ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

A CHARACTERISATION OF THE RATE OF APPROXIMATION OF THE BASKAKOV-KANTOROVICH OPERATOR

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We characterise the K-functional that was previously shown to describe the rate of approximation of the Baskakov-Kantorovich operator in the L_p-spaces with the weight $(1+x)^\gamma$ with $\gamma \le 0$. The characterisation uses the Ditzian-Totik moduli of smoothness.

Keywords: Baskakov-Kantorovich operator, modulus of smoothness, K-functional

2020 Mathematics Subject Classification: 41A25, 41A35, 41A36, 41A81

1. Introduction and main results

Let $f(x)$ be Lebesgue integrable on any finite closed subinterval of $[0,\infty)$ and $n \in \mathbb{N}_+$, $n \ge 2$. We consider the Baskakov-Kantorovich operator defined by

$$\widetilde{V}_n f(x) := \sum_{k=0}^{\infty} v_{n,k}(x)\,(n-1)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)\,du, \quad x \ge 0,$$

where

$$v_{n,k}(x) := \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Let

$$w(x) := (1+x)^\gamma, \quad \gamma \in \mathbb{R}, \tag{1.1}$$

and

$$L_p(w)[0,\infty) := \{f \in L_{1,loc}(0,\infty) \colon wf \in L_p[0,\infty)\},$$

where $1 \le p \le \infty$. Here $L_{1,loc}(0,\infty)$ stands for the space of all functions which are Lebesgue integrable on any finite closed subinterval of $(0,\infty)$. Let $\|\cdot\|_p$ denote the standard norm in $L_p[0,\infty)$. The norm in $L_p(w)[0,\infty)$ is defined by $\|f\|_{w,p} := \|wf\|_p$.

DOI: 10.60063/GSU.FMI.112.17-29

The K-functional that turns out to naturally describe the approximation rate of $\widetilde{V}_n$ in $L_p(w)[0,\infty)$ is

$$\widetilde{K}(f,t)_{w,p} := \inf_{g\in\widetilde{W}_p(w)[0,\infty)} \{\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\},$$

where $\widehat{D}g(x) := (\varphi^2(x)g'(x))'$, $\varphi(x) := \sqrt{x(1+x)}$ and the space $\widetilde{W}_p(w)[0,\infty)$ is defined, in the case $\gamma \le 0$, by

$$\widetilde{W}_p(w)[0,\infty) := \Big\{g \in AC^1_{loc}(0,\infty)\colon g, \widehat{D}g \in L_p(w)[0,\infty),\ \lim_{x\to 0+0} \varphi^2(x)g'(x) = 0\Big\},$$

and, for $\gamma > 0$, the functions in $\widetilde{W}_p(w)[0,\infty)$ are, in addition, required to satisfy the condition $\lim_{x\to\infty} \varphi^2(x)g'(x) = 0$. As usually, we denote by $AC^m_{loc}(0,\infty)$, where $m \in \mathbb{N}_0$, the space of all functions on $(0,\infty)$, which possess absolute continuous derivatives up to order m on any finite closed subinterval of $(0,\infty)$.

Gadjev [9] proved the direct estimate

$$\|\widetilde{V}_n f - f\|_p \le c\widetilde{K}(f, n^{-1})_{1,p},$$

for all $f \in L_p[0,\infty)$, $1 \le p \le \infty$, and $n \in \mathbb{N}_+$, $n \ge 2$. Here c is a positive constant whose value is independent of f and n.

That estimate was generalised in $L_p(w)[0,\infty)$, $1 \le p \le \infty$, for any $\gamma \in \mathbb{R}$ by Parvanov [24]

$$\|w(\widetilde{V}_n f - f)\|_p \le c\widetilde{K}(f, n^{-1})_{w,p}, \quad f \in L_p(w)[0,\infty), \quad n > |\gamma| + 1. \tag{1.2}$$

Gadjev [9] (for $\gamma = 0$) and Gadjev and Uluchev [17] (for $\gamma < 0$) proved a two-term strong converse inequality when $1 < p \le \infty$, which shows that (1.2) cannot be improved for these p and γ.

