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HARMONIC MAPS OF COMPACT KÄHLER MANIFOLDS TO EXCEPTIONAL LOCAL SYMMETRIC SPACES OF HODGE TYPE AND HOLOMORPHIC LIFTINGS TO COMPLEX HOMOGENEOUS FIBRATIONS

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Let M be a compact Kähler manifold and G/K be a non-Hermitian Riemannian symmetric space of Hodge type. Certain harmonic maps $f : M \to \Gamma \setminus G/K$ will be proved to admit holomorphic liftings $F_P : M \to \Gamma \setminus G/G \cap P$ to complex homogeneous fibrations, where P are parabolic subgroups of $G^{\mathbb{C}}$. The work studies whether the images $F_P(M) = \Gamma_h \setminus G_h/K_h$ are local equivariantly embedded Hermitian symmetric subspaces of $\Gamma \setminus G/G \cap P$. For each of the cases examples of harmonic maps f which do not admit holomorphic liftings are supplied.

Keywords: harmonic and holomorphic maps, exceptional Riemannian symmetric spaces of Hodge type, complex homogeneous fibrations, abelian subspaces, Levi-Civita connections, equivariant Hermitian symmetric subspaces

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1. STATEMENT OF THE RESULTS

Let M be a compact Kähler manifold and $\Gamma \setminus G/K$ be a local Riemannian symmetric space of noncompact type. The results of Eells and Sampson from [7] imply that whenever $\Gamma \setminus G/K$ is compact, any continuous map $c : M \to \Gamma \setminus G/K$ is homotopic to a harmonic map $f : M \to \Gamma \setminus G/K$. Corlette has proved in [6] that a continuous map $c : M \to \Gamma \setminus G/K$ has a unique harmonic representative $f : M \to \Gamma \setminus G/K$ in its homotopy class if and only if the image $c_*\pi_1(M)$ of the induced representation $c_* : \pi_1(M) \to \Gamma$ has reductive real Zariski closure in G.

The present article studies the harmonic maps $f : M \to \Gamma \setminus G/K$ for which there exist parabolic subgroups $P \subset G^{\mathbb{C}}$, complex homogeneous fibrations $\Pi_P : G/G \cap P \to G/K$ and holomorphic liftings $F_P : M \to \Gamma \setminus G/G \cap P$, such that $f = \Pi_P F_P$. For the compact discrete quotients $\Gamma \setminus G/K$ of the irreducible classical Hermitian symmetric spaces G/K of noncompact type and $\dim_{\mathbb{C}} G/K \geq 3$, Siu has established in [14] that the harmonic maps $f : M \to \Gamma \setminus G/K$ of maximum constant $rank_x^{\mathbb{C}} df := \dim_{\mathbb{C}} df_x^{\mathbb{C}}(T_x^{1,0}M) = \dim_{\mathbb{C}} G/K$, $\forall x \in M$, are either holomorphic or anti-holomorphic. Following Helgason's classification [8] of the irreducible Riemannian symmetric spaces G/K of noncompact type, we recall the other known results for the harmonic maps $f : M \to \Gamma \setminus G/K$ of compact Kähler manifolds M. Carlson and Toledo show in [4] that the nonconstant non-holomorphic (non-anti-holomorphic) harmonic $f : M \to \Gamma \setminus SU(n,1)/S(U_n \times U_1)$ either map to a closed geodesic $f(M)$ or factor through a holomorphic map to a Riemann surface. For a harmonic map $f : M \to \Gamma \setminus SL(2n, \mathbb{R})/SO(2n)$ with $n \geq 3$ they establish in [5] that $rank_x^{\mathbb{C}} df \leq \dfrac{n(n+1)}{2}$ for $x \in M$, and the equality is realized only by the holomorphic maps of maximum constant rank onto an equivariantly embedded discrete quotient of $Sp(n,\mathbb{R})/U_n$. In [9] is proved that the harmonic maps $f : M \to \Gamma \setminus SL(2n+1,\mathbb{R})/SO(2n+1)$, $n \geq 4$, are of $rank_x^{\mathbb{C}} df \leq \dfrac{n(n+1)}{2} + 1$ at $x \in M$. The equality is attained by the holomorphic f with $f(M) = \Gamma_h \backslash G_h/K_h$ for a Hermitian symmetric (not necessarily equivariant) subspace $G_h/K_h \subset SL(2n+1,\mathbb{R})/SO(2n+1)$ or by a non-holomorphic f with $f(M) = \Gamma_0 \setminus (Sp(n,\mathbb{R})/U_n \times T^1)$, where $Sp(n,\mathbb{R})/U_n \subset SL(2n+1,\mathbb{R})/SO(2n+1)$ is an equivariant subspace and $T^1 \subset SL(2n+1,\mathbb{R})$ is a noncompact 1-dimensional torus, centralizing $Sp(n,\mathbb{R})$. Carlson and Toledo show in [5] that the harmonic maps $f : M \to \Gamma \backslash SU^*(2n)/Sp(n)$ with $n \geq 3$ have $rank_x^{\mathbb{C}} df \leq \dfrac{n(n-1)}{2}$ at $x \in M$, and the equality is attained by the holomorphic f onto a discrete quotient of an equivariantly embedded $SO^*(2n)/U_n$. For the harmonic maps $f : M \to \Gamma \setminus SO_0(n,1)/SO(n)$, Carlson and Toledo obtain in [4] that either $f(M)$ is a closed geodesic or f factors through a holomorphic map to a Riemann surface. In [5] they establish that the harmonic maps $f : M \to \Gamma \setminus SO_0(2m,2n)/SO(2m) \times SO(2n)$ with $\min(m,n) \geq 3$, $m+n > 6$, have $rank_x^{\mathbb{C}} df \leq mn$, $x \in M$, and the equality is attained by the holomorphic f onto discrete quotients of equivariantly embedded $SU(m,n)/S(U_m \times U_n)$. For the harmonic maps $f_1 : M \to \Gamma \setminus SO_0(2m+1,2n)/SO(2m+1) \times SO(2n)$ or $f_2 : M \to \Gamma \backslash SO_0(2m+1,2n+1)/SO(2m+1) \times SO(2n+1)$ with $\min(m,n) \geq 5$, the work [9] shows that $rank_x^{\mathbb{C}} df_1 \leq mn+1$, $rank_x^{\mathbb{C}} df_2 \leq mn+2$ and the equalities are attained by the holomorphic f_i onto discrete quotients of (not necessarily equivariant) Hermitian symmetric subspaces. In the case of $f : M \to \Gamma \setminus Sp(n,1)/Sp(n) \times Sp(1)$ with $n \geq 3$, Carlson and Toledo prove in [4] that either $f(M)$ is a closed geodesic or f factors through a holomorphic map to a Riemann surface or f has a holomorphic lifting $F : M \to \Gamma \setminus Sp(n,1)/Sp(n) \times U_1$. In [5] Carlson and Toledo establish that the harmonic $f : M \to \Gamma \setminus Sp(m,n)/Sp(m) \times Sp(n)$ with $\min(m,n) \geq 2$ have

$rank_x^{\mathbb{C}} df \leq mn$ and the equality is attained by the holomorphic f onto discrete quotients of equivariantly embedded $SU(m,n)/S(U_m \times U_n)$. Carlson and Hernandez show in [3] that a harmonic $f : M \to \Gamma \setminus F_{4(-20)}/SO(9) = \Gamma \setminus \mathbf{FII}$ either maps to a closed geodesic or factors through a holomorphic map to a Riemann surface, or factors through a holomorphic map $F : M \to \Gamma_0 \setminus SU(2,1)/S(U_2 \times U_1) = S$ to a discrete quotient S of a 2-ball, followed by a geodesic immersion $S \to \Gamma \setminus \mathbf{FII}$.

Let G be a noncompact simple real Lie group and P be a parabolic subgroup of its complexification $G^{\mathbb{C}}$. A necessary condition for the existence of a fibration $G/G \cap P \to G/K$ is the inclusion of $G \cap P$ in K. First of all, that requires the presence of a Cartan subalgebra $\mathbf{h} \subset \mathbf{g} := LieG$, contained in $\mathbf{k} := LieK$. The noncompact semisimple Lie groups G, whose Lie algebras admit common Cartan subalgebras with the Lie algebras of the maximal compact subgroups K of G, are said to be of Hodge type. According to Simpson [13] or Burstall and Rawnsley [2], a noncompact semisimple Lie group G is of Hodge type exactly when the Cartan involution of G is an inner automorphism. The isometry groups G of the irreducible Hermitian symmetric spaces G/K of noncompact type are groups of Hodge type. According to Simpson [13], the remaining noncompact simple Lie groups of Hodge type are

$$SO(m,2n), \quad Sp(m,n), \quad E_{6(2)}, \quad E_{7(7)}, \quad E_{7(-5)},$$
$$E_{8(8)}, \quad E_{8(-24)}, \quad F_{4(4)}, \quad F_{4(-20)}, \quad G_{2(2)}.$$

Let G_c be the compact real form of G. For a simple Lie group G of Hodge type and a parabolic subgroup $P \subset G^{\mathbb{C}}$ the inclusion $G \cap P \subset K$ is equivalent to $G \cap P = G_c \cap P$ and happens exactly when $G_c \cap P$ is a subgroup of K.

Let us recall that $G^{\mathbb{C}}/P = G_c/G_c \cap P$ is a projective algebraic manifold when $P \subset G^{\mathbb{C}}$ is a parabolic subgroup. For G of Hodge type we claim that the orbit $G/G \cap P$ is an open subset of $G_c/G_c \cap P$. If $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ is the Cartan decomposition of $\mathbf{g}$, then the tangent space $T_{\check{o}}^{\mathbb{R}} G/K$ at the origin $\check{o} \in G/K$ can be identified with $\mathbf{p}$. The exponential map $Exp_{\check{o}}^{G/K} : \mathbf{p} \to G/K$ at $\check{o} \in G/K$ is a global diffeomorphism, due to the nonpositiveness of the sectional curvatures of G/K. Let $Exp_{\check{o}}^{G_c/K} : T_{\check{o}}^{\mathbb{R}} G_c/K = i\mathbf{p} \to G_c/K$ be the locally defined exponential map of the compact dual G_c/K at $\check{o} \in G_c/K$ and $\mu_i : \mathbf{p} \to i\mathbf{p}$ be the multiplication by the imaginary unit i. Then $Exp_{\check{o}}^{G_c/K} \mu_i \left(Exp_{\check{o}}^{G/K} \right)^{-1} : G/K \to G_c/K$ is a local diffeomorphism. Since $G/G \cap P$ and $G_c/G_c \cap P$ have coinciding fibers $K/G \cap P = K/G_c \cap P$, the homogeneous space $G/G \cap P$ is immersed in $G_c/G_c \cap P$. In particular, $G/G \cap P$ is a complex (even Kähler) manifold.

Recall also that for a group G of Hodge type and a parabolic subgroup $P \subset G^{\mathbb{C}}$ with $G \cap P \subset K$ the reductive Lie group $G \cap P$ is a centralizer of a torus $T \subset K$ in G. Conversely, any centralizer $Z \subset G$ of a torus $T \subset K$ determines uniquely the parabolic subgroup P, whose semisimple part is the complexification of the semisimple part of Z.

Let $\mathbf{h}$ be a common Cartan subalgebra of $\mathbf{k}, \mathbf{g}$ and

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C} X_\sigma + \sum_{\sigma \in \Delta^+} \mathbb{C} X_{-\sigma},$$

$$\mathbf{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta_c^+} \mathbb{C}X_\sigma + \sum_{\sigma \in \Delta_c^+} \mathbb{C}X_{-\sigma}, \qquad \Delta_c^+ \subset \Delta^+,$$

be the corresponding root decompositions of the complexified Lie algebras. A parabolic subalgebra of $\mathbf{g}^{\mathbb{C}} = LieG^{\mathbb{C}} = \mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}$ is of the form

$$LieP = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C}X_{-\sigma} + \sum_{\sigma \in \Delta^+(P)} \mathbb{C}X_\sigma$$

for an appropriate subset $\Delta^+(P) \subset \Delta_c^+$. The minimal parabolic subgroup $B \subset G^{\mathbb{C}}$ with $LieB = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C}X_{-\sigma}$ is called a Borel subgroup. The corresponding $G/G \cap B \to G/K$ is referred to as a maximal complex homogeneous fibration. The Borel subgroup $B \subset G^{\mathbb{C}}$ intersects the real form G in the common maximal torus $T = G \cap B$ of K, G with $LieT = \mathbf{h}$ and centralizes itself. The maximal complex homogeneous fibration G/T contains an equivariant Hermitian symmetric subspace G_h/K_h if and only if G_h/K_h is a polydisc. Any parabolic subgroup $T \subset P \subset G^{\mathbb{C}}$ contains the Borel subgroup $B \supset T$. That determines a fibration $G/G \cap B \to G/G \cap P$ with a holomorphic projection. The existence of a holomorphic lifting $F_B : M \to \Gamma \backslash G/T$ of a harmonic map $f : M \to \Gamma \backslash G/K$ implies the existence of holomorphic liftings $F_P : M \to \Gamma \setminus G/G \cap P$ for all parabolic subgroups $P \supset T$. The complex homogeneous fibrations $G/G \cap P$, associated with centralizers $G \cap P$ of 1-dimensional tori $T^1 \subset K$, are called minimal.

