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ON A CLASS OF MEROMORPHIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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In this paper we obtain coefficient inequalities and distortion theorems for the class $T^*(\alpha,\beta,A)$ of meromorphic functions with negative coefficients.

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1. INTRODUCTION

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured unit disk

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$$

with a simple pole at the origin and with residue 1 there. Let Σ^* denote the subclass of Σ consisting of functions $f(z)$, which are convex with respect to the origin, that is, satisfying the condition

$$\Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad z \in U^* . \tag{1.2}$$

Let $\Sigma^*(\alpha)$ denote the subclass of Σ consisting of functions $f(z)$ which are convex of order α, that is, satisfying the condition

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \quad z \in U^*,\ 0 \le \alpha < 1\ . \tag{1.3}$$

Let $\Sigma^*(\alpha, A)$ denote the class of functions $f(z) \in \Sigma$ such that

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{1 + (A - \alpha A + \alpha)w(z)}{1 + w(z)}\ . \tag{1.4}$$

Here $w(z)$ is analytic in $U = \{z : |z| < 1\}$ and satisfies the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in U.$$

The condition (1.4) is equivalent to

$$\left|\frac{\dfrac{zf''(z)}{f'(z)} + 2}{1 + \dfrac{zf''(z)}{f'(z)} + (A - \alpha A + \alpha)}\right| < 1, \quad z \in U^*. \tag{1.5}$$

We note also that

$$\Sigma^*(\alpha, -1) = \Sigma^*(\alpha).$$

Let T denote the subclass of Σ consisting of functions of the form

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n. \tag{1.6}$$

A function $f(z) \in \Sigma$ is in the class $\Sigma^*(\alpha, \beta, A)$ if it satisfies the condition

$$\left|\frac{\dfrac{zf''(z)}{f'(z)} + 2}{1 + \dfrac{zf''(z)}{f'(z)} + (A - \alpha A + \alpha)}\right| < \beta, \tag{1.7}$$

$z \in U^*$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < 1$.

Let us write

$$T^*(\alpha, \beta, A) = \Sigma^*(\alpha, \beta, A) \cap T\ .$$

We note that

$$T^*(\alpha, 1, -1) = T^*(\alpha)$$

is the class of meromorphically convex functions of order α with negative coefficients, which was studied by Uralegaddi and Ganigi [1].

In this paper we obtain coefficient inequalities and distortion theorems for the class $T^*(\alpha, \beta, A)$. We employ techniques similar to those used earlier by Silverman [2].

Theorem 2.1. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in* U^*. *If*

$$\sum_{n=1}^{\infty}\{(n+1)+\beta[n+(1-A)\alpha+A]\}n|a_n| \leq (1-A)\beta(1-\alpha)\ , \tag{2.1}$$

$0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, *then* $f(z) \in \Sigma^*(\alpha, \beta, A)$.

Proof. Let (2.1) hold true for all admissible values of α, β, A. Let us consider the expression

$$F(f, f') = |zf''(z) + 2f'(z)| - \beta|f'(z) + zf''(z) + (A - A\alpha + \alpha)f'(z)|. \tag{2.2}$$

Replacing f and f' by their series expansions, for $0 < |z| = r < 1$ we have

$$F(f, f') = \left|\sum_{n=1}^{\infty}(n+1)na_n z^{n-1}\right| - \beta\left|\frac{(1-A)(1-\alpha)}{z^2} + \sum_{n=1}^{\infty}[n+(1-A)\alpha+A]na_n z^{n-1}\right|.$$

Now

$$\begin{aligned} r^2F(f, f') &\leq \sum_{n=1}^{\infty}(n+1)n|a_n|r^{n+1} \\ &\quad - \beta\left\{(1-A)(1-\alpha) - \sum_{n=1}^{\infty}[n+(1-A)\alpha+A]n|a_n|r^{n+1}\right\} \\ &= \sum_{n=1}^{\infty}\left\{(n+1)+\beta[n+(1-A)\alpha+A]n|a_n|r^{n+1}\right\} - (1-A)\beta(1-\alpha). \end{aligned}$$

Since the above inequality holds true for all r $(0 < r < 1)$, letting $r \to 1-$, we have

$$F(f, f') \leq \sum_{n=1}^{\infty}\{(n+1)+\beta[n+(1-A)\alpha+A]\}\,n|a_n| - (1-A)\beta(1-\alpha) \leq 0$$

by (2.1).

