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A LOGARITHMIC CLASS OF SEMILINEAR WAVE EQUATIONS

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We study the global existence, long-time behaviour and blow-up of classical solutions of the equation $\Box u = u \ln^q(1+u^2)$ in $(3+1)$-space-time with arbitrary big initial data. Thus we have a case of a repellent potential energy term in the relevant energetic identity, contrary to the attractive energy case described by the well-known equation $\Box u = -u|u|^{p-1}$. The global existence result for $0 < q \leq 2$ is first established. Then special "counterdecay" (for $0 < q < 2$) and blow-up effects (for $q > 2$) are found, which show that $q = 2$ is a "critical" value. In this way it is answered, in particular, to a question that has arisen already in the pioneering works of Keller and Jörgens on the semilinear wave equation.

Keywords: global classical solutions, exponentially increasing solutions, blow-up solutions

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1. INTRODUCTION

Consider the semilinear wave equation

$$\Box u = f(u), \tag{1.1}$$

where $\Box \equiv \partial_t^2 - \Delta$, $\Delta = \Delta_x \equiv \sum_{j=1}^n \partial_{x_j}^2$, n is the spatial dimension. For the function $f(u)$ it is assumed that $f(u) = O(|u|^p)$, for $|u| \to +\infty$ or $|u| \to 0$, with a certain parameter $p > 0$.

As it is well-known, there are two critical numbers $p^*(n)$ and $p_0(n)$ that play a prominent role in the theory of Eq. (1.1). They are defined as follows:

$$p^*(n) = \frac{n+2}{n-2}, \quad p_0(n) = z^+, \tag{1.2}$$

where z^+ is the positive root of the equation

$$(n-1)z^2 - (n+1)z - 2 = 0.$$

The significance of the number $p^*(n)$ was revealed by Jörgens [6], by the results of many authors afterwards, see, e.g., [1, 9], and in the more recent papers [3, 10, 12] and the references therein. This number gives an answer to the question for the strongest nonlinearity of $f(u)$, as $|u| \to +\infty$, which admits global (in time) classical solutions to Eq. (1.1) without restrictions on the magnitude of the initial data. In the above cited papers global existence results have been obtained in the case $1 < p < p^*$, for an arbitrary magnitude of data, under the assumptions that the potential energy term

$$U_f[u] \equiv \int_0^u f(z)\,\mathrm{d}z$$

is attractive, i.e. $U_f[u] \leq \text{const}, \forall u$, within the main class of functions $f(u)$ of the form

$$f(u) = -u|u|^{p-1}. \tag{1.3}$$

The second critical number $p_0(n)$ shows up for u close to zero, in the quest for existence of global solutions with small enough data. The number $p_0(n)$ has been found by John [5] for $n = 3$, when $p_0 = 1 + \sqrt{2}$, and by Strauss [11] for an arbitrary n. It is noted that after the blow-up result in [5] for the equations $\Box u = \pm u^2$ (see also Lindblad [8]) with $p = 2$ — a number within the subcritical interval $1 < p < 1 + \sqrt{2}$, and the global existence result in [5] for $p \geq 1 + \sqrt{2}$, many authors paid a special attention to the cases of the critical value $p = p_0(n)$ and to the supercritical interval $p > p_0(n)$.

In the present paper we consider the classical solutions of the Cauchy problem

$$\Box u = u \ln^q(1+u^2), \quad x \in R^3,\ t > 0,\ q > 0, \tag{1.4}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in R^3, \tag{1.5}$$

where $\varphi \in C^3$, $\psi \in C^2$. (As usual, $C^k = C^k(R^3)$ is the space of the k-time smooth functions.) The main results below shed more light particularly on the small data problem for Eq. (1.1). (For more details see, e.g., [11, 13, 14].) When $|u|$ is small, we recover in Eq. (1.4) a particular case of Eq. (1.1) with $p = 1+2q$. However, in the subcritical interval $q < 1/\sqrt{2}$ ($1 < p < 1 + \sqrt{2}$) the problem (1.4), (1.5) possesses global solutions for arbitrary magnitudes of the data. From the view-point of the interest of many authors to the so-called supercritical interval for Eq. (1.1), let us note the following. For $q > 2$ we have in Eq. (1.4) a whole interval of supercritical powers $p = 1 + 2q$, provided $|u|$ is small; then, obviously, $p > 5 > 1 + \sqrt{2}$ (see [5], where the critical value $1 + \sqrt{2}$ is discussed in more details). Nevertheless, our blow-up results (established below in Section 4) show that classical solutions can exist for Eq. (1.1) (in the case (1.4)) with arbitrary small data, which blow up in the case of big enough supports of the data. Such an "anomaly" seems to be caused mainly by the repellent influence of the potential energy term U_f in