Let $J(G_c/K) \to G_c/K$ be the bundle of the Hermitian almost complex structures on G_c/K. Burstall and Rawnsley show in [2] that for any parabolic subgroup $P \subset G^{\mathbb{C}}$ the quotient $G^{\mathbb{C}}/P = G_c/G_c \cap P$ is a holomorphically embedded subspace of $J(G_c/K)$. Therefore the open subset $G/G \cap P$ of $G_c/G_c \cap P$ is also a holomorphically embedded subspace of the twistor fibration $J(G_c/K) \to G_c/K$. Consequently, any holomorphic lifting $F : M \to \Gamma \backslash G/G \cap P$ of a harmonic map $f : M \to \Gamma \backslash G/K$ can be regarded as a local holomorphic map to the twistor fibration.

The results of the present article are summarized in the following

Theorem 1. (i) *There are two minimal complex homogeneous fibrations $G_{2(2)}/G_{2(2)} \cap P_i \to G_{2(2)}/SO(4)$, $i = 1, 2$, with fibers $\mathbf{CP}^1$ and a maximal complex homogeneous fibration $G_{2(2)}/T^2 \to G_{2(2)}/SO(4)$ with fiber $\mathbf{CP}^1 \times \mathbf{CP}^1$. A harmonic map $f : M \to \Gamma \setminus G_{2(2)}/SO(4)$ with $df^{\mathbb{C}}(T_x^{1,0}M)$, $\forall x \in M$, consisting of nilpotents and of maximum constant $rank_x^{\mathbb{C}} df = 3$, admits a holomorphic lifting to either of the complex homogeneous fibrations. Neither of the corresponding holomorphic images is an equivariantly embedded local Hermitian symmetric subspace.*

(ii) *Any harmonic map $f : M \to \Gamma \setminus F_{4(4)}/Sp(3) \times SU(2)$ with $df^{\mathbb{C}}(T_x^{1,0}M)$, $\forall x \in M$, consisting of nilpotents and maximum constant $rank_x^{\mathbb{C}} df = 7$ admits holomorphic liftings $F_P : M \to \Gamma \setminus F_{4(4)}/F_{4(4)} \cap P$ to all complex homogeneous fibrations. The images of these F_P are not equivariant local Hermitian symmetric subspaces.*

(iii) *A harmonic map $f : M \to \Gamma \setminus E_{6(2)}/SU(6) \times SU(2)$ of maximum constant $rank_x^{\mathbb{C}} df = 10$ with $ad\mathbf{h}$-invariant $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J)) g_x^{-1}$,* *labeled by an $E_{6(2)}$-admissible index set J, has a holomorphic lifting $F_B : M \to \Gamma \backslash E_{6(2)}/T^6$ to a maximal complex homogeneous fibration. There are sufficient conditions for nonexistence of equivariant Hermitian symmetric subspaces $G_h/K_h \subset G/G \cap P$ with $F_P(M) = \Gamma_h \backslash G_h/K_h$.*

(iv) *If the harmonic map $f : M \to \Gamma \backslash E_{7(7)}/SU(8)$ has $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J,K)) g_x^{-1}$ for an $E_{7(7)}$-admissible set of indices J, K, then there is a holomorphic lifting $F_B : M \to \Gamma \backslash E_{7(7)}/T^7$ to a maximal complex homogeneous fibration. There are sufficient conditions for nonexistence of equivariant Hermitian symmetric subspaces $G_h/K_h \subset E_{7(7)}/E_{7(7)} \cap P$ with $F_P(M) = \Gamma_h \backslash G_h/K_h$.*

(v) *For any harmonic $f : M \to \Gamma \backslash E_{7(-5)}/SO(12) \times SU(2)$ with maximal $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K)) g_x^{-1}$ there is a holomorphic lifting $F_P : M \to \Gamma \backslash E_{7(-5)}/SO(12) \times T^1$ to a minimal complex homogeneous fibration. For an $E_{7(-5)}$-admissible set of indices I, K, there exists a holomorphic lifting $F_B : M \to \Gamma \backslash E_{7(-5)}/T^7$ to a maximal complex homogeneous fibration. There is a list of sufficient conditions for nonexistence of equivariantly embedded Hermitian symmetric subspaces $G_h/K_h \subset E_{7(-5)}/E_{7(-5)} \cap P$ with $F_P(M) = \Gamma_h \backslash G_h/K_h$.*

(vi) *If a harmonic map $f : M \to \Gamma \backslash E_{8(8)}/SO(16)$ has $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K)) g_x^{-1}$ for an $E_{8(8)}$-semi-admissible index set J, K of second kind, then there is a holomorphic lifting $F_P : M \to \Gamma \backslash E_{8(8)}/U_8 \times T^1$ to a minimal complex homogeneous fibration. Whenever $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i) g_x^{-1}$, $i = 1, 2$, for a commutative root system $C_1(I,J,K)$ or $C_2(J,K)$ with $E_{8(8)}$-admissible index sets of first or second kind, there is a holomorphic lifting $F_B : M \to \Gamma \backslash E_{8(8)}/T^8$ to a maximal complex homogeneous fibration. There is a set of sufficient conditions for nonexistence of equivariant Hermitian symmetric $G_h/K_h \subset E_{8(8)}/E_{8(8)} \cap P$ with $F_P(M) = \Gamma_h \backslash G_h/K_h$.*

(vii) *Any harmonic map $f : M \to \Gamma \backslash E_{8(-24)}/E_7 \times SU(2)$ with maximal $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J)) g_x^{-1}$, $i = 1, 2$, admits a holomorphic lifting $F_P : M \to \Gamma \backslash E_{8(-24)}/E_7 \times T^1$ to a minimal complex homogeneous fibration. If, moreover, I_1, I_2, J is an $E_{8(-24)}$-admissible set of indices of i-th kind, then there exists a holomorphic lifting $F_B : M \to \Gamma \backslash E_{8(-24)}/T^8$ to a maximal complex homogeneous fibration. Under certain conditions on I_1, I_2, J there is no equivariant Hermitian symmetric image $F_P(M) = \Gamma_h \backslash G_h/K_h$.*

The notions of admissible index sets and the sufficient conditions for nonexistence of equivariant locally Hermitian symmetric images will be clearified separately for each exceptional Riemannian symmetric space under consideration.

Here is an interpretation of a part of the already mentioned results on harmonic maps as existence of holomorphic liftings, whenever they exist. Since the Hermitian symmetric G/K of noncompact type are complex homogeneous spaces, Siu's result [14] can be viewed as an existence of a holomorphic lifting to the fibration with a trivial fiber. Similarly, Carlson and Toledo's article [4] specifies that a harmonic map $f : M \to \Gamma \backslash SU(n,1)/S(U_n \times U_1)$, whose image is not a closed geodesic and which does not factor through a holomorphic map to a Riemann surface, admits a holomorphic lifting to the complex homogeneous fibration with a trivial fiber. For a harmonic map $f : M \to \Gamma \setminus SO_0(2m, 2n)/SO(2m) \times SO(2n)$ with $\min(m, n) \geq 3$, $m + n > 6$, $rank_x^{\mathbb{C}} df = mn$ for all $x \in M$, Carlson and Toledo's results from [5] can be interpreted as an existence of a holomorphic lifting $F_P : M \to \Gamma \setminus SO_0(2m, 2n)/U_m \times U_n \times T^2$ to a complex homogeneous fibration. The results of [9] imply that a harmonic map $f : M \to \Gamma \setminus SO_0(2m+1, 2n)/SO(2m+1) \times SO(2n)$ with $\min(m, n) \geq 5$ and a constant $rank_x^{\mathbb{C}} df = mn + 1$ admits a holomorphic lifting $F_P : M \to \Gamma \setminus SO_0(2m+1, 2n)/U_m \times U_{n-1} \times T^3$. Concerning the harmonic maps $f : M \to \Gamma \setminus Sp(n, 1)/Sp(n) \times Sp(1)$ with $n \geq 3$, which do not map to a closed geodesic and do not factor through holomorphic maps to Riemann surfaces, Carlson and Toledo prove in [4] the existence of a holomorphic lifting $F_P : M \to \Gamma \setminus Sp(n, 1)/Sp(n) \times U_1$ to a complex homogeneous fibration. Carlson and Toledo's results from [5] reveal that a harmonic map $f : M \to \Gamma \setminus Sp(m, n)/Sp(m) \times Sp(n)$ with $\min(m, n) \geq 2$ and constant $rank_x^{\mathbb{C}} df = mn$ admits a holomorphic lifting $F_P : M \to \Gamma \setminus Sp(m, n)/U_m \times U_n \times T^2$ to a complex homogeneous fibration. In [3] Carlson and Hernandez establish that the harmonic maps $f : M \to \Gamma \setminus F_{4(-20)}/SO(9)$, whose image is not a closed geodesic and which do not factor through holomorphic maps to Riemann surfaces, admit holomorphic liftings $F_P : M \to \Gamma \setminus F_{4(-20)}/S(U_2 \times U_3)$ to complex homogeneous fibrations.

2. BASIC TECHNIQUES OF THE ARGUMENT

The proof of Theorem 1 is based on Sampson's result [11] for the harmonic maps $f : M \to \Gamma \setminus G/K$ of compact Kähler manifolds M into local Riemannian symmetric spaces $\Gamma \setminus G/K$ of noncompact type. It asserts that such f are pluriharmonic and $df^{\mathbb{C}}(T_x^{1,0} M)$ are abelian subspaces of $T_{f(x)}^{\mathbb{C}} \Gamma \setminus G/K$ for all $x \in M$.

In order to formulate precisely, let us recall few basics of the structure theory of semisimple Lie algebras. Assume that $\mathbf{g} := LieG$ for a noncompact simple Lie group G of Hodge type and fix a common Cartan subalgebra $\mathbf{h}$ of $\mathbf{k}(G) := LieK$ and $\mathbf{g}$, where K is a maximal compact subgroup of G. There is a Killing orthogonal Cartan decomposition $\mathbf{g} := \mathbf{k}(G) \oplus \mathbf{p}(G)$. Its complexification $\mathbf{g}^{\mathbb{C}} = \mathbf{k}^{\mathbb{C}}(G) \oplus \mathbf{p}^{\mathbb{C}}(G)$ is invariant under the adjoint action of $\mathbf{h}^{\mathbb{C}} := \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C}$. More precisely, $\mathbf{k}^{\mathbb{C}}(G) := \mathbf{k}(G) \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h}^{\mathbb{C}} + \sum_{\sigma \in \Delta_c(G)} \mathbb{C} X_\sigma$ and $\mathbf{p}^{\mathbb{C}}(G) := \mathbf{p}(G) \otimes_{\mathbb{R}} \mathbb{C} = \sum_{\sigma \in \Delta_{nc}(G)} \mathbb{C} X_\sigma$ for an appropriate decomposition $\Delta(G) = \Delta_c(G) \cup \Delta_{nc}(G)$ into a disjoint union of compact and noncompact roots. An arbitrary ordering on $\Delta(G)$, compatible with the Lie bracket of the corresponding root vectors, introduces splittings into disjoint unions $\Delta_c(G) = \Delta_c^+(G) \cup \Delta_c^-(G)$, $\Delta_{nc}(G) = \Delta_{nc}^+(G) \cup \Delta_{nc}^-(G)$, whereas $\Delta(G) = \Delta^+(G) \cup \Delta^-(G)$ with $\Delta^+(G) = \Delta_c^+(G) \cap \Delta_{nc}^+(G)$, $\Delta^-(G) = \Delta_c^-(G) \cup \Delta_{nc}^-(G)$. The pairs of positive and negative root vectors are complex conjugate to each other, $\overline{X_\sigma} = X_{-\sigma}$. Observe also that the root system $\Delta(G)$ and its decomposition $\Delta(G) = \Delta^+(G) \cup \Delta^-(G)$ depend only on the complexification $G^{\mathbb{C}}$ but not on the real form G. We take $\Delta(G_{2(2)}) = \Delta(G_2^{\mathbb{C}})$ from Sato and Kimura's paper [12] and borrow the other $\Delta(G) = \Delta(G^{\mathbb{C}})$ from the Table of Bourbaki's book [1]. The notation $G_{(n)}$ stands for the real form of $G^{\mathbb{C}}$ with $\dim_{\mathbb{R}} \mathbf{p}(G_{(n)}) - \dim_{\mathbb{R}} \mathbf{k}(G_{(n)}) = n$

(cf. [8]). In order to avoid the explicit matrix realization of the root vectors X_σ, $\sigma \in \Delta(G)$, and the calculation of their Lie brackets, let us introduce structure constants $N_{\sigma,\tau}$, such that $[X_\sigma, X_\tau] = N_{\sigma,\tau} X_{\sigma+\tau}$ whenever $\sigma + \tau \in \Delta(G)$.

At the origin $\breve{o} \in G/K$, the complexified tangent space $T^{\mathbb{C}}_{\breve{o}} G/K := T^{\mathbb{R}}_{\breve{o}} G/K \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{p}^{\mathbb{C}}(G)$ and the holomorphic tangent space of a Hermitian symmetric G_h/K_h is $T^{1,0}_{\breve{o}} G_h/K_h = \mathbf{p}^+_h$. At an arbitrary point $gK \in G/K$ the tangent spaces $T^{\mathbb{R}}_{gK} G/K = g\mathbf{p}(G)g^{-1}$ and $T^{\mathbb{C}}_{gK} G/K = g\mathbf{p}^{\mathbb{C}}(G)g^{-1}$.