Hence it follows that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < \beta\left|1 + \frac{zf''(z)}{f'(z)} + A - A\alpha + \alpha\right|,$$

so that $f(z) \in \Sigma^*(\alpha, \beta, A)$. $\square$

Theorem 2.2. *Let the function* $f(z)$ *defined by* (1.6) *be abalytic in* U^*. *Then* $f(z) \in T^*(\alpha, \beta, A)$ *iff* (2.1) *is satisfied.*

Proof. In view of Theorem 2.1, let us assume that the function $f(z)$ defined by (1.6) is in the class $T^*(\alpha, \beta, A)$. Then

$$\left| \frac{\dfrac{zf''(z)}{f'(z)} + 2}{1 + \dfrac{zf''(z)}{f'(z)} + (A - A\alpha + \beta)} \right|$$

$$= \left| \frac{-\sum_{n=1}^{\infty}(n+1)n|a_n|z^{n-1}}{\dfrac{(1-A)(1-\alpha)}{z^2} - \sum_{n=1}^{\infty}[n + (1-A)\alpha + A]n|a_n|z^{n-1}} \right| < \beta, \quad z \in U^*.$$

But $\Re(z) \le |z|$ for all z. Thus we have

$$\Re\left\{ \frac{\sum_{n=1}^{\infty}(n+1)n|a_n|z^{n-1}}{\dfrac{(1-A)(1-\alpha)}{z^2} - \sum_{n=1}^{\infty}[n + (1-A)\alpha + A]n|a_n|z^{n-1}} \right\} < \beta, \quad z \in U^*. \tag{2.3}$$

Now we choose the values of z on the real axis, so that $1 + \dfrac{zf''(z)}{f'(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1-$ through real values, we obtain

$$\sum_{n=1}^{\infty}(n+1)n|a_n| \le \beta\left\{(1-A)(1-\alpha) - \sum_{n=1}^{\infty}[n + (1-A)\alpha + A]n|a_n|\right\}$$

or

$$\sum_{n=1}^{\infty}\{(n+1) + \beta[n + (1-A)\alpha + A]\}\, n|a_n| \le (1-A)\beta(1-\alpha), \tag{2.4}$$

which proves the theorem. □

3. A DISTORTION THEOREM

Theorem 3.1. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(\alpha, \beta, A)$. Then for $0 < |z| = r < 1$*

$$\frac{1}{r} - \frac{(1-A)\beta(1-\alpha)}{2 + \beta[1 + A + (1-A)\alpha]}r \le |f(z)| \le \frac{1}{r} + \frac{(1-A)\beta(1-\alpha)}{2 + \beta[1 + A + (1-A)\alpha]}r, \tag{3.1}$$

$$\frac{1}{r^2} - \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]} \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]}. \tag{3.2}$$

The result is sharp. The equality holds true for the function $f(z)$ given by

$$f(z) = \frac{1}{z} - \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]} z \tag{3.3}$$

at $z = r$.

Proof. In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty} |a_n| \le \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]}.$$

Thus for $0 < |z| = r < 1$

$$|f(z)| \le \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n \le \frac{1}{r} + r\sum_{n=1}^{\infty} |a_n| \le \frac{1}{r} + \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]} r$$

and

$$|f(z)| \ge \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| r^n \ge \frac{1}{r} - r\sum_{n=1}^{\infty} |a_n| \ge \frac{1}{r} - \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]} r.$$

Furthermore, it follows from Theorem 2.2 that

$$\sum_{n=1}^{\infty} n|a_n| \le \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]}.$$

Hence

$$|f'(z)| \le \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n| r^{n-1} \le \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n| \le \frac{1}{r^2} + \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]}$$

and

$$|f'(z)| \ge \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| r^{n-1} \ge \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| \ge \frac{1}{r^2} - \frac{(1-A)\beta(1-\alpha)}{2+\beta[1+A+(1-A)\alpha]}.$$

It can easily be seen that the function $f(z)$ defined by (3.3) is extremal for Theorem 3. □

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