We will characterise the K-functional $\widetilde{K}(f,t)_{w,p}$ for $\gamma \le 0$ by the weighted Ditzian-Totik modulus of continuity [4, (6.1.5) and (3.2.1)]

$$\bar{\omega}^1_{1+\chi}(f,t)_{w,p} := \sup_{0<h\le t} \|w\vec{\Delta}_{h(1+\chi)} f\|_p,$$

where $\chi(x) := x$ and

$$\vec{\Delta}_\tau f(x) := f(x+\tau) - f(x),$$

and the weighted Ditzian-Totik modulus of smoothness of a second order $\omega^2_\varphi(f,t)_{w,p}$, defined by [4, (6.1.5)]

$$\omega^2_\varphi(f,t)_{w,p} := \sup_{0<h\le t} \|w\Delta^2_{h\varphi} f\|_p,$$

where

$$\Delta^2_\tau f(x) := \begin{cases} f(x+\tau) - 2f(x) + f(x-\tau), & x-\tau \ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

We will show that the following equivalence relation holds. Before that we need to introduce a relation. We say that the real-valued functionals $A(f,t)$ and $B(f,t)$ are equivalent and write $A(f,t) \sim B(f,t)$ for f and t in specified domains if and only if there exists a positive constant c such that $c^{-1}B(f,t) \le A(f,t) \le cB(f,t)$ for all f and t in the specified domains.

Theorem 1.1. *Let $1 < p \le \infty$ and $w(x)$ be given by (1.1) with $\gamma \le 0$. Then there exists $t_0 > 0$ such that*

$$\widetilde{K}(f,t^2)_{w,p} \sim \omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p},$$

where $f \in L_p(w)[0,\infty)$ and $0 < t \le t_0$.

Remark 1.2. As we will show in the proof of the theorem, the inequality

$$\widetilde{K}(f,t^2)_{w,p} \le c\big(\omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p}\big),$$

where $f \in L_p(w)[0,\infty)$ and $0 < t \le t_0$ with some $t_0 > 0$, holds for $p = 1$ as well. Then, in the case $\gamma \le 0$, the direct estimate (1.2) can be stated in the form

$$\|w(\widetilde{V}_n f - f)\|_p \le c\big(\omega_\varphi^2(f,n^{-1/2})_{w,p} + \bar{\omega}_{1+\chi}^1(f,n^{-1})_{w,p}\big)$$

for all $f \in L_p(w)[0,\infty)$, $1 \le p \le \infty$, and $n \ge n_0$ with some constant $n_0 \ge 2$, which is independent of f.

Relations like the one in Theorem 1.1 are not new. The first one of this type known to the author was proved by Gonska and Zhou [18, Theorem 1.2 and Remark 1.3] (see also [1, Theorem B]) for the approximation by the Kantorovich operator in $L_p[0,1]$. It is for $1 < p \le \infty$, too. Later on, the author established a similar characterisation of the rate of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums [6]. About the case $p = 1$, Ivanov [22] introduced a different kind of a modulus of smoothness to characterise the K-functional associated to the Kantorovich and Durrmeyer operator.

A very closely related result like the one in Theorem 1.1 in the unweighted case was established in [10]. In the case $p = \infty$, the characterisation in Theorem 1.1 was essentially established in [7].

Let us recall that the weighted approximation by the Baskakov operator and the Meyer-König and Zeller operator are closely related [25] (see also [8] and [12, Section 2]). This relation might turn useful to transfer results about the rate of approximation by the Baskakov-Kantorovich operator to the Kantorovich form of the Meyer-König and Zeller operator. Direct estimates about the latter were established in [11, 13]. Also, it seems quite reasonable to expect that a similar characterisation of the rate of approximation can be established for the operators considered in [15, 16, 21] as well. In this regard, the K-functional used in [14], where such operators were considered, is equivalent to the one characterised in [5, Theorem 5.1] with $r = 2$. Finally, such an approach should be effective to characterise the K-functionals associated to the approximation rate of combinations of exponential-type operators of the type introduced in [2, 23] and [4, Section 9.2]. Approximation results about such operators were established, e.g., in [26–28] (see also [19]).

2. Preliminaries

We will relate the K-functional $\widetilde{K}(f,t)_{w,p}$ to two simpler ones. They are defined by

$$K_{2,\varphi}(f,t)_{w,p} := \inf\big\{\|w(f-g)\|_p + t\|w\varphi^2 g''\|_p\colon \\ g\in AC^1_{loc}(0,\infty),\ g,\varphi^2 g''\in L_p(w)[0,\infty)\big\}$$

and

$$K_{1,1+\chi}(f,t)_{w,p} := \inf\big\{\|w(f-g)\|_p + t\|w(1+\chi)g'\|_p\colon \\ g\in AC_{loc}(0,\infty),\ g,(1+\chi)g'\in L_p(w)[0,\infty)\big\}.$$

To recall, we have set $\chi(x) := x$.