Carlson and Toledo have established in [5] that for any abelian subspace $\mathbf{a} \subset \mathbf{p}^{\mathbb{C}}$, which consists entirely of nilpotent elements, there exists a Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$ with respect to which $\mathbf{a} \subset \mathbf{p}^+$. The construction of $\mathbf{h}$ reveals that whenever G is of Hodge type, this Cartan subalgebra is contained in $\mathbf{k}(G)$. Whenever $df^{\mathbb{C}}(T^{1,0}_{x_0} M)$ is an abelian subspace of $\mathbf{p}^+$, $f(x_0) = \breve{o}$, the complexified differential of f is represented by $\theta + \overline{\theta}$ for an appropriate $\theta \in \Omega^{1,0}_M(\mathbf{p}^+)$. Let ∇ be the flat Levi-Civita connection of the locally trivial bundle $f^* T^{\mathbb{R}}(\Gamma \setminus G/K)$. It decomposes into a sum $\nabla = D + \theta + \overline{\theta}$, where D is a $\mathbf{k}(G)$-valued connection. Further decomposition into $(1,0)$- and $(0,1)$-types provides $D = D' + D''$ with $\overline{D'} = D''$. The pluriharmonic equation for f reads as

$$D''\theta = 0. \tag{1}$$

On the other hand, the $(2,0)$-component with values in $\mathbf{p}^{\mathbb{C}}$ of the flatness equation $\nabla^2 = 0$ provides

$$D'\theta = 0. \tag{2}$$

For some specific $df^{\mathbb{C}}(T^{1,0}_{x_0} M)$ or, equivalently, θ, the equations (1) and (2) reduce D to a $Lie(G \cap P)$-valued connection for an appropriate parabolic subgroup $P \subset G^{\mathbb{C}}$. That implies the existence of a lifting $F_P : M \to \Gamma \setminus G/G \cap P$ of $f : M \to \Gamma \setminus G/K$. If $f(x) = \Gamma g_x K$ and

$$dF^{\mathbb{C}}_P(T^{1,0}_x M) = df^{\mathbb{C}}(T^{1,0}_x M) \subset g_x \mathbf{p}^+ g_x^{-1} \subset g_x(\mathbf{p}^+ \oplus \sum_{\sigma \in \Delta^+_c(G) - \Delta^+(P)} \mathbb{C} X_\sigma) g_x^{-1}$$

$$= T^{1,0}_{\Gamma g_x (G \cap P)} \Gamma \setminus G/G \cap P$$

for all $x \in M$, then the lifting F_P is holomorphic. That is why, it is natural to assume that $df^{\mathbb{C}}(T^{1,0}_x M)$ consist entirely of nilpotent elements for all $x \in M$, in order to look for holomorphic liftings of $f : M \to \Gamma \setminus G/K$.

For the proof of the main Theorem 1, we have to characterize the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ of maximum $\dim_{\mathbb{C}} \mathbf{a} = 3$ and the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+(F_{4(4)})$ of maximum $\dim_{\mathbb{C}} \mathbf{a} = 7$. To this end, we apply Malcev's method of the leading root vectors for studying abelian subspaces of nilpotents in semisimple Lie algebras (cf. [10]). More precisely, Gauss-Jordan elimination on a basis $Y_1, \ldots, Y_k$

of an abelian subspace $\mathbf{a} \subset \mathbf{p}^+$ allows to represent

$$\begin{array}{llllll} Y_1 = & X_{\sigma_1} & & & + \sum\limits_{\tau \neq \sigma_1,\dots,\sigma_k} y_1^\tau X_\tau, \\ Y_2 = & & X_{\sigma_2} & & + \sum\limits_{\tau \neq \sigma_1,\dots,\sigma_k} y_2^\tau X_\tau, \\ \dots & \dots & \dots & \dots & \dots \\ Y_k = & & & X_{\sigma_k} & + \sum\limits_{\tau \neq \sigma_1,\dots,\sigma_k} y_k^\tau X_\tau \end{array}$$

by positive noncompact root vectors with $\sigma_1 < \sigma_2 < \ldots < \sigma_k$, $\sigma_i < \tau$, for all $X_\tau \in SuppY_i$ and $y_i^\tau \in \mathbb{C}$. According to the compatibility of the Lie bracket with the ordering, the equality $0 = [Y_i, Y_j] = [X_{\sigma_i}, X_{\sigma_j}] + [X_{\sigma_i}, \sum y_j^\tau X_\tau] + [\sum y_i^\tau X_\tau, X_{\sigma_j}] + [\sum y_i^\tau X_\tau, \sum y_j^\tau X_\tau]$ implies the vanishing of the minimal term $[X_{\sigma_i}, X_{\sigma_j}] = 0$. Thus, the root system $C = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is commutative, i.e.,

$$\forall \sigma_i, \sigma_j \in C \ \Rightarrow \ \sigma_i + \sigma_j \notin \Delta(G).$$

The commutative root systems $C \subset \Delta^+_{nc}(G)$ are studied up to the Weyl group action. Accordingly, the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+$ are described modulo the adjoint action of $K^{\mathbf{C}}$.

Let us assume that there exists an equivariant Hermitian symmetric subspace $G_h/K_h \subset G/G \cap P$ with $T^{1,0}_{\check{o}} G_h/K_h = \mathbf{a}$ for some parabolic subgroup $P \subset G^{\mathbf{C}}$. Then the Lie bracket of $\mathbf{g}_h := LieG_h$ is the restriction of the Lie bracket of $\mathbf{g} := LieG$. The same holds for the corresponding complexifications. If $\mathbf{a} = \mathbf{p}_h^+$, then $[\mathbf{a}, \overline{\mathbf{a}}] \subseteq \mathbf{k}_h^{\mathbf{C}}$ and $[\mathbf{a}, [\mathbf{a}, \overline{\mathbf{a}}]] \subseteq \mathbf{a}$. When $\mathbf{a} = Span_{\mathbf{C}}(X_\sigma | \sigma \in C)$, the presence of $\sigma_1, \sigma_2, \sigma_3 \in C$ with $\sigma_2 - \sigma_3 \in \Delta_c(G)$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(G) - C$ rejects the existence of an equivariant Hermitian symmetric $G_h/K_h \subset G/G \cap P$ with $T^{1,0}_{\check{o}} G_h/K_h = \mathbf{a}$.

For each of the noncompact exceptional simple Lie groups $G \neq E_{6(-14)}, E_{7(-25)}, F_{4(-20)}$ of Hodge type are constructed examples of harmonic maps $f : M \to \Gamma \setminus G/K$, which do not admit holomorphic liftings. Let $df^{\mathbf{C}}(T^{1,0}_x M) = g_x Span_{\mathbf{C}}(X_\sigma, X_{-\sigma}, X_\tau | \sigma \in S_1, \tau \in S_2) g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x K$, where $S_1 \neq \varnothing$ and the disjoint union $S = S_1 \cup S_2 \subset \Delta^+_{nc}(G)$ is strongly commutative, i.e.,

$$\forall \sigma, \tau \in S \ \Rightarrow \ \sigma + \tau \notin \Delta(G) \text{ and } \sigma - \tau \notin \Delta(G).$$

Then a lifting $F_P : M \to \Gamma \setminus G/G \cap P$ to a complex homogeneous fibration is not holomorphic, according to $dF_P^{\mathbf{C}}(T^{1,0}_x M) = df^{\mathbf{C}}(T^{1,0}_x M) \not\subset T^{1,0}_{\Gamma g_x (G \cap P)} \Gamma \setminus G/G \cap P$. For specific examples of strongly commutative $S \subset \Delta^+_{nc}(G)$, we refer to the next sections.

3. $\mathbf{G} = G_{2(2)}/SO(4)$

The complexified Lie algebra $\mathbf{g}_2^{\mathbf{C}}$ admits a representation by (7×7)-matrices and can be identified with the derivations of the Cayley numbers (cf. [12]). We use the system of the positive roots $\Delta^+(G_2^{\mathbf{C}}) = \{e_1, e_2, e_1 + e_2, e_1 - e_2, e_1 + 2e_2, 2e_1 + e_2\}$, borrowed from [12]. The Lie algebra $\mathbf{g}_{2(2)}$ of Hodge type admits a 2-dimensional

Cartan subalgebra $\mathbf{h} \subset so(4)$. The complexified isotropy subalgebra $so(4,\mathbb{C}) = \mathbf{h}^{\mathbb{C}} + \sum_{i=1}^{2} \mathbb{C}X_{\sigma_i} + \sum_{i=1}^{2} \mathbb{C}X_{-\sigma_i}$, where the compact roots σ_1, σ_2 have one and the same length and $\sigma_1 + \sigma_2 \notin \Delta(G_2^{\mathbb{C}})$. Bearing in mind that $\sigma, \tau \in \Delta^+_{nc}(G_{2(2)}), \sigma + \tau \in \Delta^+(G_{2(2)}) \Rightarrow \sigma + \tau \in \Delta^+_c(G_{2(2)})$, one specifies that $\Delta^+_c(G_{2(2)}) = \{e_1 - e_2, e_1 + e_2\}$, whereas $\Delta^+_{nc}(G_{2(2)}) = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\}$. This choice is also subject to $\sigma \in \Delta^+_c(G_{2(2)}), \tau \in \Delta^+_{nc}(G_{2(2)}), \sigma + \tau \in \Delta^+(G_{2(2)}) \Rightarrow \sigma + \tau \in \Delta^+_{nc}(G_{2(2)})$.

The only restriction to which a commutative root system $C \subset \Delta^+_{nc}(G_{2(2)})$ obeys is not to contain simultaneously e_1 and e_2. Up to the action of the Weyl group of $SO(4,\mathbb{C})$, which is generated by the permutation of e_1 with e_2 and their simultaneous sign changes, one can assume that the maximal commutative root system $C = \{e_1, e_1 + 2e_2, 2e_1 + e_2\}$.

Lemma 2. *The 3-dimensional abelian subspaces* $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ *are* $SO(4,\mathbb{C})$-*conjugate to*

$$\mathbf{a}_4 = Span_{\mathbb{C}}(X_{e_1}, \quad X_{e_1+2e_2}, \quad X_{2e_1+e_2}).$$

Proof. An abelian $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ with a leading root system $C = \{e_1, e_1 + 2e_2, 2e_1 + e_2\}$ has generators

$$Y'_1 = X_{e_1} + a_1 X_{e_2}, \quad Y'_2 = X_{e_1+2e_2} + a_2 X_{e_2}, \quad Y'_3 = X_{2e_1+e_2} + a_3 X_{e_2}.$$

After the action of

$$AdExp\left(\frac{-a_1}{N_{-e_1+e_2,e_1}} X_{-e_1+e_2}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} ad^k \left(\frac{-a_1}{N_{-e_1+e_2,e_1}} X_{-e_1+e_2}\right),$$

followed by an elimination of $X_{e_1+2e_2}$ from the image of Y'_3, one gets $Y''_1 = X_{e_1}$, $Y''_2 = X_{e_1+2e_2} + a_2 X_{e_2}$, $Y''_3 = X_{2e_1+e_2} + a_3 X_{e_2}$. The commutations $[Y''_1, Y''_2] = 0$ and $[Y''_1, Y''_3] = 0$ reveal the vanishing of a_2 and a_3, Q.E.D.

Let us describe the parabolic subgroups $P \subset G_2^{\mathbb{C}}$. According to $card\Delta^+_c = 2$, there are a Borel subgroup $B \subset G_2^{\mathbb{C}}$ with $LieB = \mathbf{h} + \sum_{\sigma \in \Delta^+(G_2^{\mathbb{C}})} \mathbb{C}X_{-\sigma}$ and two maximal parabolic subgroups $P_1, P_2 \subset G_2^{\mathbb{C}}$ with $LieP_1 = LieB + \mathbb{C}X_{e_1-e_2}$ and $LieP_2 = LieB + \mathbb{C}X_{e_1+e_2}$. Clearly, $G_{2(2)} \cap B = T^2 = Exp_1^{G_{2(2)}}(\mathbb{R}H_1 + \mathbb{R}H_2)$ centralizes itself, $G_{2(2)} \cap P_1 \simeq SU(2) \times T^1_+$ centralizes the 1-dimensional torus $T^1_+ = Exp_1^{G_{2(2)}}(\mathbb{R}(H_1 + H_2))$ and $G_{2(2)} \cap P_2 \simeq SU(2) \times T^1_-$ centralizes the 1-dimensional torus $T^1_- = Exp_1^{G_{2(2)}}(\mathbb{R}(H_1 - H_2))$. Bearing in mind that $SO(4) = SU(2) \times SU(2)$, one observes that the complex homogeneous fibrations $G_{2(2)}/G_{2(2)} \cap P_i \to G_{2(2)}/SO(4)$, $i = 1, 2$, have fibers $SU(2)/S^1 = SU(2)/S(U_1 \times U_1) = \mathbf{CP}^1$ and $G_{2(2)}/T^2 \to G_{2(2)}/SO(4)$ has a fiber $\mathbf{CP}^1 \times \mathbf{CP}^1$. There are also fibrations $G_{2(2)}/T^2 \to G_{2(2)}/G_{2(2)} \cap P_i$ with fibers $\mathbf{CP}^1$ and holomorphic projections.