Eq. (1.4), when $U_f[u] \geq \text{const}, \forall\, u$. Compared with (1.3), the nonlinearity of the source term is much weaker than that in the right-hand side of (1.4), as $|u| \to +\infty$, but the potential energy term $U_f[u]$ is repellent. Taking into account the Keller's pioneering result [7] (see also that of Glassey [4] and the counter-example in [9, 12] for (1.1) with $f = u|u|^{p-1}$), one can expect an absence of global solutions of (1.1) for repellent U_f, when the test life-span T_f^0,

$$T_f^0 \equiv \int_0^{+\infty} \left(1 + 2\int_0^z f(s)\,\mathrm{d}s\right)^{-1/2} \mathrm{d}z,$$

is finite. However, it remains an open question in general whether Eq. (1.1) possesses global classical solutions for arbitrary data (1.5) if $U_f \geq \text{const}$, but $T_f^0 = +\infty$. For Eq. (1.4) we answer this question in Section 4 below. More precisely, we show that there exists a unique global solution of (1.4), (1.5) for $0 < q \leq 2$ (when $T_f^0 = +\infty$). In Section 3 the behaviour of the positive solutions of the Newton equation $\ddot{v} = v\ln^q(1+v^2)$, associated with Eq. (1.4), is studied. In Section 4 it is proven that the classical solutions of Eq. (1.4) increase exponentially ("counterdecay") if $0 < q < 2$, and blow up if $q > 2$, for data, either positive or negative, which produce the so-called space-destinated waves. Thus it is shown, in particular, that $q = 2$ is a critical value for the global classical solutions of Eq. (1.4).

2. TWO BASIC PRINCIPLES

Consider the following Cauchy problem for the semilinear wave equation:

$$\Box u = f(u), \quad x \in R^3,\ t > 0, \tag{2.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),\ x \in R^3, \tag{2.2}$$

where $f \in C^2(R^1), f(0) = 0$, and $\varphi \in C^3(R^3), \psi \in C^2(R^3)$. Recall that a solution u of Eq. (2.1) is called classical in a set $G \subset R^3 \times [0, +\infty)$ such that $G \cap \{t = 0\}$ contains some domain in R^3 if $u \in C^2(G) \cap C^1(\overline{G})$, $\overline{G}$ being the closure of G. For a given compact $K \subset R^3$ we denote by K_T^- the part of the backward light cone contained in the strip $0 \leq t \leq T$ and based on K at $t = 0$. Similarly, K_T^+ denotes the forward light cone issued from K for $0 \leq t \leq T$.

We suppose known the local existence theorem for (2.1), (2.2), and the theorems for uniqueness and the continuous dependence on the data.

Below we shall use the integral equation

$$u = u^0 + E * f(u^+). \tag{2.3}$$

Here u^0 is the solution of the free wave equation $\Box u = 0$, given by the classical Kirchhoff formula

$$u^0(x,t) = \partial_t \left(\frac{t}{4\pi} \int_{|\omega|=1} \varphi(x + t\omega)\,\mathrm{d}s_\omega \right) + \frac{t}{4\pi} \int_{|\omega|=1} \psi(x + t\omega)\,\mathrm{d}s_\omega, \quad \omega \in R^3, \tag{2.4}$$

where E is the fundamental solution of the wave operator, $u^+ = u$ $(t > 0)$, $u^+ = 0$ $(t < 0)$, and $E * f(u^+)$ is the convolution in R^4 in the usual distribution sense. Recall also that the following known formula is valid for $E * f(u^+)$, namely,

$$E * f(u^+) = \frac{1}{4\pi}\int_0^t \tau \int_{|\omega|=1} f\left(u(x+\tau\omega, t-\tau)\right)\, \mathrm{d}s_\omega \mathrm{d}\tau, \tag{2.5}$$

for $x \in R^3$ and $t > 0$.

2.1. COMPARISON PRINCIPLE

We realize here, in a new case, a classical comparison idea known, e.g., from the theory of the finite-dimensional dynamic systems. The essence is to compare the solution $u(x,t)$ of (2.1), (2.2) with the solutions of the Newton equation $\ddot{v} = f(v)$ from below or above. In the next theorem we impose the following requirements upon the function f:

$$f \in C^2(R^1), \quad f(0) = 0,\ f' \geq 0, \tag{2.6}$$

in R^1, and use backward light cones C_B^- in $\{0 \leq t\}$, based on closed balls $B \subset R^3$. (Obviously, a cone C_B^- can be represented as K_T^- with $K = B$ and big enough T.) Now the comparison principle reads:

Theorem 2.1. *Suppose the condition* (2.6) *holds for* $f(u)$. *Let the functions* $u(x,t)$ *and* $u^0(x,t)$ *be the solution of* (2.1), (2.2) *and the function from* (2.4), *respectively. Let* C_B^- *be a given backward light cone. Suppose also that* u^0 *satisfies the inequality*

$$u^0(x,t) \geq a_1 + b_1 t, \quad (x,t) \in C_B^-, \tag{2.7}$$

with certain constants a_1, $b_1 \geq 0$ *depending on* B. *Then the following estimate holds for the solution* u:

$$u(x,t) \geq v_1(t), \quad (x,t) \in C_B^- \cap G \cap \{0 \leq t < T_1\}, \tag{2.8}$$

where $v_1(t)$ *is the solution of the Cauchy problem*

$$\ddot{v} = f(v), \quad v(0) = a,\ \dot{v}(0) = b, \tag{2.9}$$

defined for $t \in [0, T_1)$, *for a positive* T_1, *with* $a = a_1$, $b = b_1$. *Similarly, we have the estimate*

$$|u(x,t)| \leq v_2(t), \quad (x,t) \in C_B^- \cap G \cap \{0 \leq t < T_2\}, \tag{2.10}$$

if $v_2(t)$ *is the solution of* Eq. (2.9), *defined for* $t \in [0, T_2)$, $T_2 > 0$, *with data* $a = a_2$, $b = b_2$, *where*

$$|u^0(x,t)| \leq a_2 + b_2 t, \quad (x,t) \in C_B^-, \tag{2.11}$$

and $a_2, b_2 \geq 0$ *are certain constants depending on* B.