By [4, Theorem 6.1.1], there exists $t_0 > 0$ such that for all $f \in L_p(w)(0,\infty)$ and $t \in (0,t_0]$ there hold

$$K_{2,\varphi}(f,t^2)_{w,p} \sim \omega^2_\varphi(f,t)_{w,p} \tag{2.1}$$

and

$$K_{1,1+\chi}(f,t)_{w,p} \sim \bar{\omega}^1_{1+\chi}(f,t)_{w,p}. \tag{2.2}$$

We will also make use of the K-functional

$$K_{2,\phi}(f,t)_{p,[0,1]} := \inf\big\{\|f-g\|_{p,[0,1]} + t\|\phi^2 g''\|_{p,[0,1]}\colon \\ g\in AC^1_{loc}(0,1),\ g,\phi^2 g''\in L_p[0,1]\big\},$$

where $\|\cdot\|_{p,J}$ stands for the standard L_p-norm on the interval J and $\phi(x) := \sqrt{x(1-x)}$. Similarly, we have (see [4, Theorem 2.1.1]) that there exists $t_0 > 0$ such that for all $f \in L_p[0,1]$ and $t \in (0,t_0]$ there holds

$$K_{2,\phi}(f,t^2)_{p,[0,1]} \sim \omega^2_\phi(f,t)_{p,[0,1]}. \tag{2.3}$$

Here $\omega^2_\phi(f,t)_{p,[0,1]}$ is the Ditzian-Totik modulus of smoothness of order 2, defined in [4, (2.1.2)] by

$$\omega^2_\phi(f,t)_{p,[0,1]} := \sup_{0<h\le t}\|\bar{\Delta}^2_{h\phi}f\|_{p,[0,1]},$$

where

$$\bar{\Delta}^2_\tau f(x) := \begin{cases} f(x+\tau)-2f(x)+f(x-\tau), & x\pm\tau\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

The last two K-functionals which will play an auxiliary role in the proof of the main result are given by

$$K_{2,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \inf\big\{\|\chi^\gamma(f-g)\|_{p,[1/4,\infty)} + t\|\chi^{\gamma+2}g''\|_{p,[1/4,\infty)}\colon \\ g\in AC^1_{loc}(1/4,\infty),\ g,\chi^2 g''\in L_p(\chi^\gamma)[1/4,\infty)\big\}$$

and

$$K_{1,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \inf\big\{\|\chi^\gamma(f-g)\|_{p,[1/4,\infty)} + t\|\chi^{\gamma+1}g'\|_{p,[1/4,\infty)} : \\ g \in AC_{loc}(1/4,\infty),\ g, \chi g' \in L_p(\chi^\gamma)[1/4,\infty)\big\}. \quad (2.4)$$

The first K-functional is related to the weighted Ditzian-Totik modulus of smoothness of second order, defined by [4, (6.1.5)]

$$\omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \sup_{0<h\le t} \|\chi^\gamma \Delta^2_{h\chi} f\|_{p,[1/4,\infty)},$$

and the second K-functional to the weighted Ditzian-Totik modulus of continuity, defined by [4, (6.1.5)]

$$\bar\omega^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \sup_{0<h\le t} \|\chi^\gamma \overrightarrow{\Delta}_{h\chi} f\|_{p,[1/4,\infty)},$$

where

$$\Delta^2_\tau f(x) := \begin{cases} f(x+\tau) - 2f(x) + f(x-\tau), & x-\tau \ge 1/4, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\overrightarrow{\Delta}_\tau f(x) := f(x+\tau) - f(x).$$

We have, by [4, Theorem 6.1.1], that there exists $t_0 > 0$ such that for all $f \in L_p(w)(0,\infty)$ and $t \in (0, t_0]$ there hold

$$K_{2,\chi}(f,t^2)_{\chi^\gamma,p,[1/4,\infty)} \sim \omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} \quad (2.5)$$

and

$$K_{1,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} \sim \bar\omega^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}. \quad (2.6)$$

In all the above instances, the modulus of continuity is defined by the forward finite difference rather than by the symmetric one as it was done in the cited definitions in [4]. We do that for technical convenience only. We still have (2.2) and (2.6) for the forward finite difference modulus of continuity. That was shown in [4, Theorem 3.2.1] for the case $\gamma = 0$, but the proof can be readily extended to any real γ.

In order to estimate the K-functional $\widetilde{K}(f,t)_{w,p}$ from below by means of the K-functionals $K_{2,\varphi}(f,t)_{w,p}$ and $K_{1,1+\chi}(f,t)_{w,p}$, hence, in view of (2.1) and (2.2), by $\omega^2_\varphi(f,t)_{w,p}$ and $\bar\omega^1_{1+\chi}(f,t)_{w,p}$, we will use the embedding inequalities below. They are known (see [17, Lemma 4]), we include their short proof for the reader's convenience.