Lemma 3. *Let* $f : M \to \Gamma \setminus G_{2(2)}/SO(4)$ *be a harmonic map of a compact Kähler manifold* M *with maximum constant* $rank_x^{\mathbb{C}} df = 3$ *and* $df^{\mathbb{C}}(T_x^{1,0}M)$, $\forall x \in M$, *consisting entirely of nilpotent elements. Then there exists a holomorphic lifting*

$F_B : M \to \Gamma \setminus G_{2(2)}/T^2$ to a maximal complex homogeneous fibration. Neither of $f(M)$, $F_B(M)$, $F_{P_1}(M)$ or $F_{P_2}(M)$ is a local equivariantly embedded Hermitian symmetric subspace.

Proof. The (0,1)-part of the $so(4)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{2} \overline{\xi_i} \otimes H_i + \overline{p} \otimes X_{e_1-e_2} + \overline{r} \otimes X_{-e_1+e_2} + \overline{\zeta} \otimes X_{e_1+e_2} + \overline{z} \otimes X_{-e_1-e_2}$$

for some $\xi_i, \eta_{ij}, \zeta, z \in \Omega^{1,0}_M$. The holomorphic differential of f is represented by the 1-form

$$\theta = dx^1 \otimes X_{e_1} + dx^2 \otimes X_{e_1+2e_2} + dx^3 \otimes X_{2e_1+e_2}.$$

Wedging the forms and computing the Lie bracket of the root vectors and Cartan generators, one obtains $D''\theta = (\overline{\xi_1} \wedge dx^1 + N_{-e_1-e_2,2e_1+e_2}\overline{z} \wedge dx^3) \otimes X_{e_1} + (N_{-e_1+e_2,e_1}\overline{r} \wedge dx^1 + N_{-e_1-e_2,e_1+2e_2}\overline{z} \wedge dx^2) \otimes X_{e_2} + N_{-e_1-e_2,e_1}\overline{z} \wedge dx^1 \otimes X_{-e_2} + [(\overline{\xi_1}+2\overline{\xi_2}) \wedge dx^2 + N_{-e_1+e_2,2e_1+e_2}\overline{r} \wedge dx^3] \otimes X_{e_1+2e_2} + [(2\overline{\xi_1}+\overline{\xi_2}) \wedge dx^3 + N_{e_1-e_2,e_1+2e_2}\overline{p} \wedge dx^2 + N_{e_1+e_2,e_1}\overline{\zeta} \wedge dx^1] \otimes X_{2e_1+e_2} = 0$. Bearing in mind the $\mathbb{C}$-linear independence of the root vectors and the functional independence of dx^1, dx^2, dx^3, one derives that $D'' = \overline{\partial}$. Therefore $D = \partial + \overline{\partial} = d$ takes values in $\mathbf{h} = Lie(G_{2(2)} \cap B)$ and there is a holomorphic lifting $F_B : M \to \Gamma \setminus G_{2(2)}/T^2$.

The only 3-dimensional Hermitian symmetric spaces of noncompact type are $SU(3,1)/S(U_3 \times U_1) \simeq SO^*(6)/U_3$ with 9-dimensional complexified isotropy subalgebra and $SO(3,2)/SO(3) \times SO(2) \simeq Sp(2)/U_2$ with 3-dimensional complex isotropy subalgebra. They both satisfy $\mathbf{k}_h^{\mathbb{C}} = [\mathbf{p}_h^{\mathbb{C}}, \mathbf{p}_h^{\mathbb{C}}]$. For the abelian subspace $\mathbf{a} = Span_{\mathbb{C}}(X_{e_1}, X_{e_1+2e_2}, X_{2e_1+e_2})$ it is straightforward that $[\mathbf{a}+\overline{\mathbf{a}}, \mathbf{a}+\overline{\mathbf{a}}] = so(4,\mathbb{C})$ is of dimension 6. That would contradict an assumption $\mathbf{a} = \mathbf{p}_h^+ = \mathbf{p}^+(G_h)$ for an equivariant Hermitian symmetric subspace G_h/K_h, Q.E.D.

The strongly commutative subsets $S \subset \Delta^+_{nc}(G_{2(2)})$ are commutative. Therefore, one can assume that $S \subset \{e_1, e_1+2e_2, 2e_1+e_2\}$. Bearing in mind that $(2e_1 + e_2) - e_1 = e_1 + e_2$, $(2e_1+e_2) - (e_1+2e_2) = e_1 - e_2$, one determines $S = \{e_1, e_1+2e_2\}$ of maximal cardinality, up to $Weyl(SO(4,\mathbb{C}))$-action. The harmonic maps $f : M \to \Gamma \setminus G_{2(2)}/SO(4)$ with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{-e_1}, X_{e_1+2e_2}, X_{-e_1-2e_2})g_x^{-1}$,

$$df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{-e_1}, X_{e_1+2e_2})g_x^{-1}$$

or

$$df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{e_1+2e_2}, X_{-e_1-2e_2})g_x^{-1}$$

have no holomorphic liftings to complex homogeneous fibrations.

4. $\mathbf{FI} = F_{4(4)}/Sp(3) \times SU(2)$

Let us recall from Bourbaki's Table [1]

$$\Delta^+(F_4^{\mathbb{C}}) = \Big\{ e_i (1 \leq i \leq 4), \quad e_i \pm e_j (1 \leq i < j \leq 4),$$

$$\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \mu e_4)(\varepsilon, \nu, \mu \in \{\pm 1\}) \Big\}.$$

One needs to decompose into a disjoint union $\Delta^+(F_4^{\mathbb{C}}) = \Delta_c^+(F_{4(4)}) \cup \Delta_{nc}^+(F_{4(4)})$, where $\Delta_c^+(F_{4(4)}) = \Delta^+(Sp(3,\mathbb{C})) \cup \Delta^+(SL(2,\mathbb{C}))$. Observe that the positive roots of $Sp(3,\mathbb{C})$ can be expressed by two short simple roots α_1, α_2 and a long simple root α_3. Among the short roots e_i, $\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \mu e_4)$ and the long roots $e_i \pm e_j$ of $F_4^{\mathbb{C}}$, the only possible choices are $\alpha_1 = e_i$, $\alpha_2 = \frac{1}{2}(e_1 - e_i - e_j \mp e_k)$, $\alpha_3 = e_j \pm e_k$. Up to the action of the Weyl group of $F_4^{\mathbb{C}}$, let us specify $\alpha_1 = e_2$, $\alpha_2 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_3 = e_3 + e_4$. As far as $Sp(3)$ and $SU(2)$ are in a direct product in the isotropy subgroup, for any $\sigma \in \Delta^+(Sp(3,\mathbb{C}))$ and the only positive root τ of $SL(2,\mathbb{C})$ the sum $\sigma + \tau$ is not a root of $F_4^{\mathbb{C}}$. That determines $\tau = e_3 - e_4$, so that

$$\Delta_c^+(F_{4(4)})$$

$$= \left\{ e_i (1 \leq i \leq 2),\ \ e_1 \pm e_2,\ \ e_3 \pm e_4,\ \ \frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \nu e_4)(\varepsilon, \nu \in \{\pm 1\}) \right\},$$

$$\Delta_{nc}^+(F_{4(4)}) = \Big\{ e_i (3 \leq i \leq 4),\ \ e_i \pm e_j (1 \leq i \leq 2, 3 \leq j \leq 4),$$
$$\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4)(\varepsilon, \nu \in \{\pm 1\}) \Big\}.$$

A maximal commutative root system $C \subset \Delta_{nc}^+(F_{4(4)})$ decomposes into a disjoint union of the commutative root systems $C_1 := C \cap \{e_i | 3 \leq i \leq 4\}$, $C_2 := C \cap \{\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4) | \varepsilon, \nu \in \{\pm 1\}\}$, $C_3 := C \cap \{e_i \pm e_3 | 1 \leq i \leq 2\}$ and $C_4 := C \cap \{e_i \pm e_4 | 1 \leq i \leq 2\}$. Therefore $C_1 \subseteq \{e_i\}$ for some fixed $i = 3$ or 4, $C_2 \subseteq \{\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4) | \varepsilon = \pm 1\}$ for some fixed $\nu \in \{\pm 1\}$, and $C_j \subseteq \{e_i \pm e_j\}$ for some fixed $1 \leq i \leq 2$ or $C_j \subseteq \{e_i + \varepsilon e_j | 1 \leq i \leq 2\}$ for some fixed $\varepsilon \in \{\pm 1\}$ whenever $3 \leq j \leq 4$. Preventing the presence of $\rho_i \in C_i$, $\rho_j \in C_j$ with $\rho_i + \rho_j \in C$, $i \neq j$, one obtains the following commutative root systems $C \subset \Delta_{nc}^+(F_{4(4)})$ of maximal $card C = 7$, up to the action of the Weyl group of $Sp(3,\mathbb{C}) \times SL(2,\mathbb{C})$:

$$C' = \left\{ e_3,\ \ e_1 + e_3,\ \ e_2 + e_3,\ \ e_1 \pm e_4,\ \ \frac{1}{2}(e_1 \pm e_2 + e_3 - e_4) \right\}$$

and

$$C'' = \left\{ e_3,\ \ e_1 + e_3,\ \ e_2 + e_3,\ \ e_1 - e_4,\ \ e_2 - e_4,\ \ \frac{1}{2}(e_1 \pm e_2 + e_3 - e_4) \right\}.$$

Lemma 4. *The abelian subalgebras* $\mathbf{a} \subset \mathbf{p}^+(F_{4(4)})$ *of maximal* $\dim_{\mathbb{C}} \mathbf{a} = 7$ *are* $Sp(3,\mathbb{C}) \times SL(2,\mathbb{C})$*-conjugate to* $\mathbf{a}' = Span_{\mathbb{C}}(X_\sigma | \sigma \in C')$ *or* $\mathbf{a}'' = Span_{\mathbb{C}}(X_\sigma | \sigma \in C'')$*, where* $C', C'' \subset \Delta_{nc}^+(F_{4(4)})$ *are the aforementioned commutative root systems of maximal cardinality* 7.

Proof. On $Y_\sigma = X_\sigma + \sum_{\tau\in\Delta^+_{nc}(F_{4(4)})-C} y^\tau_\sigma X_\tau$, for $\sigma \in C$, $C = C'$ or C'', one applies the adjoint action of $Exp\left(-\frac{y^{e_4}_{e_3}}{N_{-e_3+e_4,e_3}} X_{-e_3+e_4}\right)$, in order to annihilate the coefficient of X_{e_4} from Y_{e_3}. After eliminating X_{σ_i} from the expressions of Y_{σ_j} for $\sigma_i, \sigma_j \in C$, $\sigma_i \neq \sigma_j$, one calculates the commutators $[Y_{\sigma_i}, Y_{\sigma_j}] = 0$ for all different $\sigma_i, \sigma_j \in C$ and concludes the vanishing of y^τ_σ except $y^{e_1+e_4}_{e_2-e_4}$ for $C = C''$. If $Y_{e_2-e_4} = X_{e_2-e_4} + y^{e_1+e_4}_{e_2-e_4} X_{e_1+e_4}$ with $y^{e_1+e_4}_{e_2-e_4} \neq 0$, one can reduce the considerations to the leading root system C', Q.E.D.

Lemma 5. *If $f : M \to \Gamma \setminus F_{4(4)}/Sp(3) \times SU(2)$ is a harmonic map with 7-dimensional abelian spaces of nilpotents $df^{\mathbb{C}}(T^{1,0}_x M)$, $\forall x \in M$, then there is a holomorphic lifting $F_B : M \to \Gamma \setminus F_{4(4)}/T^4$ to a maximal complex homogeneous fibration. There are no parabolic subgroup $P \subset F_4^{\mathbb{C}}$ and equivariant Hermitian symmetric subspace $G_h/K_h \subset F_{4(4)}/F_{4(4)} \cap P$ such that $F_P(M) = \Gamma_h \setminus G_h/K_h$.*

Proof. The (0,1)-part of the $sp(3) \oplus su(2)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{4} \overline{\xi_i} \otimes H_i + \sum_{\sigma\in\Delta^+_c(F_{4(4)})} \overline{\eta_\sigma} \otimes X_\sigma + \sum_{\sigma\in\Delta^+_c(F_{4(4)})} \overline{\zeta_\sigma} \otimes X_{-\sigma}.$$

The differential of a harmonic map f with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x \mathbf{a}' g_x^{-1}$ or $df^{\mathbb{C}}(T^{1,0}_x M) = g_x \mathbf{a}'' g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x(Sp(3) \times SU(2))$ is represented, respectively, by $\theta_1 = \sum_{\tau\in C'} dx^\tau \otimes X_\tau$ or $\theta_2 = \sum_{\tau\in C''} dx^\tau \otimes X_\tau$. The pluriharmonic equations $D''\theta_i = 0$ imply $D'' = \overline{\partial} + \overline{\eta_{e_1+e_2}} \otimes X_{e_1+e_2}$ in both cases. Then the consequences $D'\theta_i = 0$ of the flatness equation force $D = d$.

Let us assume that there is an equivariant Hermitian symmetric subspace $G_h/K_h \subset F_{4(4)}/F_{4(4)} \cap P$ with $\mathbf{p}^+_h = \mathbf{a}'$ or $\mathbf{a}''$. Then $[X_{e_1+e_3}, X_{-e_3}] = N_{e_1+e_3,-e_3} X_{e_1} \in \mathbf{k}^{\mathbb{C}}_h$, whereas $[X_{-e_1}, X_{e_1-e_4}] = N_{-e_1,e_1-e_4} X_{-e_4} \in \mathbf{p}^{\mathbb{C}}_h$, which is not true in either case, Q.E.D.