Proof. Take an arbitrary backward light cone K_T^- such that

$$K_T^- \subset C_B^- \cap G \cap \{0 \le t \le T_1\}$$

and set

$$w_1(t,T) = \min_y u(y,t), \quad (y,t) \in K_T^-, \; t \in [0,T),$$

with fixed t. Note that the solution u is non-negative in K_T^-: this follows from the condition (2.7) and the positivity of the operator in (2.5). For a fixed $t \in [0,T)$, in virtue of Eq. (2.3) and the inequality (2.7), we have the estimate

$$w_1(t,T) \ge a_1 + b_1 t + \min_x E * f(u^+)(x,t), \quad (x,t) \in K_T^-. \tag{2.12}$$

Next, we can easily verify the inequality

$$E * f(u^+)(x,t) \ge E_0 * f(w_1^+)(t), \quad (x,t) \in K_T^-, \tag{2.13}$$

where E_0 is the fundamental solution of the operator d^2/dt^2, and

$$E_0 * f(w_1^+)(t) = \int_0^t s f\left(w_1(t-s,T)\right) \mathrm{d}s. \tag{2.14}$$

Indeed, the formula (2.5) shows that the fundamental solution E represents a positive measure and because of the monotonicity of function f we have

$$E * f(u^+) \ge E * f(w_1^+) = E_0 * f(w_1^+), \quad (x,t) \in K_T^-.$$

Then from (2.12), (2.13) we obtain

$$w_1(t,T) \ge a_1 + b_1 t + E_0 * f(w_1^+)(t,T), \quad t \in [0,T). \tag{2.15}$$

Now it is natural to compare the function $w_1(t,T)$ with the solution $v_1(t)$, using that $v_1(t)$ solves the equation

$$v_1(t) = a_1 + b_1 t + E_0 * f(v_1^+)(t), \quad t \in [0,T). \tag{2.16}$$

Employing familiar arguments, it is not difficult to show that

$$w_1(t,T) \ge v_1(t), \quad \forall\, t \in [0,T),$$

which proves (2.8), because K_T^- is an arbitrary cone.

To prove the estimate (2.10), we utilize fully similar arguments. We now set

$$w_2(t,T) = \max_y |u(y,t)|, \quad (y,t) \in K_T^-, \; t \in [0,T),$$

with fixed t. Then

$$w_2(t,T) \le a_2 + b_2 t + E_0 * f(w_2^+)(t,T)$$

as a consequence of (2.3), (2.11) and the properties of convolutions $E * f(u^+)$, $E_0 * f(w^+)$. Next it remains to compare the function $w_2(t,T)$ with the solution $v_2(t)$, repeating the arguments from the comparison of w_1 and v_1. This completes the proof of Theorem 2.1. □

This principle gives an affirmative answer to the natural question, concerning time extension of the classical solutions of Eq. (2.1), dominated by certain "super-solutions."

Theorem 2.2. *Suppose* (2.6) *is valid for* $f(u)$ *and the following inequality holds for the function* u^0:

$$u^0 : |u^0(x,t)| \leq a + bt, \quad (x,t) \in K^-_{T_0},$$

for a given cone $K^-_{T_0}$, *with constants* $a, b \geq 0$ *depending on* T_0. *Let* $v(t)$ *be the solution of the problem* (2.9) *defined for* $t \in [0,T]$, $T \leq T_0$. *Then there exists a unique classical solution* u_T *of* (2.1), (2.2) *in* K^-_T, *which coincides with* u *in* $G \cap K^-_T$, *where*

$$K^-_T = K^-_{T_0} \cap \{0 \leq t \leq T\}.$$

The proof, being known, is omitted (see, e.g., [5] and the references cited therein). □

2.3. POSITIVE AND NEGATIVE SOLUTIONS

In this section we establish the existence of positive and negative solutions of Eq. (2.1). We introduce a class of solutions called space-destinated waves and note that they are positive or negative solutions, for $t > 0$, if the initial data are positive or negative, respectively. To this end we use the Lorentz pseudometric m_L,

$$m_L = dt^2 - |dx|^2, \quad |dx|^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad x = (x^1, x^2, x^3).$$

Following Friedlander [2], we recall the relations

$$m_L(\xi,\eta) = \xi^0\eta^0 - \sum_{j=1}^{3} \xi^j\eta^j, m_L(\xi,\xi) = (\xi^0)^2 - |\xi'|^2, \tag{2.17}$$

where $\xi' = (\xi^1, \xi^2, \xi^3)$, $|\xi'| = \sqrt{\sum_{j=1}^3 (\xi^j)^2}$, and $\xi = (\xi', \xi^0)$, $\eta = (\eta', \eta^0)$ are arbitrary vectors in R^{3+1}.