Proposition 2.1. *Let $1 < p \le \infty$ and $w(x)$ be given by (1.1) with $\gamma \le 0$. Let $g \in \widetilde{W}_p(w)[0,\infty)$. Then*

$$\|w(1+\chi)g'\|_p \le \frac{p}{p-1}\,\big\|w\widehat{D}g\big\|_p \quad (2.7)$$

and

$$\|w\varphi^2 g''\|_p \le \frac{3p-1}{p-1}\,\big\|w\widehat{D}g\big\|_p. \quad (2.8)$$

For $p = \infty$, the expressions $p/(p-1)$ and $(3p-1)/(p-1)$ are to be interpreted as their limit at infinity.

Proof. Since $|w(x)\varphi^2(x)g''(x)| \le 2|w(x)(1+x)g'(x)| + |w(x)\widehat{D}g(x)|$, it is enough to show (2.7).

We let $\varepsilon \to 0+0$ in

$$\int_{\varepsilon}^{x} \widehat{D}g(u)\, du = \varphi^2(x)g'(x) - \varphi^2(\varepsilon)g'(\varepsilon), \quad x > 0,$$

to arrive at

$$x(1+x)g'(x) = \int_0^x \widehat{D}g(u)\, du, \quad x > 0. \tag{2.9}$$

We applied the Dominated Convergence Theorem and $\lim_{\varepsilon\to 0+0} \varphi^2(\varepsilon)g'(\varepsilon) = 0$. Regarding the former, we have $\widehat{D}g \in L_1[0,x]$ for any $x > 0$ by virtue of Hölder's inequality.

Next, (2.9) yields

$$|w(x)(1+x)g'(x)| \le \frac{1}{x}\int_0^x |w(u)\widehat{D}g(u)|\, du, \quad x > 0; \tag{2.10}$$

hence, for $1 < p < \infty$,

$$\left(\int_0^\infty |w(x)(1+x)g'(x)|^p\, dx\right)^{1/p} \le \left(\int_0^\infty \left(\frac{1}{x}\int_0^x |w(u)\widehat{D}g(u)|\, du\right)^p dx\right)^{1/p}.$$

Now, by virtue of Hardy's inequality (see [20, p. 245])

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |F(u)|\, du\right)^p dx\right)^{1/p} \le \frac{p}{p-1}\,\|F\|_p,$$

with $F(u) = w(u)\widehat{D}g(u)$, we arrive at (2.7) for $1 < p < \infty$.

In the case $p = \infty$, we readily derive from (2.10) the inequality

$$\|w(1+\chi)g'\|_\infty \le \left\|w\widehat{D}g\right\|_\infty.$$

Thus (2.7) is established. □

3. Proof of Theorem 1.1

We denote by c and t_0 positive constants, whose value is independent of the functions involved, the function variable and t. Their value can vary at each occurrence.

First, we will show that there exist a positive constant t_0 such that for all $f \in L_p(w)[0,\infty)$ and $t \in (0, t_0]$, there holds

$$\omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p}. \tag{3.1}$$

Let $g \in \widetilde{W}_p(w)[0,\infty)$. By Proposition 2.1, there hold

$$\|w(1+\chi)g'\|_p \le c\|w\widehat{D}g\|_p$$

and

$$\|w\varphi^2 g''\|_p \le c\|w\widehat{D}g\|_p.$$

Consequently, $(1+\chi)g', \varphi^2 g'' \in L_p(w)[0,\infty)$,

$$K_{1,1+\chi}(f,t)_{w,p} \le \|w(f-g)\|_p + t\|w(1+\chi)g'\|_p \le c\big(\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\big)$$

and

$$K_{2,\varphi}(f,t)_{w,p} \le \|w(f-g)\|_p + t\|w\varphi^2 g''\|_p \le c\big(\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\big).$$

Next, we take the infimum on $g \in \widetilde{W}_p(w)[0,\infty)$ to arrive at

$$K_{1,1+\chi}(f,t)_{w,p} \le c\widetilde{K}(f,t)_{w,p}, \quad t>0,$$

and

$$K_{2,\varphi}(f,t)_{w,p} \le c\widetilde{K}(f,t)_{w,p}, \quad t>0.$$

Now, (2.1) and (2.2) imply

$$\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p},$$

and

$$\omega^2_\varphi(f,t)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p},$$

where $0<t\le t_0$ with some t_0 independent of f; hence we get (3.1).

To establish the reverse relation, we use the same approach as in the proof of [7, Theorem 1.2], where the case $p=\infty$ was considered.