After detecting the pairs σ, τ from C' or C'' with $\sigma - \tau \in \Delta_c(F_{4(4)})$, one drops out at least one member of these pairs and obtains the strongly commutative root systems

$$S_1 = \{e_3, \ \ e_1 \pm e_4\} \quad \text{and} \quad S_2 = \left\{\frac{1}{2}(e_1 + e_2 + e_3 - e_4), \ \ e_2 - e_3, \ \ e_1 + e_4\right\}$$

in $\Delta^+_{nc}(F_{4(4)})$, up to $Weyl(Sp(3,\mathbb{C}) \times SL(2,\mathbb{C}))$-action.

5. $\mathbf{EII} = E_{6(2)}/SU(6) \times SU(2)$

Let us recall that there is a chain of subgroups $E_6^{\mathbb{C}} \subset E_7^{\mathbb{C}} \subset E_8^{\mathbb{C}}$. In terms of the root decompositions of the corresponding Lie algebras, if $H_1, \dots H_8$ generate a

Cartan subalgebra $\mathbf{h}_8^{\mathbb{C}}$ of $LieE_8^{\mathbb{C}}$, then

$$\mathbf{h}_7^{\mathbb{C}} = \Big\{ \sum_{i=1}^{8} x_i H_i | x_i \in \mathbb{C}, x_7 + x_8 = 0 \Big\} \subset \mathbf{h}_8^{\mathbb{C}}$$

is a Cartan subalgebra of $LieE_7^{\mathbb{C}}$ and

$$\mathbf{h}_6^{\mathbb{C}} = \Big\{ \sum_{i=1}^{8} x_i H_i | x_i \in \mathbb{C}, x_6 - x_7 = 0, x_7 + x_8 = 0 \Big\} \subset \mathbf{h}_7^{\mathbb{C}}$$

is a Cartan subalgebra of $LieE_6^{\mathbb{C}}$. The positive root system

$$\Delta^+(E_8^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 8),\ e_i + e_j (1 \le i < j \le 8),$$
$$\frac{1}{2}\Big(\sum_{i=1}^{7} \varepsilon_i e_i + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{7} \varepsilon_i = 1 \Big) \Big\}$$

contains

$$\Delta^+(E_7^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 6),\ \ e_i + e_j (1 \le i < j \le 6),\ \ -e_7 + e_8,$$
$$\frac{1}{2}\Big(\sum_{i=1}^{6} \varepsilon_i e_i - e_7 + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{6} \varepsilon_i = -1 \Big) \Big\},$$

which, in turn, contains

$$\Delta^+(E_6^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 5),\ \ e_i + e_j (1 \le i < j \le 5),$$
$$\frac{1}{2}\Big(\sum_{i=1}^{5} \varepsilon_i e_i - e_6 - e_7 + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{5} \varepsilon_i = 1 \Big) \Big\}$$

(cf. [1]).

For the study of the Riemannian symmetric spaces **EII**, **EV**, **EVI**, **EVIII** and **EIX**, let us introduce the notations

$$\lambda_{ij} := -e_i + e_j \quad (1 \le i < j \le 8), \qquad \mu_{ij} := e_i + e_j \quad (1 \le i < j \le 8),$$

$$\alpha := \frac{1}{2}\Big(\sum_{i=1}^{8} e_i \Big),$$

$$\beta_{ij} := \frac{1}{2}(-e_i - e_j + e_k + e_l + e_m + e_n + e_p + e_8) \quad (1 \le i < j \le 7),$$

$$\gamma_{ijk} := \frac{1}{2}(e_i + e_j + e_k - e_l - e_m - e_n - e_p + e_8) \quad (1 \le i < j < k \le 7),$$

$$\delta_i := \frac{1}{2}(e_i - e_j - e_k - e_l - e_m - e_n - e_p + e_8) \quad (1 \le i \le 7),$$

where i, j, k, l, m, n, p stand for a permutation of $1, 2, \ldots, 7$. It is convenient to put also $\lambda_{ji} := -\lambda_{ij}$, $\mu_{ji} := \mu_{ij}$, $\beta_{ji} := \beta_{ij}$ for $i < j$ and $\gamma_{ikj} = \gamma_{jik} = \gamma_{jki} = \gamma_{kij} = \gamma_{kji} := \gamma_{ijk}$ for $i < j < k$.

In order to recognize the subset $\Delta_c^+(E_{6(2)}) \subset \Delta^+(E_6^{\mathbb{C}})$, let us observe that $sl(6, \mathbb{C}) = su(6) \otimes_{\mathbb{R}} \mathbb{C}$ has 5 simple roots α_i, such that $\sum_{k=i}^{j} \alpha_k$ is a root for any $1 \leq i \leq j \leq 5$. Looking at the Dynkin diagram of $E_6^{\mathbb{C}}$, one notes that the only unramified path with 5 vertices corresponds to the simple roots δ_1, λ_{12}, λ_{23}, λ_{34}, λ_{45}. Therefore $\Delta^+(SL(6, \mathbb{C})) = \{\lambda_{ij}(1 \leq i < j \leq 5), \delta_i(1 \leq i \leq 5)\}$. Since $SU(6)$ and $SU(2)$ are in a direct product in the isotropy group of **EII**, the only positive root σ of $sl(2, \mathbb{C})$ is such that $\sigma + \tau \notin \Delta^+(E_6^{\mathbb{C}})$ for all $\tau \in \Delta^+(SL(6, \mathbb{C}))$. That specifies $\Delta^+(SL(2, \mathbb{C})) = \{\beta_{67}\}$. Thus,

$$\Delta_c^+(E_{6(2)}) = \{\lambda_{ij}(1 \leq i < j \leq 5), \quad \delta_i(1 \leq i \leq 5), \quad \beta_{67}\}$$

and

$$\Delta_{nc}^+(E_{6(2)}) = \{\mu_{ij}(1 \leq i < j \leq 5), \quad \gamma_{ijk}(1 \leq i < j < k \leq 5)\}.$$

The maximal commutative $C \subset \Delta_{nc}^+(E_{6(2)})$ are of the form $C(J) = \{\mu_{ij}((i,j) \in J), \gamma_{klm}((i,j) \notin J)\}$ for some subsets J of unordered pairs $1 \leq i, j \leq 5$ .In particular, $card C(J) = 10$.

A generic abelian subspace $\mathbf{a} \subset \mathbf{p}^+(E_{6(2)})$ of maximal $\dim_{\mathbb{C}} \mathbf{a} = 10$ is not invariant under the adjoint action of the Cartan subalgebra. Therefore, the pluriharmonic equation $D''\theta = 0$ and the consequence $D'\theta = 0$ of the flatness $\nabla^2 = 0$ do not force a reduction of D to a $Lie(G \cap P)$-valued connection.

Definition 6. The set J of unordered pairs is $E_{6(2)}$-*admissible* if there hold simultaneously the following conditions:

(i) for an arbitrary $i \neq j$ with $(j, k) \in J$ for all $k \notin \{i, j\}$ there exists $(i, k) \in J$;

(ii) for an arbitrary i there exists $(j, k) \in J$ with different i, j, k;

(iii) for an arbitrary i there exists $(j, k) \notin J$ with different i, j, k.

Lemma 7. *Let $f : M \to \Gamma \backslash E_{6(2)}/SU(6) \times SU(2)$ be a harmonic map of maximum dimension with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J)) g_x^{-1}$ for a commutative root system $C(J)$, labeled by an $E_{6(2)}$-admissible set of indices J. Then f lifts to a holomorphic map $F_B : M \to \Gamma \backslash E_{6(2)}/T^6$ to a maximal complex homogeneous fibration.*

Proof. In general,

$$D'' = \overline{\partial} + \sum_{i=1}^{5} \overline{\xi_i} \otimes H_i + \overline{\xi_6} \otimes (H_6 + H_7 - H_8) + \sum_{1 \leq i \neq j \leq 5} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$

$$+ \sum_{i=1}^{5} \overline{\zeta_i} \otimes X_{\delta_i} + \sum_{i=1}^{5} \overline{z_i} \otimes X_{-\delta_i} + \overline{\rho} \otimes X_{\beta_{67}} + \overline{r} \otimes X_{-\beta_{67}}$$

and

$$\theta = \sum_{(i,j)\in J} dx^{ij} \otimes X_{\mu_{ij}} + \sum_{(i,j)\notin J} dx^{klm} \otimes X_{\gamma_{klm}}.$$

Making use of the table of $\sigma \in \Delta_c(E_{6(2)})$, $\tau \in \Delta^+_{nc}(E_{6(2)})$ with $\sigma + \tau \in \Delta_{nc}(E_{6(2)})$, one derives from the pluriharmonic equation $D''\theta = 0$ that $\eta_{ij} = 0$ if $\exists(i,k) \in J$ or $\exists(j,k) \notin J$, $\zeta_i = 0$ if $\exists(j,k) \in J$, $z_i = 0$ if $\exists(j,k) \notin J$ and $r = 0$. For an $E_{6(2)}$-admissible index set J there follows

$$D'' = \overline{\partial} + \sum_{i=1}^{5} \overline{\xi_i} \otimes H_i + \overline{\xi_6} \otimes (H_6 + H_7 - H_8) + \overline{\rho} \otimes X_{\beta_{67}}.$$

Then the consequence $D'\theta = 0$ of the flatness equation $\nabla^2 = 0$ reveals the vanishing of ρ, Q.E.D.

Lemma 8. *Each of the following conditions is sufficient for the nonexistence of a Hermitian symmetric G_h/K_h, where G_h is a subgroup of $E_{6(2)}$ and $\mathbf{p}^+_h = Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J))$ is associated with a maximal commutative root system $C(J) \subset \Delta^+_{nc}(E_{6(2)})$:*

(a) *the existence of different i,j,k,l with $(i,j),(i,k),(k,l) \in J$ and $(j,l) \notin J$;*
(b) *the existence of different i,j,k,l with $(i,k) \in J$ and $(i,j),(j,l),(k,l) \notin J$;*
(c) *the existence of different i,j,k,l with $(i,j),(i,k),(j,l) \in J$ and $(k,l) \notin J$.*

Proof. In either case, it suffices to exhibit $\sigma_1, \sigma_2, \sigma_3 \in C(J)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{6(2)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{6(2)}) - C(J)$. Namely,

(a) $\gamma_{ikm} + (\mu_{ij} - \mu_{ik}) = \gamma_{ikm} + \lambda_{kj} = \gamma_{ijm}$;
(b) $\mu_{ik} + (\gamma_{ijm} - \gamma_{ikm}) = \mu_{ik} + \lambda_{kj} = \mu_{ij}$;
(c) $\mu_{ik} + (\gamma_{ijm} - \mu_{ij}) = \mu_{ik} + \delta_m = \gamma_{ikm}$, Q.E.D.

Towards the construction of strongly commutative root systems $S \subset \Delta^+_{nc}(E_{6(2)})$, let us associate them to graphs with 5 vertices. For $\mu_{ij} \in S$ draw a "blue" edge, connecting the i-th and the j-th vertices. When $\gamma_{klm} \in S$, the complementing vertices i and j are connected by a "red" edge. According to $\mu_{ij} - \mu_{ik} = \lambda_{kj} \in \Delta_c(E_{6(2)})$ and $\gamma_{klm} - \gamma_{jlm} = \lambda_{jk} \in \Delta_c(E_{6(2)})$, no edges of one and the same color have a common vertex. Further, $\gamma_{klm} - \mu_{kl} = \delta_m \in \Delta_c(E_{6(2)})$ requires the nonexistence of disjoint "blue" and "red" edges. Putting all together, one obtains $S = \{\gamma_{345},\ \mu_{23},\ \gamma_{125},\ \mu_{45}\}$ up to $Weyl(SL(6,\mathbb{C}))$-action.