Definition 2.1. A classical solution $u(x,t)$ of (2.1) is called a *space-destinated wave* if the set

$$\Sigma_0 = \{(x,t) \in R^{3+1} : u(x,t) = u(x_0, 0)\}$$

is a non-degenerated smooth 3D-hypersurface in a neighbourhood of $(x_0, 0)$, $\forall x_0 \in R^3$, and the following inequality is fulfilled:

$$(-1)m_L(\xi,\xi) \geq 0, \quad \xi \in T_{(x_0,0)}(\Sigma_0), \ \xi \neq 0, \tag{2.18}$$

where $T_{(x_0,0)}(\Sigma_0)$ is the tangent hyperplane to Σ_0 at the point $(x_0, 0)$. A classical solution u is called a *strongly space-destinated wave* if the inequality (2.18) is strong.

Remark 2.1 (existence of space-destinated waves). It is not difficult to verify (see, e.g., [2]) that $(-1)m_L \geq 0$ on the tangent space from (2.18) if and only if $m_L(u'(x_0,0), u'(x_0,0)) \geq 0$, where $u'(x,t) \in R^{3+1}$ is the gradient of $u(x,t)$ at the point (x,t). Now, using the coordinate representation $u'(x,0) = (\nabla\varphi(x), \psi(x))$, where $u(x,0) = \varphi(x), u_t(x,0) = \psi(x)$, and $\nabla\varphi$ is the gradient of φ, we apply the formula (2.17) for $\xi = \eta = u'$ and reach the following conclusion. A classical solution $u(x,t)$ is a space-destinated wave if and only if the data $\varphi(x), \psi(x)$ have the properties

$$(\nabla\varphi(x), \psi(x)) \neq 0, \quad |\psi(x)| - |\nabla\varphi(x)| \geq 0, \ x \in R^3.$$

Proposition 2.1. *Suppose the function f satisfies the condition*

$$f(u) \in C^2(R^1), \quad uf(u) \geq 0, \ u \in R^1,$$

and the solution $u(x,t)$ of (2.1), (2.2), *defined on a set G, is a space-destinated wave. Then $u(x,t)$ is positive in G, for $t > 0$, if the initial data $u(x,0), u_t(x,0)$ are non-negative, for $x \in R^3$, and at least one of the following two conditions hold:*

(i) $$u(x,0) \neq 0, \ x \in R^3;$$

(ii) $u(x,t)$ *is a strongly space-destinated wave.*

Similarly, the solution $u(x,t)$ is negative in G, for $t > 0$, if the initial data are non-positive and at least one of the conditions (i), (ii) *holds.*

The proof is omitted, because the key ideas can be taken from the proof of Theorem 2.1. □

Since the nonlinear part $f(u)$ of Eq. (2.1) is odd, the following connection between positive and negative solutions of this equation is obvious.

Proposition 2.2. *Suppose the function $f(u)$ satisfies the condition*

$$f(u) \in C^2(R^1), \quad f(-u) = -f(u), \ \forall u.$$

Then the map $u(x,t) \to -u(x,t)$ is a bijection between the sets of the positive and the negative classical solutions of (2.1).

3. POSITIVE SOLUTIONS OF A NEWTON EQUATION

The so-called automodel solutions of the problem (1.4), (1.5), when the initial data are constants, satisfy the Cauchy problem for the Newton equation:

$$\ddot{u} = u \ln^q(1+u^2), \quad t > 0, \tag{3.1}$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1\,, \tag{3.2}$$

where u_0, u_1 are constants. We study here the behaviour of the positive solutions $u(t)$ of (3.1), (3.2), provided $u_0 \geq 0$, $u_1 \geq 0$ and $u_0 + u_1 > 0$. The positive solutions of (3.1), (3.2) play the role of sub- and super-solutions for the wave equation (1.4), estimating the solutions of (1.4), (1.5) under the assumptions of the comparison principle (Theorem 2.1). For the solution $u(t)$ of (3.1), (3.2) we shall use the following well-known formulae:

$$u(t) = u_0 + u_1 t + \int_0^t (t-s) f\left(u(s)\right)\,\mathrm{d}s, \tag{3.3}$$

$$t = \int_{u_0}^{u(t)} \left(u_1^2 + 2\int_{u_0}^{z} f(s)\,ds\right)^{-1/2} \mathrm{d}z, \tag{3.4}$$

where $f(u) = u\ln^q(1+u^2)$.

A basic property of the positive solutions under discussion is given by the next lemma.

Lemma 3.1. *If $u_0 \geq 0, u_1 \geq 0$, $u_0 + u_1 > 0$, then the solution $u(t)$ of the problem (3.1), (3.2) is positive and defined in a maximal interval $[0, T^0)$. Moreover, $u(t)$ is a monotonically increasing function in $[0, T^0)$ such that*

$$\lim u(t) = +\infty, \quad t \to T^0. \tag{3.5}$$

Proof. Let us write the relation (3.4) in the form $F(u) = t$, where

$$F(u) \equiv \int_{u_0}^{u} \left(u_1^2 + 2\int_{u_0}^{z} f(s)\,ds\right)^{-1/2} \mathrm{d}z, \quad u \geq u_0.$$

Due to the monotonicity of the function F we have, for the solution $u(t)$ of the problem (3.1), (3.2), $u(t) = F^{-1}(t)$, where F^{-1} is the inverse function of F. The statements of the lemma directly follow from the classical theory of the Newton equation. □

In the next lemma we study a general estimate for the solution $u(t)$.