Let $1\le p\le\infty$. For any $t\in(0,t_0]$ with some $t_0\in(0,1]$ to be specified in the course of the proof, we will define a function $g_t \in \widetilde{W}_p(w)[0,\infty)$ such that

$$\|w(f-g_t)\|_p \le c\,\omega^2_\varphi(f,t)_{w,p}, \tag{3.2}$$

and

$$t^2\big\|w\widehat{D}g_t\big\|_p \le c\left(\omega^2_\varphi(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right). \tag{3.3}$$

Then we readily get

$$\widetilde{K}(f,t)_{w,p} \le c\left(\omega^2_\varphi(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0<t\le t_0.$$

To establish (3.3), we will show that

$$t^2\|w(1+\chi)g_t'\|_p \le c\left(\omega^2_\varphi(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right) \tag{3.4}$$

and

$$t^2\|w\varphi^2 g_t''\|_p \le c\left(\omega^2_\varphi(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right). \tag{3.5}$$

We will define g_t by patching smoothly two approximants of f on $[0,3/4]$ and $[1/2,\infty)$, respectively, which satisfy (3.2), (3.4) and (3.5).

We begin with the approximant on $[0,3/4]$. We use the same argument as in [6, p. 132] and [7, pp. 25–26]. It is not long and we include it for the sake of completeness. Let $m \in \mathbb{N}_+$ be such that $t \in (1/(m+1), 1/m]$ and let $Q_t(x) := Q_T(f)(x)$ be the quasi-interpolant spline operator of order $r+1$ with knots t_j, $j = -m+1, \dots, m-1$, used in the proof of [3, Chapter 6, Theorem 6.2] with $r = 2$ for the interval $[0,1]$ instead of $[-1,1]$.

We have, by [3, Chapter 5, Proposition 4.6, and Chapter 6, Theorem 4.2, (6.22) and (6.24)]), that $Q_t \in AC^1[0,1]$ and

$$\begin{aligned} \|f - Q_t\|_{p,[0,1]} &\le c\,\omega_\phi^2(f,t)_{p,[0,1]}, \\ t^2\|Q_t'\|_{p,[0,1]} &\le c\,\omega(f,t^2)_{p,[0,1]}, \\ t^2\|\phi^2 Q_t''\|_{p,[0,1]} &\le c\,\omega_\phi^2(f,t)_{p,[0,1]} \end{aligned} \tag{3.6}$$

for $0 < t \le 1/4$. Here $\omega(f,t)_{p,[0,1]}$ denotes the classical modulus of continuity in $L_p[0,1]$.

Next, since $w(x) \ge c > 0$ and $\phi(x) \le \varphi(x)$ on $[0,1]$, then

$$K_{2,\phi}(f,t)_{p,[0,1]} \le c\,K_{2,\varphi}(f,t)_{w,p};$$

hence, by virtue of and (2.1) and (2.3) (cf. [4, Theorem 4.1.1])

$$\omega_\phi^2(f,t)_{p,[0,1]} \le c\,\omega_\varphi^2(f,t)_{w,p} \tag{3.7}$$

for $0 < t \le t_0$ with some $t_0 \in (0,1)$. In addition, similarly,

$$\omega(f,t^2)_{p,[0,1]} \le c\,\bar\omega_{1+\chi}^1(f,t^2)_{w,p}, \quad 0 < t \le t_0.$$

We combine the last two estimates with (3.6) to deduce

$$\|f - Q_t\|_{p,[0,3/4]} \le c\,\omega_\varphi^2(f,t)_{w,p}, \tag{3.8}$$

$$t^2\|Q_t'\|_{p,[0,3/4]} \le c\,\bar\omega_{1+\chi}^1(f,t^2)_{w,p}, \tag{3.9}$$

$$t^2\|\varphi^2 Q_t''\|_{p,[0,3/4]} \le c\,\omega_\varphi^2(f,t)_{w,p} \tag{3.10}$$

for $0 < t \le t_0$.

We proceed to the definition of the approximant on $[1/2,\infty)$. We will use the same one as in the second half of the proof of [7, Theorem 1.2], where the case $p = \infty$ was only considered. The prove below include all $\gamma \in \mathbb{R}$ and $1 \le p \le \infty$.

Let $1 \le p < \infty$. Following [7, p. 27], we introduce the Steklov-type function

$$H_t(x) := \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2} \left[f\big(x + x(u_1+u_2)\big) + f\big(x - x(u_1+u_2)\big)\right] du_1\, du_2,$$

where $0 < t \le 1/2$ and $x \ge 1/2$.

By Minkowski's integral inequality, we arrive at

$$\begin{aligned}\|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)} &\le \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2}\|\chi^\gamma\Delta^2_{(u_1+u_2)\chi}f\|_{p,[1/2,\infty)}\,du_1\,du_2\\ &\le \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2}\omega^2_\chi(f,t)_{\chi^\gamma,[1/4,\infty)}\,du_1\,du_2\\ &\le \omega^2_\chi(f,t)_{\chi^\gamma,[1/4,\infty)}.\end{aligned}$$

Then similarly to (3.7), we use (2.1) and (2.5) to further deduce

$$\|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)} \le c\,\omega^2_\varphi(f,t)_{w,p},\quad 0<t\le t_0. \tag{3.11}$$

We set

$$\widetilde{H}_t(x) := \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2} f\big(x+x(u_1+u_2)\big)\,du_1\,du_2.$$

We have $H_t(x)=\widetilde{H}_t(x)+\widetilde{H}_{-t}(x)$.