6. EV $= E_{7(7)}/SU(8)$

The elements of $\Delta^+_c(E_{7(7)})$ are expressed by simple roots $\alpha_1, \ldots, \alpha_7$, such that $\sum_{k=i}^{j} \alpha_k \in \Delta^+_c(E_{7(7)})$ for all $1 \leq i \leq j \leq 7$. From the Dynkin diagram of $E_7^{\mathbb{C}}$ (cf. [1]) one recognizes $\alpha_1 := \delta_1$ and $\alpha_i := \lambda_{i-1,i}$ for $2 \leq i \leq 6$ with

$$\Delta'_c(E_7^{\mathbb{C}}) = \left\{ \sum_{k=i}^{j} \alpha_k | 1 \leq i \leq j \leq 6 \right\}$$
$$= \{\delta_i (1 \leq i \leq 6), \lambda_{ij} (1 \leq i < j \leq 6)\} \subset \Delta^+(E_7^{\mathbb{C}}).$$

The existence of $\alpha_0 \in \Delta^+(E_7^{\mathbf{C}}) - \Delta'_c(E_7^{\mathbf{C}})$ with $\alpha_0 + \delta_i \in \Delta^+(E_7^{\mathbf{C}})$ for all $1 \leq i \leq 6$ is contradicted by $\mu_{ij} + \delta_i \notin \Delta(E_7^{\mathbf{C}})$, $\lambda_{78} + \delta_i \notin \Delta(E_7^{\mathbf{C}})$, $\beta_{i7} + \delta_j \notin \Delta(E_7^{\mathbf{C}})$ for $i \neq j$ and $\gamma_{ijk} + \delta_i \notin \Delta(E_7^C)$ for different i, j, k. Therefore, there is $\alpha_7 \in \Delta^+(E_7^{\mathbf{C}}) - \Delta'_c(E_7^{\mathbf{C}})$ with $\alpha_7 + \lambda_{i6} \in \Delta^+(E_7^{\mathbf{C}})$ for $1 \leq i \leq 5$ and $\alpha_7 + \delta_6 \in \Delta^+(E_7^{\mathbf{C}})$. For any $1 \leq i < j \leq 5$ there exists $k \in \{1, \ldots, 5\} - \{i, j\}$ such that $\mu_{ij} + \lambda_{k6} \notin \Delta^+(E_7^{\mathbf{C}})$. Clearly, $\lambda_{78} + \lambda_{i6} \notin \Delta^+(E_7^{\mathbf{C}})$, $\beta_{i7} + \lambda_{i6} \notin \Delta^+(E_7^{\mathbf{C}})$ for $1 \leq i \leq 5$ and $\gamma_{ijk} + \delta_6 \notin \Delta^+(E_7^{\mathbf{C}})$, regardless whether $6 \in \{i, j, k\}$ or $6 \notin \{i, j, k\}$. Finally, $\beta_{67} + \lambda_{i6} = \beta_{i7}$ for $1 \leq i \leq 5$ and $\beta_{67} + \delta_6 = \lambda_{78}$ reveal that $\alpha_7 = \beta_{67}$, whereas

$$\Delta_c^+(E_{7(7)}) = \{\lambda_{ij}(1 \leq i < j \leq 6), \;\; \lambda_{78}, \;\; \beta_{i7}(1 \leq i \leq 6), \;\; \delta_i(1 \leq i \leq 6)\}$$

and

$$\Delta_{nc}^+(E_{7(7)}) = \{\mu_{ij}(1 \leq i < j \leq 6), \;\; \gamma_{ijk}(1 \leq i < j < k \leq 6)\}.$$

After listing the pairs $\sigma, \tau \in \Delta_{nc}^+(E_{7(7)})$ with $\sigma + \tau \in \Delta_c^+(E_{7(7)})$, one characterizes the commutative root systems

$$C(J, K) := \{\mu_{ij}((i, j) \in J), \gamma_{ijk}((i, j, k) \in K)\} \subset \Delta_{nc}^+(E_{7(7)})$$

by the conditions $(i, j, k) \in K \Rightarrow (l, m) \notin J$ and $(i, j, k) \in K \Rightarrow (l, m, n) \notin K$ for different i, j, k, l, m, n.

A generic abelian subspace $\mathbf{a} \subset \mathbf{p}^+(E_{7(7)})$ with a leading root system $C(J, K)$ is not invariant under the adjoint action of the Cartan subalgebra $\mathbf{h}_7^{\mathbf{C}}$. The associated $su(8)$-valued connection D is not reduced to a $Lie(E_{7(7)} \cap P)$-valued one. Even when $T_x^{1,0}M$, $x \in M$, map to $ad\mathbf{h}_7^{\mathbf{C}}$-invariant abelian subspaces of $g_x \mathbf{p}^+(E_{7(7)}) g_x^{-1}$, $f(x) = \Gamma g_x SU(8)$, the existence of a holomorphic lifting to a complex homogeneous fibration is not clear.

Definition 9. The set of indices $J \subseteq \{(i, j) | 1 \leq i \neq j \leq 6\}$, $K \subset \{(i, j, k) \mid 1 \leq i, j, k \leq 6\}$, labeling a commutative root system $C(J, K) \subset \Delta_{nc}^+(E_{7(7)})$, is $E_{7(7)}$-*admissible* if there hold simultaneously the conditions:

(i) for an arbitrary $i \neq j$ with $(i, k) \notin J$ for all $k \notin \{i, j\}$ there exists $(i, k, l) \in K$;

(ii) for an arbitrary i with $(j, k) \notin J$ for all j, k different from i there exists $(j, k, l) \in K$ with $l \notin \{i, j, k, \}$;

(iii) for an arbitrary i there exists $(i, j, k) \in K$.

Lemma 10. *Let us suppose that for the harmonic map $f : M \to \Gamma \backslash E_{7(7)}/SU(8)$ there holds $df^{\mathbf{C}}(T_x^{1,0}M) = g_x Span_{\mathbf{C}}(X_\sigma | \sigma \in C(J, K)) g_x^{-1}$ for some $E_{7(7)}$-admissible index set J, K. Then there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(7)}/T^7$ to a maximal complex homogeneous fibration.*

Proof. In general, the $(0,1)$-component of the $su(8)$-valued connection D is

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \sum_{1 \leq i \neq j \leq 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \overline{\eta_{78}} \otimes X_{\lambda_{78}}$$

$$+ \overline{\eta_{87}} \otimes X_{\lambda_{87}} + \sum_{i=1}^{6} \overline{\zeta_i} \otimes X_{\beta_{i7}} + \sum_{i=1}^{6} \overline{z_i} \otimes X_{-\beta_{i7}} + \sum_{i=1}^{6} \overline{\rho_i} \otimes X_{\delta_i} + \sum_{i=1}^{6} \overline{r_i} \otimes X_{-\delta_i}.$$

The $(1,0)$-component of $df^{\mathbf{C}}$ is represented by

$$\theta = \sum_{(i,j)\in J} dx^{ij} \otimes X_{\mu_{ij}} + \sum_{(i,j,k)\in K} dx^{ijk} \otimes X_{\gamma_{ijk}}.$$

The pluriharmonic equation $D''\theta = 0$ implies that $\eta_{ij} = 0$ if there exist $(i,k) \in J$ or $(i,k,l) \in K$, $\eta_{87} = 0$, $z_i = 0$ if there exist $(j,k) \in J$ or $(j,k,l) \in K$, and $r_i = 0$ if there exists $(i,j,k) \in K$. For $E_{7(7)}$-admissible J, K there follows

$$D' = \partial - \sum_{i=1}^{6} \xi_i \otimes H_i - \xi_7 \otimes (H_7 - H_8) + \eta_{78} \otimes X_{\lambda_{87}} + \sum_{i=1}^{6} \zeta_i \otimes X_{-\beta_{i7}} + \sum_{i=1}^{6} \rho_i \otimes X_{-\delta_i}.$$

The consequence $D'\theta = 0$ of the flatness equation $\nabla^2 = 0$ forces

$$D = d + \sum_{i=1}^{6} (\overline{\xi_i} - \xi_i) \otimes H_i + (\overline{\xi_7} - \xi_7) \otimes (H_7 - H_8),$$

which suffices for the existence of a holomorphic F_B, Q.E.D.

Lemma 11. *Each of the following conditions is sufficient for the nonexistence of an equivariant Hermitian symmetric $G_h/K_h \subset E_{7(7)}/E_{7(7)} \cap P$ with $\mathbf{p}_h^+ = Span_{\mathbf{C}}(X_\sigma | \sigma \in C(J,K))$:*

(a) *the existence of $(i,k), (j,k), (i,l) \in J$ with $(j,l) \notin J$;*

(b) *the existence of $(i,k,l), (j,k,l), (i,p,q) \in K$ with $(j,p,q) \notin K$, regardless of $\{k,l\} \cap \{p,q\}$;*

(c) *the existence of $(i,j,k), (i,p,q) \in K$ and $(j,k) \in J$ with $(p,q) \notin J$, regardless of $\{j,k\} \cap \{p,q\}$.*

Proof. In either case, we choose $\sigma_1, \sigma_2, \sigma_3 \in C(J,K)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{7(7)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{7(7)}) - C(J,K)$. More precisely,

(a) $\mu_{il} + (\mu_{jk} - \mu_{ik}) = \mu_{il} + \lambda_{ij} = \mu_{jl}$;

(b) $\gamma_{ipq} + (\gamma_{jkl} - \gamma_{ikl}) = \gamma_{ipq} + \lambda_{ij} = \gamma_{jpq}$;

(c) $\gamma_{ipq} + (\mu_{jk} - \gamma_{ijk}) = \gamma_{ipq} - \delta_i = \mu_{pq}$, Q.E.D.

The root system

$$S(J,K) = \{\mu_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\} \subset \Delta_{nc}^+(E_{7(7)})$$

is *strongly commutative* if its index set J, K satisfies the following conditions:

(i) any two pairs from J are disjoint;

(ii) any two triples from K intersect in exactly one index;

(iii) any triple from K intersects any pair from J in exactly one index.

7. EVI = $E_{7(-5)}/SO(12) \times SU(2)$

The complexified isotropy subalgebra

$$so(12,\mathbb{C}) = \mathbf{h}_7^{\mathbf{C}} + \sum_{1\le i\ne j\le 6} \mathbb{C}X_{\lambda_{ij}} + \sum_{1\le i<j\le 6} (\mathbb{C}X_{\mu_{ij}} + \mathbb{C}X_{-\mu_{ij}}).$$

Since $SU(2)$ is in a direct product with $SO(12)$, the only positive root σ of $sl(2,\mathbb{C})$ satisfies $\sigma + \tau \notin \Delta(E_7^{\mathbb{C}})$ for all $\tau \in \Delta(so(12,\mathbb{C}))$. That specifies $\sigma = \lambda_{78}$. Consequently,

$$\Delta_c^+(E_{7(-5)}) = \{\lambda_{ij}(1 \le i < j \le 6),\ \ \mu_{ij}(1 \le i < j \le 6),\ \ \lambda_{78}\}$$

and

$$\Delta_{nc}^+(E_{7(-5)}) = \{\beta_{i7}(1 \le i \le 6),\ \ \gamma_{ijk}(1 \le i < j < k \le 6),\ \ \delta_i(1 \le i \le 6)\}.$$

The maximal commutative subsets $C \subset \Delta_{nc}^+(E_{7(-5)})$ are of the form $C(I,K) = \{\beta_{i7}(i \in I), \gamma_{ijk}((i,j,k) \in K), \delta_i(i \notin I)\}$ for $I \subseteq \{1,\ldots,6\}$ and a subset K of unordered triples, subject to $(i,j,k) \in K \Rightarrow (l,m,n) \notin K$. In particular, the maximum cardinality of a commutative root system $C \subset \Delta_{nc}^+(E_{7(-5)})$ is 16.

In order to study the holomorphic liftings to a maximal complex homogeneous fibration, let us introduce

Definition 12. An index set I, K of a commutative root system $C(I,K)$ is $E_{7(-5)}$-*admissible* if it satisfies the following conditions:

(i) for any $i \in I$, $j \notin I$ there exists $(i,k,l) \in K$ with different i,j,k,l;

(ii) if $\{1,\ldots,6\} - \{i,j\} \subseteq I$, then there exists $(k,l,m) \in K$ with different i,j,k,l,m;

(iii) if $I \subseteq \{i,j\}$, then there exists $k \notin \{i,j\}$ with $(i,j,k) \in K$.

For example, $I = \{1\}$, $K = \{(1,i,j) | 2 \le i < j \le 6\}$ and $I = \{1,2\}$, $K \supset \{(1,2,i) | 3 \le i \le 6\}$ are $E_{7(-5)}$-admissible.

Lemma 13. *Let $f : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times SU(2)$ be a harmonic map with maximum dimensional $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K)) g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x(SO(12) \times SU(2))$. Then there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times T^1$ to the minimal complex homogeneous fibration, whose associated parabolic subgroup P has semisimple part $SO(12,\mathbb{C})$. If, moreover, the index set I, K is $E_{7(-5)}$-admissible, then f admits a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(-5)}/T^7$ to a maximal complex homogeneous fibration.*

Proof. The $(0,1)$-part of the $so(12) \times su(2)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \sum_{1 \le i \ne j \le 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$
$$+ \overline{\eta_{78}} \otimes X_{\lambda_{78}} + \overline{\eta_{87}} \otimes X_{\lambda_{87}} + \sum_{1 \le i < j \le 6} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i < j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}}.$$

Under the assumptions of the lemma, the $(1,0)$-component of $df^{\mathbb{C}}$ is represented by

$$\theta = \sum_{i \in I} dx^i \otimes X_{\beta_{i7}} + \sum_{(i,j,k) \in K} dx^{ijk} \otimes X_{\gamma_{ijk}} + \sum_{i \notin I} dx^i \otimes X_{\delta_i}.$$

The pluriharmonic equation $D''\theta = 0$ reveals that $\eta_{87} = 0$, $\eta_{ij} = 0$ for $i \notin I$ or $j \in I$ or if there is $(i,k,l) \in K$, $\zeta_{ij} = 0$ if $\exists (k,l,m) \in K$ or $\exists k \notin I$ and $z_{ij} = 0$ if $\exists k \in I$ or $\exists (i,j,k) \in K$ for different i,j,k,l,m. Further, $D'\theta = 0$ implies that $\eta_{78} = 0$.

Consequently, $D = \overline{D''} + D''$ takes values in $so(12)$ and there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times T^1$.

For $E_{7(-5)}$-admissible I, K, the pluriharmonic equation forces

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \overline{\eta_{78}} \otimes X_{\lambda_{78}}$$

and $D'\theta = 0$ specifies that D takes values in $\mathbf{h}_7$. In other words, there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(-5)}/T^7$, Q.E.D.