Lemma 3.2. *Let $u(t)$ be the solution from Lemma 3.1, defined for $t \in [0, T^0)$ and $\psi(z) = z(z-u_0)\ln^q(1+z^2)$. Then the solution $u(t)$ satisfies the estimate*

$$\int_{u_0}^{u} \left(u_1^2 + 2\psi(z)\right)^{-1/2} \mathrm{d}z \leq t \leq \sqrt{1+q}\int_{u_0}^{u} \left((1+q)u_1^2 + \psi(z)\right)^{-1/2} \mathrm{d}z \tag{3.6}$$

for all $t \in [0, T^0)$, $u = u(t)$.

Proof. We begin by studying the function $\varphi(u) = 2\displaystyle\int_{u_0}^{u} f(s)\,\mathrm{d}s$:

$$\varphi(u) = 2\int_{u_0}^{u} s\ln^q(1+s^2)\,\mathrm{d}s = \int_{1+u_0^2}^{1+u^2} \ln^q t\,\mathrm{d}t.$$

Then

$$\varphi'(z) = 2z\ln^q(1+z^2),$$

$$\psi'(z) = (2z-u_0)\ln^q(1+z^2) + \frac{2qz^2}{1+z^2}(z-u_0)\ln^{q-1}(1+z^2).$$

Next we use the estimate for $\psi'(z)$:

$$\psi'(z) \le 2z\ln^q(1+z^2) + 2qz\ln^q(1+z^2), \quad z \ge u_0 .$$

It follows from the above formula for $\psi'(z)$, together with the inequality $z^2(1+z^2)^{-1} \le \ln(1+z^2)$. Estimating $\psi'(z)$ from below, we obtain

$$\psi'(z) \ge z\ln^q(1+z^2) = \frac{1}{2}\varphi'(z), \quad z \ge u_0.$$

Hence

$$\frac{1}{2}\varphi'(z) \le \psi'(z) \le (1+q)\varphi'(z), \quad z \ge u_0,$$

and because of the initial data $\varphi(u_0) = \psi(u_0) = 0$ we obtain the inequality

$$\frac{1}{2}\varphi(z) \le \psi(z) \le (1+q)\varphi(z), \quad z \ge u_0. \tag{3.7}$$

From (3.4) we see that

$$t = \int_{u_0}^{u} \left(u_1^2 + \varphi(z)\right)^{-1/2} \mathrm{d}z, \quad u = u(t),$$

and applying (3.7) we establish the estimate

$$u_1^2 + \frac{\psi(z)}{1+q} \le u_1^2 + \varphi(z) \le u_1^2 + 2\psi(z),$$

which yields (3.6). □

The next lemma is a direct consequence of Theorems 2.1 and 2.2.

Lemma 3.3 (comparison and continuation principles). *Suppose $u(t)$, $v(t)$ are the solutions of* (3.1), (3.2) *with data (u_0, u_1), (v_0, v_1), defined in the maximal intervals $[0, T_u)$, $[0, T_v)$, respectively. Then:*

(i) $$u(t) \le v(t), \quad t \in [0, T_u) \cap [0, T_v),$$

if $0 \le u_j \le v_j$, $j = 0, 1$;

(ii) $$T_u \ge T_v,$$

if $u(t) \le v(t)$ *in* $[0, T_u) \cap [0, T_v)$.

Below we shall employ systematically the general estimate (3.6) in order to study the global solutions of the problem (3.1), (3.2) ($0 < q \le 2$), their large time behaviour and the blow-up phenomena ($2 < q$).

Lemma 3.4 (global solutions). *Let $0 < q \le 2$ and $u_0 \ge 0$, $u_1 \ge 0$, $u_0 + u_1 > 0$. Then the solution $u(t)$ of the problem (3.1), (3.2) is global, i.e. $T^0 = +\infty$, and for $0 < q < 2$ satisfies the estimate*

$$u(t) \le \exp\left((4+u_1)t\right)^{2/(2-q)}, \quad t \ge 3 + u_0 + \frac{2}{d_{0,1}}, \quad 0 < q < 2, \tag{3.8}$$

where $d_{0,1} = \ln^{q/2}(1+u_0^2)$ if $u_1 = 0$, $d_{0,1} = u_1$ if $u_0 = 0$, and $d_{0,1} = \min\{\ln^{q/2}(1+u_0^2), u_1\}$ if $u_1 u_0 > 0$.

Proof. Suppose $0 < q < 2$. Consider firstly the case $u_0 \ge 0$, $u_1 > 0$. In virtue of (3.5) we can chose $t_1 \in (0, T^0)$ such that $u(t) \ge 2 + u_0$ for $t \ge t_1$. Then, due to (3.6), we have

$$t \ge \int_{2+u_0}^{u} \left(u_1^2 + 2(1+z)^2 2^q \ln^q(1+z)\right)^{-1/2} \mathrm{d}z \quad (t \ge t_1).$$

Therefore

$$(u_1^2+8)^{-1/2} \int_{2+u_0}^{u} (1+z)^{-1} \ln^{-q/2}(1+z)\,\mathrm{d}z \le t, \quad t \ge t_1. \tag{3.9}$$

But

$$\int_{2+u_0}^{+\infty} (1+z)^{-1} \ln^{-q/2}(1+z)\,\mathrm{d}z = +\infty$$

and (3.9) shows that the solution $u(t)$ is global. The inequality $u(t) \ge u_0 + u_1 t \ge 2 + u_0$, valid for $t \ge 2/u_1$, and (3.9) yield

$$\ln^{1-q/2}(1+u) \le (3+u_1)t + \ln^{1-q/2}(3+u_0) \le (4+u_1)t \quad (t \ge 3+u_0).$$