We write $\widetilde{H}_t$ in the form

$$\widetilde{H}_t(x) = \frac{2}{x^2t^2}\int_x^{x(1+t/2)}(u-x)f(u)\,du + \frac{2}{x^2t^2}\int_{x(1+t/2)}^{x(1+t)}(x(1+t)-u)f(u)\,du.$$

Therefore, $\widetilde{H}_t\in AC^1[1/2,\infty)$ and straightforward calculations yield

$$\begin{aligned}\widetilde{H}'_t(x) &= \frac{2}{xt^2}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\\ &\qquad -\frac{4}{xt^2}\int_0^{t/2}\int_0^{t/2}\left[f\big(x+x(u_1+u_2)\big)-f\left(x+x\left(u_1+\frac{t}{2}\right)\right)\right]du_1\,du_2.\end{aligned}$$

To estimate the $L_p(\chi^{\gamma+1})[1/2,\infty)$-norm of each of the integrals above, we subtract and add $f(x)$, split the integral into two terms by the triangle inequality, and apply Minkowski's integral inequality. For the first integral, we have

$$\begin{aligned}&\left(\int_{1/2}^\infty\left|x^{\gamma+1}\frac{1}{x}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\right|^p dx\right)^{1/p}\\ &\quad\le \left(\int_{1/2}^\infty\left|x^\gamma\int_0^{t/2}\left|f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x)\right|du\right|^p dx\right)^{1/p}\\ &\qquad+\left(\int_{1/2}^\infty\left|x^\gamma\int_0^{t/2}|f(x+xu)-f(x)|\,du\right|^p dx\right)^{1/p}\\ &\quad\le \int_0^{t/2}\big\|\chi^\gamma\overrightarrow{\Delta}_{(u+t/2)\chi}f\big\|_{p,[1/2,\infty)}\,du+\int_0^{t/2}\big\|\chi^\gamma\overrightarrow{\Delta}_{u\chi}f\big\|_{p,[1/2,\infty)}\,du\\ &\quad\le t\,\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}.\end{aligned}$$

We estimate the $L_p(\chi^{\gamma+1})[1/2,\infty)$-norm of the second integral in the above expression of $\widetilde{H}'_t$ in a similar way. Thus, we arrive at

$$t^2\|\chi^{\gamma+1}\widetilde{H}'_t\|_{p,[1/2,\infty)} \le ct\,\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} \le c\,\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}, \quad 0<t\le t_0,$$

as for the second estimate we took into account (2.2) and (2.6).

We prove the analogue of this estimate for $\widetilde{H}_{-t}$ in a similar way, as we take into account the fact that if we replace the forward finite difference in the definition of $\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}$ with the backward one, we get a modulus, which is still equivalent to the same K-functional; hence to $\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}$ (see [4, Section 3.2]).

Consequently,

$$t^2\|\chi^{\gamma+1}H'_t\|_{p,[1/2,\infty)} \le c\,\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}, \quad 0<t\le t_0. \tag{3.12}$$

By means of straightforward calculations, we arrive at

$$\begin{aligned}\widetilde{H}''_t(x) = &-\frac{8}{x^2t^2}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\\ &-\frac{2}{x^2t}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f\left(x+x\frac{t}{2}\right)\right]du\\ &+\frac{12}{x^2t^2}\int_0^{t/2}\int_0^{t/2}\left[f(x+x(u_1+u_2))-f\left(x+x\left(u_1+\frac{t}{2}\right)\right)\right]du_1\,du_2\\ &+\frac{2(t+2)}{x^2t}\left[f(x+xt)-f\left(x+x\frac{t}{2}\right)\right]\\ &+\frac{2}{x^2t^2}\left[f(x+xt)-2f\left(x+x\frac{t}{2}\right)+f(x)\right].\end{aligned}$$

We estimate the $L_p(\chi^{\gamma+2})[1/2,\infty)$-norm, multiplied by t^2, of all the terms on the first four lines on the right-hand side above by $\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}$ as we did for $\widetilde{H}'_t$. Likewise, we estimate the norm of the corresponding terms in $\widetilde{H}''_{-t}$. As for the terms on the last line, we consider their sum with the corresponding terms in $\widetilde{H}''_{-t}$ and write this sum in the form

$$\begin{aligned}G_t(x) :=& \frac{2}{x^2t^2}\left[f(x+xt)-2f\left(x+x\frac{t}{2}\right)+f(x)\right]\\ &+\frac{2}{x^2t^2}\left[f(x-xt)-2f\left(x-x\frac{t}{2}\right)+f(x)\right]\\ =& \frac{2}{x^2t^2}\,\Delta^2_{xt}f(x)-\frac{4}{x^2t^2}\,\Delta^2_{xt/2}f(x);\end{aligned}$$

hence

$$t^2\|\chi^{\gamma+2}G_t\|_{p,[1/2,\infty)} \le c\,\omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}.$$