Lemma 14. *Each of the following conditions is sufficient for the nonexistence of a Hermitian symmetric space G_h/K_h with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K))$, equivariantly embedded in some complex homogeneous fibration $E_{7(-5)}/E_{7(-5)} \cap P$:*

(a) *$k \in I$, $l \notin I$, $(i,j,l) \in K$ and $(l,m,n) \notin K$ for different i,j,k,l,m,n;*

(b) *$(i,j,k), (l,m,n) \in K$ for different i,j,k,l,m,n;*

(c) *$k, l \notin I$, $(i,j,l) \in K$ and $(i,j,k) \notin K$ for different i,j,k,l.*

Proof. The aforementioned conditions provide the following $\sigma_1, \sigma_2, \sigma_3 \in C(I,K)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{7(-5)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{7(-5)}) - C(I,K)$:

(a) $\beta_{k7} + (\delta_l - \gamma_{ijl}) = \beta_{k7} - \mu_{ij} = \gamma_{lmn}$;

(b) $\gamma_{ijk} + (\gamma_{lmn} - \beta_{k7}) = \gamma_{ijk} - \mu_{ij} = \delta_k$ for $k \in I$ or $\gamma_{lmn} + (\gamma_{ijk} - \delta_k) = \gamma_{lmn} + \mu_{ij} = \beta_{k7}$ for $k \notin I$;

(c) $\delta_k + (\gamma_{ijl} - \delta_l) = \delta_k + \mu_{ij} = \gamma_{ijk}$, Q.E.D.

The strongly commutative root systems $S \subset \Delta_{nc}^+(E_{7(-5)})$ of maximum cardinality are equivalent to $S = \{\beta_{17}, \gamma_{134}, \gamma_{156}, \delta_2\}$ modulo the action of the Weyl group of $SO(12,\mathbb{C}) \times SL(2,\mathbb{C})$.

8. EVIII = $E_{8(8)}/SO(16)$

As far as

$$\Delta^+(so(16,\mathbb{C})) = \Delta_c^+(E_{8(8)}) = \{\lambda_{ij}(1 \leq i < j \leq 8), \quad \mu_{ij}(1 \leq i < j \leq 8)\},$$

there follows

$$\Delta_{nc}^+(E_{8(8)}) = \{\alpha, \quad \beta_{ij}(1 \leq i < j \leq 7), \quad \gamma_{ijk}(1 \leq i < j < k \leq 7), \quad \delta_i(1 \leq i \leq 7)\}.$$

The commutative root systems $C \subset \Delta_{nc}^+(E_{8(8)})$ are of the form

$$C_1(I,J,K) = \{\delta_i(i \in I), \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

or

$$C_2(J,K) = \{\alpha, \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

with $(i,j) \in J \Rightarrow p \notin I$ for $p \in \{i,j\}$; $(i,j,k) \in K \Rightarrow (p,q) \notin J$ for $p,q \in \{i,j,k\}$; $(i,j,k) \in K \Rightarrow (l,m,n) \notin K$ for different i,j,k,l,m,n.

Definition 15. For a commutative root system $C_1(I,J,K)$ one says that I,J,K is an $E_{8(8)}$-admissible index set of first kind whenever there hold the following conditions:

(i) if $i \notin I$, $i \notin SuppK$, then there exists $(j,k) \in J$ for different i,j,k;

(ii) if $(i,j) \notin J$, $I \subseteq \{i,j\}$, then there exists $(k,l,m) \in K$ with different i,j,k,l,m;

(iii) if $(i,j,k) \notin K$ for fixed $i \neq j$ and all $k \notin \{i,j\}$, then there exists $(k,l) \in J$.

Definition 16. If $C_2(J,K)$ is a commutative root system, then J,K is an $E_{8(8)}$-semi-admissible set of indices of second kind whenever for any unordered pair $(i,j) \notin J$ there exists an unordered triple $(k,l,m) \in K$ with different i,j,k,l,m.

Definition 17. For a commutative root system $C_2(J,K)$ the pair J,K is an $E_{8(8)}$-admissible index set of second kind if it is $E_{8(8)}$-semi-admissible and for any $i \neq j$ with $(j,k) \notin J$ for all $k \notin \{i,j\}$ there exists $l \notin \{i,j,k\}$ with $(i,k,l) \in K$.

Lemma 18. *Let $f : M \to \Gamma \setminus E_{8(8)}/SO(16)$ be a harmonic map of a compact Kähler manifold M with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K))g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x SO(16)$ and an $E_{8(8)}$-semi-admissible index set J,K of second kind. Then f admits a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(8)}/U_8 \times T^1$ to a minimal complex homogeneous fibration $\Gamma \setminus E_{8(8)}/U_8 \times T^1 \to \Gamma \setminus E_{8(8)}/SO(16)$ with fiber* **DIII**$_c(8)/T^1$. *For a harmonic map $f : M \to \Gamma \setminus E_{8(8)}/SO(16)$ with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_1(I,J,K))g_x^{-1}$, where I,J,K is an $E_{8(8)}$-admissible index set of first kind, or $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K))g_x^{-1}$, where J,K is an $E_{8(8)}$-admissible index set of second kind, there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$ to the maximal complex homogeneous fibration.*

Proof. The $(0,1)$-component of the $so(16)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \neq j \le 8} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \sum_{1 \le i < j \le 8} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i < j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}}.$$

The $(1,0)$-component of the differential $df^{\mathbb{C}}$, associated with a commutative root system $C_2(J,K)$, is

$$\theta_2 = dx^0 \otimes X_\alpha + \sum_{(k,l) \in J} dx^{kl} \otimes X_{\beta_{kl}} + \sum_{(p,q,r) \in K} dx^{pqr} \otimes X_{\gamma_{pqr}}.$$

The pluriharmonic equation $D''\theta_2 = 0$ implies that $z_{ij} = 0$ for all $1 \le i < j \le 7$, $z_{i8} = 0$ for all $1 \le i \le 7$, $\eta_{ij} = 0$ if $\exists (j,k) \in J$ or $\exists (i,k,l) \in K$, $\zeta_{ij} = 0$ if $(i,j) \in J$ or $\exists (k,l,m) \in K$ for different i,j,k,l,m. For $E_{8(8)}$-semi-admissible J,K of second kind that provides

$$D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \neq j \le 8} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$

and an existence of a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(8)}/U_8 \times T^1$. Here $U_8 \times T^1$ is the centralizer of $Exp_1^{E_{8(8)}}\left(\mathbb{R}\left(\sum_{i=1}^{8} H_i\right)\right)$. Whenever J,K is an $E_{8(8)}$-admissible index set of second kind, there follows $D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i$, which suffices for the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$.

If the harmonic map f is associated with a commutative root system $C_1(I,J,K)$ of first kind, then

$$\theta_1 = \sum_{i\in I} dx^i \otimes X_{\delta_i} + \sum_{(i,j)\in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j,k)\in K} dx^{ijk} \otimes X_{\gamma_{ijk}}.$$

The pluriharmonic equation $D''\theta_1 = 0$ implies that $\eta_{ij} = 0$ if $i \in I$ or there exist $(j,k) \in J$ or $(i,k,l) \in K$, $\zeta_{ij} = 0$ for $(i,j) \in J$ or $\exists(k,l,m) \in K$ or $\exists k \in I$, $z_{ij} = 0$ for $1 \leq i < j \leq 7$ if $\exists(k,l) \in J$ or $\exists(i,j,k) \in K$, $z_{i8} = 0$ if $i \in I$ or $\exists(j,k) \in J$ or $\exists(i,j,k) \in K$. The $E_{8(8)}$-admissibility of the index set I, J, K of first kind suffices for $D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i$ and the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$, Q.E.D.

Lemma 19. *Each of the following conditions implies the nonexistence of an equivariant Hermitian symmetric $G_h/K_h \subset E_{8(8)}/E_{8(8)} \cap P$ with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_1(I,J,K))$ or $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K))$:*

(a) *the existence of $(i,k),(j,k) \in J$, $(i,k,l) \in K$ with $(j,k,l) \notin K$;*

(b) *the existence of $i,j \in I$, $(j,k) \in J$ with $(i,k) \notin J$ in the case of $C_1(I,J,K)$;*

(c) *the existence of $(i,j) \in J$, $(i,j,k) \in K$ in the case of $C_2(J,K)$.*

Proof. Below are exhibited $\sigma_1, \sigma_2, \sigma_3 \in C_i$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{8(8)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \notin C_i$:

(a) $\gamma_{ikl} + (\beta_{ik} - \beta_{jk}) = \gamma_{ikl} + \lambda_{ij} = \gamma_{jkl}$;

(b) $\beta_{jk} + (\delta_j - \delta_i) = \beta_{jk} + \lambda_{ij} = \beta_{ik}$;

(c) $\gamma_{ijk} + (\beta_{ij} - \alpha) = \gamma_{ijk} - \mu_{ij} = \delta_k$, Q.E.D.

In order to describe the strongly commutative root systems $S \subset \Delta_{nc}^+(E_{8(8)})$, one lists $\sigma, \tau \in \Delta_{nc}^+(E_{8(8)})$ with $\sigma - \tau \in \Delta_c(E_{8(8)})$ and observes that

$$S = S_1(I,J,K) = \{\delta_i(i \in I), \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

are subject to the conditions:

(i) $cardI \leq 1$; (ii) the pairs from J are disjoint; (iii) any two triples from K intersect in a single element; (iv) $(i,j) \in J \Rightarrow i,j \notin I$; (v) $(i,j,k) \in K \Rightarrow (i,j),(i,k),(j,k) \notin J$; (vi) $(i,j,k) \in K \Rightarrow (l,m) \notin J$ for different i,j,k,l,m; (vii) $(i,j,k) \in K \Rightarrow i,j,k \notin I$.

The strongly commutative root systems $S_2(J = \varnothing, K) = \{\alpha, \gamma_{ijk}((i,j,k) \in K)\}$ are characterized by the fact that any pair of triples from K intersect in a single index.

9. EIX $= E_{8(-24)}/E_7 \times SU(2)$

The positive roots of E_7 are listed at the beginning of Section 5. The only positive root τ of $sl(2,\mathbb{C})$ is subject to the property $\sigma+\tau \notin \Delta(E_8^{\mathbb{C}})$ for all $\sigma \in \Delta(E_7^{\mathbb{C}})$. Thus, $\tau = \mu_{78}$ and

$$\Delta_c^+(E_{8(-24)}) = \{\lambda_{ij}(1 \le i < j \le 6), \mu_{ij}(1 \le i < j \le 6), \lambda_{78}, \mu_{78},$$
$$\beta_{i7}(1 \le i \le 6), \gamma_{ijk}(1 \le i < j < k \le 6), \delta_i(1 \le i \le 6)\},$$
$$\Delta_{nc}^+(E_{8(-24)}) = \{\lambda_{ij}, \mu_{ij}(1 \le i \le 6, 7 \le j \le 8),$$
$$\alpha, \beta_{ij}(1 \le i < j \le 6), \gamma_{ij7}(1 \le i < j \le 6), \delta_7\}.$$

For a commutative root system $C \subset \Delta_{nc}^+(E_{8(-24)})$ of maximal cardinality, let I_1 be the set of the indices i with $\lambda_{i7} \in C$, I_2 be the set of the indices i with $\lambda_{i8} \in C$, and J be the set of the unordered pairs (i,j) with $\beta_{ij} \in C$. Then either

$$C = C_1(I_1, I_2, J) = \{\alpha, \lambda_{i7}(i \in I_1), \mu_{i8}(i \notin I_1), \lambda_{i8}(i \in I_2),$$
$$\mu_{i7}(i \notin I_2), \beta_{ij}((i,j) \in J), \gamma_{ij7}((i,j) \notin J)\}$$

or

$$C = C_2(I_1, I_2, J) = \{\delta_7, \lambda_{i7}(i \in I_1), \mu_{i8}(i \notin I_1), \lambda_{i8}(i \in I_2),$$
$$\mu_{i7}(i \notin I_2), \beta_{ij}((i,j) \in J), \gamma_{ij7}((i,j) \notin J)\}.$$

Definition 20. A maximal commutative root system $C_1(I_1, I_2, J)$ is labeled by $E_{8(-24)}$-admissible indices of first kind if there hold the following conditions:

(i) for any $i \in I_1 \cap I_2$ and $j \notin I_1 \cup I_2$ there exist $(j,k) \in J$ or $(i,k) \notin J$;

(ii) if $I_1 = \varnothing$, then $I_2 \neq \{1, \ldots, 6\}$;

(iii) if $I_2 = \varnothing$, then $I_1 \neq \{1, \ldots, 6\}$;

(iv) if $(i,j) \notin J$ and $i \notin I_1 \cup I_2$, then there exists $(k,l) \notin J$;

(v) if $i \in I_2$ and $I_1 \subseteq \{i\}$, then there exists $(i,j) \notin J$;

(vi) if $\{i,j,k\} \cap I_1 = \varnothing$, $l, m, n \in I_2$ and $(l,m), (l,n), (m,n) \in J$, then at least one of the pairs $(i,j), (j,k)$ and (i,k) belongs to J;

(vii) if $\{i,j,k\} \cap I_1 = \varnothing$, $\{l,m,n\} \cap I_2 = \varnothing$ and $(i,j), (i,k), (j,k) \in J$, then at least one of the pairs $(l,m), (l,n), (m,n)$ belongs to J;

(viii) if $i \in I_1$ and $I_2 \subseteq \{i\}$, then there exists $(i,j) \notin J$.