The estimate (3.8) obviously follows now with $d_{0,1} \le u_1$. In the case $u_0 > 0$, $u_1 \ge 0$ we argue in a similar manner — noticing that $u(t)$ is global and employing the formula (3.3) to get

$$u(t) \ge u_0 + (t^2/2) f(u_0) \ge 2 + u_0 \quad \left(t \ge 2/\sqrt{f(u_0)},\ f(u_0) = u_0 \ln^q(1+u_0^2)\right).$$

Then (3.8) also follows but with $d_{0,1} \le \sqrt{f(u_0)}$. Suppose now $q = 2$. In the case $u_0 \ge 0, u_1 > 0$ we reach again (3.9) and the solution $u(t)$ is thus global. □

Lemma 3.5 (lower estimates). *Let $0 < q < 2$ and $T \ge 1$ be a parameter. Suppose the initial data (3.2) depend on T, i.e. $u_0 = u_{0,T}$, $u_1 = u_{1,T}$, satisfying the assumptions*

$$0 \le u_{0,T}, \quad \frac{c_1}{1+T} \le u_{1,T}, \tag{3.10}$$

$c_1 > 0$ being a constant. Then the solution $u(t)$ of the problem (3.1), (3.2) satisfies the estimate

$$u(t) \ge (4/c_1) \exp\left(\frac{2-q}{16} h^* t\right)^{2/(2-q)}, \quad t \in [2T/3, T] \quad (0 < q < 2), \tag{3.11}$$

when $T \geq 24/(2-q)h^*$ *and* $h^* = c_1^2/(64+c_1^2)$.

The proof uses arguments fully similar to those from the proof of Lemma 3.4. That is why it is omitted. □

Lemma 3.6 (blow-up time). *Suppose* $q > 2$ *in* Eq. (3.1) *and* $u_0 u_1 \geq 0$, $u_0 + u_1 > 0$. *Then the life-span* T^0 *is finite for the solution* $u(t)$ *of* (3.1), (3.2), *which blows up at the moment* T^0. *Moreover, the blow-up time* T^0 *can be estimated as follows:*

$$\frac{1}{2^{q/2}(q-2)(1+u_1)(1+u_0)^{q/2-1}} \leq T^0 \leq \sqrt{1+q}\left(\frac{\sqrt{3}}{u_1} + \frac{2}{2^{q/2}(q-2)}\right) \quad (u_1 > 0); \tag{3.12}$$

$$\frac{1}{2^{q/2}(q-2)(1+u_1)(1+u_0)^{q/2-1}} \leq T^0 \leq \sqrt{1+q}\left(2\sqrt{\frac{3+u_0}{u_0 \ln^q(1+u_0^2)}} + \frac{4}{2^{q/2}(q-2)}\right) \quad (u_0 > 0). \tag{3.13}$$

Proof. To establish the left-hand sides of (3.12), (3.13), we shall use the left-hand side of (3.6). For the sum $u_1^2 + 2\psi(z)$ we have

$$u_1^2 + 2\psi(z) \leq u_1^2 + 2z^2 \ln^q(1+z^2) \leq (1+u_1)^2(2z)^2 \ln^q(2z)^2,$$

when $z \geq 1+u_0$. From (3.6) it follows

$$T^0 \geq \int_{1+u_0}^{+\infty} \frac{\mathrm{d}z}{(1+u_1)2^{q/2}2z\ln^{q/2}(2z)} = \frac{\ln^{1-q/2}(2+2u_0)}{2^{q/2}(q-2)(1+u_1)}.$$

Afterwards we use the inequality $\ln 2(1+u_0) \leq 1+u_0$, which yields the upper estimates of (3.12), (3.13). In the case of $u_1 > 0$ we can consider $u_0 = 0$ (due to the comparison principle). Then from (3.6), taking the appropriate limit, it follows

$$T^0 \leq \sqrt{1+q}\int_0^{+\infty} \frac{\mathrm{d}z}{\sqrt{3u_1^2 + \psi(z)}} \quad \left(\psi(z) = z^2\ln^q(1+z^2)\right).$$

But for the just defined function $\psi(z)$ we have $\psi(z) \geq 2^q z^2 \ln^q z$ for $z \geq 3$. Hence

$$T^0 \leq \sqrt{1+q}\int_0^3 \frac{\mathrm{d}z}{\sqrt{3}u_1} + \frac{\sqrt{1+q}}{2^{q/2}}\int_3^{+\infty} \frac{\mathrm{d}z}{z\ln^{q/2} z}$$

and the right-hand side of (3.12) immediately follows.

If $u_0 > 0$, we can similarly consider $u_1 = 0$ (again recalling the comparison principle). Next, employing the estimate

$$\psi(z) \geq (z-u_0)^2 \ln^q(1+z^2) \geq 2^{q-2} z^2 \ln^q z$$

for $z \geq 2u_0$, we obtain from (3.6)

$$T^0 \leq \sqrt{1+q}\int_{u_0}^{3+2u_0} \frac{\mathrm{d}z}{\sqrt{u_0 \ln^q(1+u_0^2)(z-u_0)}} + \sqrt{1+q}\int_{3+2u_0}^{+\infty} \frac{\mathrm{d}z}{2^{q/2-1} z \ln^{q/2} z},$$

which gives the right-hand side of (3.13). The blow-up property $\lim u(t) = +\infty$ for $t \to T^0$ has been already established in Lemma 3.1. □

4. GLOBALITY, COUNTER-DECAY AND EXPLOSION OF SOLUTIONS

In this section the main results of the paper are formulated and proven for the class (1.4), where the potential energy contains a repellent term. It becomes clear, in particular, that for Eq. (1.4) $q = 2$ is a critical value when the global existence problem for arbitrary big data is considered.