Further, we use (2.1) and (2.5) to get

$$t^2\|\chi^{\gamma+2}G_t\|_{p,[1/2,\infty)} \le c\,\omega^2_\varphi(f,t)_{w,p}.$$

Thus, we show that

$$t^2\|\chi^{\gamma+2}H_t''\|_{p,[1/2,\infty)} \le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0 < t \le t_0. \tag{3.13}$$

We are ready to define $g_t(x)$ and derive that it satisfies (3.2), (3.4) and (3.5). We apply a standard argument (see, e.g., [3, p. 176]). Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi(x) = 0$ for $x \le 1/2$ and $\psi(x) = 1$ for $x \ge 3/4$. Clearly, there exists a positive constant c such that $|\psi^{(i)}(x)| \le c$, $x \in \mathbb{R}$, where $i = 0, 1, 2$. We set

$$g_t(x) := (1-\psi(x))Q_t(x) + \psi(x)H_t(x), \quad x \ge 0.$$

Then $g_t \in AC^1_{loc}(0,\infty)$ and $\lim_{x\to 0+0}\varphi^2(x)g_t'(x) = \lim_{x\to 0+0} x(1+x)Q_t'(x) = 0$.

Next, we take into account that $|\psi(x)| \le c$ for $x \in \mathbb{R}$, $\psi(x) = 0$ for $x \le 1/2$ and $\psi(x) = 1$ for $x \ge 3/4$, and (3.8) and (3.11) to get

$$\begin{aligned}\|w(f-g_t)\|_p &\le c\big(\|f-Q_t\|_{p,[0,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)}\big)\\ &\le c\,\omega_\varphi^2(f,t)_{w,p}, \quad 0 < t \le t_0.\end{aligned}$$

Thus, (3.2) is established.

We represent the first derivative of g_t in the form

$$g_t'(x) = \psi'(x)[f(x)-Q_t(x)] - \psi'(x)[f(x)-H_t(x)] + (1-\psi(x))Q_t'(x) + \psi(x)H_t'(x).$$

Then, similarly to the last estimate, but using also (3.9) and (3.12), we get

$$\begin{aligned}t^2\|w(1+\chi)g_t'\|_p &\le c\big(\|f-Q_t\|_{p,[1/2,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,3/4]}\\ &\qquad + \|Q_t'\|_{p,[0,3/4]} + \|\chi^{\gamma+1}H_t'\|_{p,[1/2,\infty)}\big)\\ &\le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0 < t \le t_0.\end{aligned}$$

Thus, (3.4) is established.

Finally, for the second derivative of g_t, we have

$$\begin{aligned}g_t''(x) &= \psi''(x)[f(x)-Q_t(x)] - \psi''(x)[f(x)-H_t(x)]\\ &\qquad - 2\psi'(x)Q_t'(x) + 2\psi'(x)H_t'(x) + (1-\psi(x))Q_t''(x) + \psi(x)H_t''(x).\end{aligned}$$

Then, similarly to the last two estimates, but taking into account all relations (3.8)–(3.10) and (3.11)–(3.13), we arrive at

$$\begin{aligned}t^2\|w\varphi^2 g_t''\|_p &\le c\big(\|f-Q_t\|_{p,[1/2,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,3/4]}\\ &\qquad + \|Q_t'\|_{p,[1/2,3/4]} + \|\chi^{\gamma+1}H_t'\|_{p,[1/2,3/4]}\big)\\ &\qquad + \|Q_t''\|_{p,[0,3/4]} + \|\chi^{\gamma+2}H_t''\|_{p,[1/2,\infty)}\big)\\ &\le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0 < t \le t_0.\end{aligned}$$

Thus, we have shown (3.5) and completed the proof of $g_t \in \widetilde{W}_p(w)[0,\infty)$ and the second part of the proof of Theorem 1.1 for $1 \le p < \infty$. The validity of that part for $p = \infty$ was shown in [7, pp. 27–29].

Acknowledgements

The research is supported by the Bulgarian National Science Fund under Contract KP-06-N62/4 from 2022. I am thankful to the reviewers for their remarks which improved the exposition and corrected several mistakes.