Definition 21. A maximal commutative root system $C_2(I_1, I_2, J)$ has an $E_{8(-24)}$-admissible index set I_1, I_2, J of second kind when there hold the following conditions:

(i) if $j \notin I_1 \cup I_2$ and $i \in I_1 \cap I_2$, then there exists $(j,k) \in J$;

(ii) if $I_1 = \varnothing$, then $I_2 \neq \{1, \ldots, 6\}$;

(iii) if $I_2 = \varnothing$, then $I_1 \neq \{1, \ldots, 6\}$;

(iv) if $i \in I_1 \cap I_2$, then there exists $(i,j) \notin J$;

(v) if $i \notin I_2$, then there exists $(i,j) \in J$;

(vi) for any permutation i, j, k, l, m, n of $1, \ldots, 6$ with $\{l,m,n\} \subseteq I_2$ and $(l,m), (l,n), (m,n) \in J$ there exists $(i,j), (j,k)$ or (i,k) from J;

(vii) for $\{i,j,k\} \subseteq I_1$ with $(i,j), (j,k), (i,k) \in J$ there exists $(l,m) \in J$;

(viii) if $i \notin I_1$, then there exists $(i,j) \in J$.

Lemma 22. *If $f : M \to \Gamma \setminus E_{8(-24)}/E_7 \times SU(2)$ is a harmonic map with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J)) g_x^{-1}$ for a maximal commutative root system $C_i(I_1, I_2, J) \subset \Delta_{nc}^+(E_{8(-24)})$, then there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$ to a minimal complex homogeneous fibration. If $C_i(I_1, I_2, J)$ is* *labeled by an $E_{8(-24)}$-admissible index set of i-th kind, $i = 1$ or 2, then f admits a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$ to a maximal complex homogeneous fibration.*

Proof. The $(0,1)$-component of the $\mathbf{e}_7 \oplus su(2)$-valued connection D has the form

$$D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \ne j \le 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \overline{\eta_{78}} \otimes X_{\lambda_{78}} + \overline{\eta_{87}} \otimes X_{\lambda_{87}}$$
$$+ \sum_{1 \le i < j \le 6} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i < j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}} + \overline{\zeta_{78}} \otimes X_{\mu_{78}} + \overline{z_{78}} \otimes X_{-\mu_{78}}$$
$$+ \sum_{i=1}^{6} \overline{\sigma_i} \otimes X_{\beta_{i7}} + \sum_{i=1}^{6} \overline{s_i} \otimes X_{-\beta_{i7}} + \sum_{1 \le i < j < k \le 6} \overline{\tau_{ijk}} \otimes X_{\gamma_{ijk}}$$
$$+ \sum_{1 \le i < j < k \le 6} \overline{t_{ijk}} \otimes X_{-\gamma_{ijk}} + \sum_{i=1}^{6} \overline{\rho_i} \otimes X_{\delta_i} + \sum_{i=1}^{6} \overline{r_i} \otimes X_{-\delta_i}.$$

The $(1,0)$-component of the differential of f, associated with $C_1(I_1, I_2, J)$, is

$$\theta_1 = dx^0 \otimes X_\alpha + \sum_{i \in I_1} dx^i \otimes X_{\lambda_{i7}} + \sum_{i \notin I_1} dy^i \otimes X_{\mu_{i8}} + \sum_{i \in I_2} du^i \otimes X_{\lambda_{i8}}$$
$$+ \sum_{i \notin I_2} dv^i \otimes X_{\mu_{i7}} + \sum_{(i,j) \in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j) \notin J} dy^{ij} \otimes X_{\gamma_{ij7}}.$$

The pluriharmonic equation $D''\theta_1 = 0$ implies that $\eta_{ij} = 0$ for $i \notin I_1 \cap I_2$ or $j \in I_1 \cup I_2$ or $\exists (j,k) \in J$ or $\exists (i,k) \notin J$, $\eta_{78} = 0$ if $I_1 \ne \emptyset$ or $I_2 \ne \{1, \dots, 6\}$, $\eta_{87} = 0$ for $I_2 \ne \emptyset$ or $I_1 \ne \{1, \dots, 6\}$, $\zeta_{ij} = 0$ if $(i,j) \in J$ or $\exists (k,l) \notin J$ or $i \in I_1 \cup I_2$, $z_{ij} = 0$ for all $1 \le i < j \le 6$, $z_{78} = 0$, $\sigma_i = 0$ if $i \notin I_2$ or $\exists (i,j) \notin J$ or $\exists j \in I_1$, $s_i = 0$ for all $1 \le i \le 6$, $\tau_{ijk} = 0$ if $\exists (l,m) \notin J$ or $\exists (i,j) \in J$ or $\exists i \in I_1$ or $\exists l \notin I_2$, $t_{ijk} = 0$ if $\exists (i,j) \notin J$ or $\exists (l,m) \in J$ or $\exists i \in I_1$ or $\exists l \in I_2$, $\rho_i = 0$ for all $1 \le i \le 6$ and $r_i = 0$ if $i \notin I_1$ or $\exists j \in I_2$ or $\exists (i,j) \notin J$. Then the consequence $D'\theta_1 = 0$ of the flatness equation $\nabla^2 = 0$ yields $\zeta_{78} \wedge dx^i = 0$ for $i \in I_1$ and $\zeta_{78} \wedge dy^i = 0$ for $i \notin I_1$. If $card I_1 \ge 2$, then $\zeta_{78} \in \langle dx^{i_1} \rangle \cap \langle dx^{i_2} \rangle$ for $i_1, i_2 \in I_1$ forces the vanishing of ζ_{78}. Otherwise, $card(\{1, \dots, 6\} - I_1) \ge 5$ and the containment of ζ_{78} in at least two different differential ideals $\langle dy^{i_1} \rangle$, $\langle dy^{i_2} \rangle$, where $i_1, i_2 \notin I_1$, leads to $\zeta_{78} = 0$. That suffices for the existence of a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$. Observe that $E_7 \times T^1$ is the centralizer of the torus $T^1 = Exp_1^{E_{8(-24)}}(H_7 + H_8)$. If, moreover, the index set I_1, I_2, J is $E_{8(-24)}$-admissible, then $D = d + \sum_{i=1}^{8} (\overline{\xi_i} - \xi_i) \otimes H_i$ justifies the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$.

For a harmonic map f with an associated commutative root system $C_2(I_1, I_2, J)$ one has

$$\theta_2 = dx^0 \otimes X_{\delta_7} + \sum_{i \in I_1} dx^i \otimes X_{\lambda_{i7}} + \sum_{i \notin I_1} dy^i \otimes X_{\mu_{i8}} + \sum_{i \in I_2} du^i \otimes X_{\lambda_{i8}}$$
$$+ \sum_{i \notin I_2} dv^i \otimes X_{\mu_{i7}} + \sum_{(i,j) \in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j) \notin J} dy^{ij} \otimes X_{\gamma_{ij7}}.$$

The pluriharmonic equation provides $\eta_{ij} = 0$ for $j \in I_1 \cup I_2$ or $i \notin I_1 \cap I_2$ or $\exists (j,k) \in J$, $\eta_{78} = 0$ for $I_1 \neq \varnothing$ or $I_2 \neq \{1, \ldots, 6\}$, $\eta_{87} = 0$ for $I_2 \neq \varnothing$ or $I_1 \neq \{1, \ldots, 6\}$, $\zeta_{ij} = 0$ for all $1 \leq i < j \leq 6$, $z_{ij} = 0$ for $i \notin I_1 \cap I_2$ or $(i,j) \notin J$, $z_{78} = 0$, $\sigma_i = 0$ for all $1 \leq i \leq 6$, $s_i = 0$ if $i \in I_2$ or $\exists (i,j) \in J$, $\tau_{ijk} = 0$ if $\exists l \notin I_2$ or $\exists (i,j) \in J$ or $\exists (l,m) \notin J$, $t_{ijk} = 0$ if $\exists i \notin I_1$ or $\exists (i,j) \notin J$ or $\exists (l,m) \in J$, $\rho_i = 0$ if $i \in I_1$ or $\exists (i,j) \in J$ and $r_i = 0$ for all $1 \leq i \leq 6$. The consequence $D'\theta_2 = 0$ of the flatness equation specifies $\zeta_{78} = 0$. Therefore, f admits a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$. If I_1, I_2, J is an $E_{8(-24)}$-admissible index set of second kind, then D takes values in $\mathbf{h}$ and there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$, Q.E.D.

Lemma 23. *Each of the following conditions on the index set I_1, I_2, J implies the nonexistence of an equivariant Hermitian symmetric subspace $G_h/K_h \subset E_{8(-24)}/E_{8(-24)} \cap P$ with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J))$ for $i = 1$ or 2:*

(a) *the existence of $i \in I_1$, $j \in I_2$, $(k,l) \in J$ with $(m,n) \notin J$;*

(b) *the existence of $i \in I_1$, $j \notin I_2$ with $(i,j) \notin J$ in the case of $C_1(I_1, I_2, J)$;*

(c) *the existence of $i \notin I_2$, $j \in I_1$ with $(i,j) \in J$ in the case of $C_2(I_1, I_2, J)$.*

Proof. Here are the appropriate $\sigma_1, \sigma_2, \sigma_3 \in C_i(I_1, I_2, J)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{8(-24)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \notin C_i(I_1, I_2, J)$:

(a) $\lambda_{i7} + (\beta_{kl} - \gamma_{mn7}) = \lambda_{i7} + \mu_{ij} = \mu_{j7}$;

(b) $\alpha + (\lambda_{i7} - \mu_{j7}) = \alpha - \mu_{ij} = \beta_{ij}$;

(c) $\mu_{i7} + (\beta_{ij} - \lambda_{j7}) = \mu_{i7} + \beta_{i7} = \alpha$, Q.E.D.

Applying the very definition, one observes that the strongly commutative root systems $S \subset \Delta_{nc}^+(E_{8(-24)})$ of the form

$$S = \{\lambda_{i7}(i \in I_1), \lambda_{i8}(i \in I_2), \mu_{i8}(i \in I_3), \mu_{i7}(i \in I_4),$$
$$\beta_{ij}((i,j) \in J_1), \gamma_{ij7}((i,j) \in J_2)\}$$

are $S = \{\lambda_{17}, \lambda_{28}, \beta_{13}, \beta_{24}, \gamma_{127}\}$, $S = \{\mu_{18}, \mu_{27}, \beta_{12}, \gamma_{137}, \gamma_{247}\}$, $S = \{\lambda_{17}, \lambda_{28}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{17}, \mu_{28}, \beta_{23}, \gamma_{137}\}$, $S = \{\lambda_{17}, \mu_{18}, \beta_{12}, \gamma_{137}\}$, $S = \{\lambda_{17}, \mu_{18}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{18}, \mu_{17}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{18}, \mu_{17}, \beta_{12}, \gamma_{137}\}$ and $S = \{\mu_{18}, \mu_{27}, \beta_{12}, \gamma_{127}\}$ up to $Weyl(E_7^{\mathbb{C}} \times SL(2, \mathbb{C}))$-action.

REFERENCES

1. Bourbaki, N. Groupes et Algèbres de Lie. Chap. 4, 5 et 6, Eléments de mathématique, fascicule XXXIV, Hermann, 1968.
2. Burstall, F. E., J. H. Rawnsley. Twistor theory for Riemannian Symmetric Spaces. *Lecture Notes in Mathematics*, **1424**, Springer, 1990.

3. Carlson, J., L. Hernandez. Harmonic maps of compact Kähler manifolds to exceptional hyperbolic spaces. *J. Geom. Anal.*, **1**, 1991, 339–357.
4. Carlson, J., D. Toledo. Harmonic mappings of Kähler manifolds to locally symmetric spaces. *Publ. Math. I.H.E.S.*, **69**, 1989, 173–201.
5. Carlson, J., D. Toledo. Rigidity of harmonic maps of maximum rank. *J. Geom. Anal.*, **3**, 1993, 99–140.
6. Corlette, K. Flat G-bundles with canonical metrics. *J. Diff. Geom.*, **28**, 1988, 361–382.
7. Eells, J., J. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. Jour. Math.*, **86**, 1964, 109–160.
8. Helgason, S. Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, Orlando-Florida, 1978.
9. Kasparian, A. Hermitian symmetry of infinitesimally trivial harmonic images of compact Kähler manifolds. In: *Aspects of Complex Analysis, Differential Geometry, Mathematical Physics and Applications*, World Scientific, 1999, 121–134.
10. Malcev, A. Commutative subalgebras of semisimple Lie algebras. *Izv. Akad. Nauk USSR*, **9**, 1945, 125–133 (in Russian).
11. Sampson, J. H. Applications of harmonic maps to Kähler geometry. *Contemp. Math.*, **49**, 1986, 125–133.
12. Sato, M., T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. Jour.*, **65**, 1977, 1–155.
13. Simpson, C. Higgs bundles and local systems. *Publ. Math. I.H.E.S.*, **75**, 1992, 5–95.
14. Siu, Y. Complex analyticity of harmonic maps and strong rigidity of compact Kähler manifolds. *Ann. Math.*, **112**, 1980, 73–111.

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