Theorem 4.1 (global solutions). *Let* $0 < q \leq 2$ *in* Eq. (1.4) *and* $\varphi(x) \in C^3$, $\psi(x) \in C^2$ *are arbitrary functions. Then the classical solution* $u(x,t)$ *of the problem* (1.4), (1.5) *is global.*

Proof. By the local theory the classical solution of (1.4), (1.5) exists in a set $G \subset R^3 \times [0, +\infty)$. Let $T > 0$ be arbitrary big. Choose an arbitrary ball $B \subset R^3$ with a radius $R \geq T$ and let K_T^- be the backward light cone (see Section 2) based on the compact $K = B$, $K_T^- \subset \{0 \leq t \leq T\}$. Let

$$a = \max|\varphi(x)|, \quad b = \max\left(|\nabla\varphi(x)| + |\psi(x)|\right), \ x \in B.$$

Then for the free wave $u^0(x,t)$ (see formula (2.4)) we have

$$|u^0(x,t)| \leq a + bt, \quad (x,t) \in C_B^-,$$

where C_B^- is the backward light cone based on B and $K_T^- = C_B^- \cap \{0 \leq t \leq T\}$ for $K = B$. Moreover, the solution $v(t)$ of the problem (2.9) is global in the case of $f(v) = v\ln^q(1+v^2)$, $q \in (0,2]$, according to Lemma 3.4, and $v(t)$ is defined, in particular, for $t \in [0,T]$. Now the continuation principle (Theorem 2.2) assures that the solution $u(x,t)$ is indeed defined in K_T^-. This proves the theorem. □

Theorem 4.2 (exponential counter-decay). *Let* $0 < q < 2$ *and let the classical solution* $u(x,t)$ *of the problem* (1.4), (1.5) *be a strongly space-destinated wave, with either non-positive or non-negative initial data, satisfying the inequality*

$$\inf_x (1+|x|)\left(|\psi(x)| - |\nabla\varphi(x)|\right) > 0, \quad x \in R^3. \tag{4.1}$$

Then for each compact $K \subset R^3$ *there exist positive constants* c',h', $t' = t'(K)$ *such that*

$$|u(x,t)| \geq (4/c')\exp\left(\frac{2-q}{16}h't\right)^{2/(2-q)} \quad (0 < q < 2) \tag{4.2}$$

on $K^+_\infty \cap \{t \geq t'\}$, where $K^+_\infty = \cup K^+_T$, $T \in (0, +\infty)$, is the forward light cone issued from K. If, moreover, we have

$$\sup_x \left(|\varphi(x)| + |\nabla\varphi(x)| + |\psi(x)|\right) < +\infty, \; x \in R^3, \tag{4.3}$$

then the solution $u(x,t)$ satisfies, besides (4.2), the estimate

$$|u(x,t)| \leq \exp\left((4 + c'')t\right)^{2/(2-q)}, \forall t \geq t'', \tag{4.4}$$

as well, with certain positive constants c'' and $t'' = t''(c_0, c'')$.

Proof. For a given compact $K \subset R^3$ let us fix a ball $B_0 \subset R^3$ with a radius r_0 such that $K \subset B_0$. Translate next the origin in R^3 at the center of B_0. Obviously, it suffices to consider only the case of non-negative initial data, due to Proposition 2.2. Now (4.1) shows that the constant c_1, defined as

$$c_1 = \inf_x (1 + |x|)\left(\psi(x) - |\nabla\varphi(x)|\right), \tag{4.5}$$

is positive.

Let us take an arbitrary $T \geq 1 + r_0$ and an arbitrary point $(x_T, T) \in K^+_\infty$. Denote by $C^-(x_T)$ the backward light cone with the top at (x_T, T). From the Kirchhoff formula we obtain the following inequality for the free wave $u^0(x,t)$:

$$u^0(x,t) \geq \frac{1}{4\pi t}\int_{|x-y|=t} \left(\psi(y) - |\nabla\varphi(y)|\right) \, ds_y. \tag{4.6}$$

When (x,t) varies in the cone $C^-(x_T)$, we have obviously

$$|y| \leq r_0 + 2T, \quad y \in R^3 : \; |x - y| = t,$$

and from (4.5) we find

$$\psi(y) - |\nabla\varphi(y)| \geq \frac{c_1}{1 + r_0 + 2T} \geq \frac{c_1}{1 + 3T} \geq \frac{c'}{1+T}$$

with $c' = c_1/3$. Then (4.6) yields

$$u^0(x,t) \geq \frac{c'}{1+T}t, \quad (x,t) \in C^-(x_T).$$

Now we can apply the comparison principle (Theorem 2.1) to find

$$u(x,t) \geq v_T(t), \quad (x,t) \in C^-(x_T),$$

where $v_T(t)$ is the solution of the problem

$$\ddot{v} = v \ln^q(1 + v^2), \quad v(0) = 0, \; \dot{v}(0) = \frac{c'}{1+T}\cdot$$