References

[1] H. Berens and Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights: The cases $p = 1$ and $p = \infty$, in: Approximation, interpolation, and summation, ed. by S. Baron and D. Leviatan, Israel Mathematical Conference Proceedings, vol. 4, Jerusalem, Weizmann Science Press of Israel, 1991, 51–62.

[2] P. L. Butzer, Linear combinations of Bernstein polynomials, Canad. J. Math. 5 (1953) 559–567.

[3] R. A. DeVore and G. G. Lorentz, Constructive approximation, Springer-Verlag, Berlin, 1993.

[4] Z. Ditzian and V. Totik, Moduli of smoothness, Springer-Verlag, New York, 1987.

[5] B. R. Draganov, On simultaneous approximation by iterated Boolean sums of Bernstein operators, Results Math. 66 (2014) 21–41.

[6] B. R. Draganov, Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums, J. Approx. Theory 200 (2015) 92–135.

[7] B. R. Draganov, Direct estimates of the weighted simultaneous approximation by the Baskakov operator, Results Math. 75 (2020) 156.

[8] I. Gadjev, Strong converse result for uniform approximation by Meyer-König and Zeller operator, J. Math. Anal. Appl. 428 (2015) 32–42.

[9] I. Gadjev, Approximation of functions by Baskakov-Kantorovich operator, Results Math. 70 (2016) 385–400.

[10] I. Gadjev, About characterization of one K-functional, J. Math. Anal. Appl. 450 (2017) 1076–1082.

[11] I. Gadjev, A direct theorem for MKZ-Kantorovich operator, Anal. Math. 45 (2019) 25–38.

[12] I. Gadjev and P. E. Parvanov, Weighted approximation in uniform norm by Meyer-König and Zeller operators, Ann. Sofia Univ. Fac. Math and Inf. 104 (2017) 77–87.

[13] I. Gadjev, P. E. Parvanov and R. Uluchev, Weighted approximation by Kantorovich type modification of Meyer-König and Zeller operator, Ann. Sofia Univ. Fac. Math and Inf. 105 (2018) 75–95.

[14] I. Gadjev, P. E. Parvanov and R. Uluchev, Higher order approximation of functions by modified Goodman-Sharma operators, Constr. Math. Anal. 7(4) (2024) 180–195.

[15] I. Gadjev, P. E. Parvanov and R. Uluchev, On a new modification of Baskakov operators with higher order of approximation, Mathematics 13 (2025) 64.

[16] I. Gadjev, P. E. Parvanov and R. Uluchev, Higher order approximation of continuous functions by a modified Meyer-König and Zeller-Type operator, arXiv preprint, arXiv:2506.14392, 2025.

[17] I. Gadjev and R. Uluchev, Strong converse result for weighted approximation by Baskakov-Kantorovich operator, in: Proc. Internat. Conf. “Constructive Theory of Functions”, Sozopol, Bulgaria, June 2–8, 2019, ed. by B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev, Bulgarian Acad. Sci., Sofia, 2020, 53–71.

[18] H. Gonska and X.-l. Zhou, The strong converse inequality for Bernstein-Kantorovich operators, Comput. Math. Appl. 30 (1995) 103–128.
[19] V. Gupta and G. Tachev, Approximation with positive linear operators and linear combinations, Springer, Cham, 2017.
[20] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, 1951.
[21] M. Heilmann and M. Wagner, The genuine Bernstein-Durrmeyer operators and quasi-interpolants, Results. Math. 62 (2012) 319–335.
[22] K. G. Ivanov, A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators, in: Approximation theory, A volume dedicated to Borislav Bojanov, ed. by D. K. Dimitrov, G. Nikolov and R. Uluchev, Marin Drinov Academic Publishing House, Sofia, 2004, 112–125.
[23] C. P. May, Saturation and inverse theorems for combinations of a class of exponential-type operators, Canad. J. Math. 28 (1976) 1224–1250.
[24] P. E. Parvanov, Weighted approximation of functions in L_p-norm by Baskakov-Kantorovich operator, Anal. Math. 46 (2020) 821–842.
[25] V. Totik, Approximation by Meyer-König and Zeller type operators, Math. Z. 182 (1983) 425–446.
[26] S. Wang and L. Xie, Strong converse inequality for linear combinations of Szász-Mirakjan operators, J. Approx. Theory 273 (2022) 105651.
[27] S. Wang and L. Xie, The L_p-saturation of linear combinations of Szász-Mirakjan-Kantorovich operators, Int. J. Wavelets Multiresolut. Inf. Process. 23(1) (2025) 2450046.
[28] L. Xie, The saturation class for linear combinations of Bernstein operators, Arch. Math. 91 (2008) 86–96.

Received on October 11, 2025
Revised on November 6, 2025
Accepted on November 7, 2025

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