2,$$

$$L_1^-(t) = 1.1,\ L_2^-(t) = 1.2, F_1^+(t) = -2.1,\ F_2^+(t) = -2.2, t \in [0,1],$$

and K is sufficiently small; to say $K = 10^{-10}$. It is easy to see that $f_q = -50(2-t) - 5q^4 t < 0$ for $t \in [0,1]$ and each q. Thus $(\mathbf{B})$ holds for $L(t) = 2.1$, $F(t) = -2.1$, $t \in [0,1]$ and, $M = 3.1$. Moreover, we can choose

$$G_1^+(t) = 6.5, G_2^+(t) = 6.6,\ G_1^-(t) = 10, G_2^-(t) = 11,$$

$$H_1^-(t) = -6.5,\ H_2^-(t) = -6.6,\ H_1^+(t) = -10,\ H_2^+(t) = -11,\ t \in [0,1].$$

Finally, from

$$10.01(\lambda - 1)K + \lambda\left[p^2 - 4 - 50(2-t)10.01 - (10.01)^5 t\right] \le 0$$

and

$$-10.01(\lambda - 1)K + \lambda\left[p^2 - 4 - 50(2-t)(-10.01) - (-10.01)^5 t\right] \ge 0$$

for $\lambda, t \in [0,1]$ and $p \in [-3.11, 3.11]$ we conclude that (**C**) holds for $L(t) = 2.11$, $F(t) = -2.11$, $t \in [0,1]$, $Q = 3.11$, $Q_1 = -10.01$ and $Q_2 = 10.01$; $\varepsilon = 0.01$. Thus the problem considered has a $C^2[0,1]$-solution by Theorem 5.3.

Acknowledgements. The research of N. Popivanov was partially supported by the Bulgarian NSC under Grant MM-904/99.

REFERENCES

1. Erbe, L. H. Existence of solutions to boundary value problems for second order differential equations. *Nonlinear Analysis*, **6**, 1982, 1155–1162.
2. Vasilev, N., I. Klokov. Bases of the Theory of Boundary Value Problems for Ordinary Differential Equations. Zinatne, Riga, 1978 (in Russian).
3. Rachuncova, I. Topological degree methods in BVPs with nonlinear condiions. *Ann. Math. Sil.*, **10**, 1996, 103–111.
4. Jackson, L. K., P. K. Palamides. An existence theorem for a nonlinear two point boundary value problem. *J. Differential Equations*, **53**, 1984, 48–66.
5. Baxly, J. V., S. E. Brown. Existence and uniqueness for two-point boundary value problems. *Proc. Roy. Soc. Edinburg*, **88A**, 1981, 219–234.
6. Fabry, C., P. Habets. Upper and lower solutions for second order boundary value problems with nonlinear boundary conditions. In: Seminaire de mathematique, Louvain-La-Neuve, Belgique, 1984.
7. Thompson, H. B. Second order ordinary differential equations with fully nonlinear two-point boundary conditions. II. *Pacific J. Math.*, **172**, 1996, 279–297.
8. Granas, A., R. B. Guenther, J. W. Lee. Nonlinear boundary value problems for ordinary differential equations. Dissnes Math., Warszawa, 1985.
9. Mao, Y., J. Lee. Two point boundary value problems for nolinear differential equations. *Rocky Mount. J. Math.*, **10**, 1980, 35–58.
10. Petryshyn, W. V., Z. S. Yu. Periodic solutions of nonlinear second-order differential equations which are not solvable for the highest-order derivative. *J. Math. Anal. Appl.*, **89**, 1982, 462–488.
11. Petryshyn, W. V., Z. S. Yu. Solvability of Neumann BV problems for nonlinear second order ODE's which need not be solvable for the highest order derivative. *J. Math. Anal. Appl.*, **91**, 1983, 244–253.
12. Fitzpatrick, P. M., W. V. Petryshyn. Galerkin method in the constructive solvability of nonlinear Hammerstein equations with applications to differential equations. *Trans. Amer. Math. Soc.*, **238**, 1978, 321–340.
13. Fitzpatrick, P. M. Existence results for equations involving noncompact perturbation of Fredholm mappings with applications to differential equations. *J. Math. Anal. Appl.*, **66**, 1978, 151–177.
14. Petryshyn, W. V. Solvability of various boundary value problems for the equation $x'' = f(t, x, x', x'') - y$. *Pacific J. Math.*, **122**, 1986, 169–195.
15. Petryshyn, W. V. Approximation-solvability of nonlinear functional and differential equations. Marcel Dekker, Inc., New York, 1993.
16. Tineo, A. Existence of solutions for a class of boundary value problems for the equation $x'' = F(t, x, x', x'')$. *Comment. Math. Univ. Carolin*, **29**, 1988, 285–291.

17. Marano, S. A. On a boundary value problem for the differential equation $f(t,x,x',x'')=0$. *J. Math. Anal. Appl.*, **182**, 1994, 309–319.

18. Ricceri, B. On the Cauchy problem for the differential equation $f(t,x,x',\dots,x^{(k)})=0$. *Glasgow Math. J.*, **33**, 1991, 343–348.

19. Mao, Y., J. Lee. Two point boundary value problems for nonlinear differential equations. *Rocky Maunt. J. Math.*, **26**, 1996, 1499–1515.

20. Kelevedjiev, P. Existence results for singular inilial value problems for first order differential equations. *Nonlinear Analysis*, **39**, 2000, 65–77.

Received November 17, 2000
Revised February 15, 2001

PETIO KELEVEDJIEV
Technical University of Sliven
59 Bourgasko Chaussee str.
8800 Sliven, Bulgaria
E-mail: `keleved@mailcity.com`

NEDYU POPIVANOV
"St. Kl. Ohridski" University of Sofia
5 James Bourchier Blvd 5, POBox 48
1164 Sofia, Bulgaria
E-mail: `nedyu@fmi.uni-sofia.bg`