Afterwards it remains to apply Lemma 3.5 and to set $(x,t) = (x_T, T)$, then

$$u(x_T, T) \geq v_T(T) \geq (4/c') \exp(\frac{2-q}{16} h'T)^{2/(2-q)} \quad (0 < q < 2).$$

This proves the estimate (4.2). When (4.3) holds, we set

$$c_0 = \sup_x |\varphi(x)|, \quad c'' = \sup_x \left(|\nabla\varphi(x)| + |\psi(x)|\right),$$

and using the Kirchhoff formula we get

$$|u^0(x,t)| \leq c_0 + c''t, \quad (x,t) \in R^3 \times [0, +\infty).$$

Then, in virtue of Theorem 2.1, we conclude that

$$|u(x,t)| \leq U(t), \quad (x,t) \in R^3 \times [0, +\infty),$$

with $U(t)$ solving the equation

$$\ddot{U} = U \ln^q(1 + U^2), \quad U(0) = c_0, \ \dot{U}(0) = c''.$$

It is clear now that the estimate (4.4) follows from Lemma 3.4. □

In the next theorem we shall deal with the classical solutions $u(x,t)$ called space-destinated waves on a given compact $K \subset R^3$, which satisfy the requirements of Definition 2.1 for each $x \in K$. The notations

$$m_0 = \min_x |u(x,0)|, \ x \in K, \tag{4.7}$$

$$m_1 = \min_x \left(|u_t(x,0)| - |\nabla_x u(x,0)|\right), \ x \in K, \tag{4.8}$$

$$\tau_0 = \lim_{\varepsilon \to 0} \min\left(\frac{\sqrt{3}}{\varepsilon^2 + m_1}, 2\sqrt{\frac{3 + m_0}{\varepsilon^2 + m_0 \ln^q(1 + m_0^2)}}\right), \ m_0 + m_1 \neq 0, \tag{4.9}$$

$$M_0 = \max_x |u(x,0)|, x \in K; \tag{4.10}$$

$$M_1 = \max_x \left(|u_t(x,0)| + |\nabla_x u(x,0)|\right), \ x \in K, \tag{4.11}$$

shall be used for a given space-destinated wave $u(x,t)$ on K. By $T_q = T_q(u, K)$ we shall denote the supremum of all $T > 0$ such that a given classical solution u of (1.4) exists in the light cone K_T^-; T_q is usually called the life-span of u for the compact K.

Theorem 4.3 (blow-up of the solution). *Suppose $q > 2$ and $\varphi \in C^3$, $\psi \in C^2$ are arbitrary initial data, either non-negative or non-positive on a ball $B_r \subset R^3$, such that the solution $u(x,t)$ of (1.4), (1.5) is a space-destinated wave on B_r (r is the radius of the ball). If the numbers $m_j(r)$, $j = 0, 1$, and $\tau_0(r)$ satisfy the inequalities*

$$m_0(r) + m_1(r) > 0, \quad \sqrt{1+q}\left(\tau_0(r) + \frac{2^{2-q/2}}{q-2}\right) \leq r,$$

then the solution $u(x,t)$ blows up in a finite time and the life-span $T_q(r) = T_q(u, B_r)$ satisfies the estimates

$$\frac{2^{-q/2}\left(1+M_0(r)\right)^{1-q/2}}{(q-2)\left(1+M_1(r)\right)} \le T_q(r) \le \sqrt{1+q}\left(\tau_0(r) + \frac{2^{2-q/2}}{q-2}\right),$$

where $m_j(r)$, $M_j(r)$, $j = 0, 1$, and $\tau_0(r)$ are the constants from (4.7)–(4.11) with $K = B_r$.

Proof. By the Kirchhoff formula we obtain the estimates

$$m_0(r) + tm_1(r) \le |u^0(x,t)| \le M_0(r) + tM_1(r)$$

for $u^0(x,t)$ on the backward light cone $C^-(r) = K_T^-$, where $K = B_r, T = r$. In addition, $u^0(x,t)$ is either positive or negative on $C^-(r) \cap \{t > 0\}$. Then the comparison principle yields

$$v(t) \le |u(x,t)|, \quad (x,t) \in C^-(r) : 0 \le t < \min\left(r, T_v^0, T_q(r)\right), \tag{4.12}$$

$$|u(x,t)| \le U(t), \quad (x,t) \in C^-(r) : 0 \le t < \min\left(r, T_q(r), T_U^0\right), \tag{4.13}$$

where $v(t)$ and $U(t)$ satisfy the equation $\ddot{z} = z\ln^q(1+z^2)$ with the initial data (m_0, m_1) and (M_0, M_1) and the life-spans T_v^0, T_U^0, respectively. Next, from Lemma 3.6 we see that

$$T_U^0 \ge \frac{2^{-q/2}(1+M_0)^{1-q/2}}{(q-2)(1+M_1)}, \quad T_v^0 \le \sqrt{1+q}(\tau_0 + \frac{2^{2-q/2}}{q-2}). \tag{4.14}$$

Then the estimates (4.12)–(4.14), together with the inequalities

$$T_U^0 \le T_q(r) \le T_v^0,$$

prove the theorem. □

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