NORMALLY GENERATED SUBSPACES OF LOGARITHMIC CANONICAL SECTIONS

BORIS KOTZEV, AZNIV KASPARIAN

The logarithmic-canonical bundle $\Omega^2_{A'}(T')$ of a smooth toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a ball quotient $\mathbb{B}/\Gamma$ is known to be sufficiently ample over the Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$. The present work develops criteria for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated over $\widehat{A}$, i.e., to determine a regular immersive projective morphism of $\widehat{A}$ with normal image. These are applied to a specific example $A'_1 = (\mathbb{B}/\Gamma_1)'$ over the Gauss numbers. The first section organizes some preliminaries. The second one provides two sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$.

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1. PRELIMINARIES

Throughout, let $\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 < 1\} = SU_{2,1}/S(U_2 \times U_1)$ be the complex two dimensional ball and $\Gamma \subset SU_{2,1}$ be a lattice, acting freely on $\mathbb{B}$. The compact $\mathbb{B}/\Gamma$ are of general type. The non-compact $\mathbb{B}/\Gamma$ admit smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ by a disjoint union $T' = \cup_{i=1}^h T'_i$ of smooth irreducible elliptic curves T'_i. From now on, we concentrate on $A' = (\mathbb{B}/\Gamma)'$ with abelian minimal model A. In such a case, the lattice Γ, the ball quotient $\mathbb{B}/\Gamma$ and its compactifications are said to be co-abelian.

The contraction $\xi : A' \to A$ of the rational (-1)-curves on A' restricts to a biregular morphism $\xi : T_i' \to \xi(T_i') = T_i$, as far as an abelian surface A does not support rational curves. In such a way, ξ produces the multi-elliptic divisor $T = \xi(T') = \sum\limits_{i=1}^{h} T_i \subset A$, i.e., a divisor with smooth elliptic irreducible components T_i. According to Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$, any irreducible component of the exceptional divisor of ξ intersects T' in at least two points. Therefore $\xi : A' \to A$ is the blow-up of A at the singular locus $T^{\text{sing}} = \sum\limits_{1 \le i < j \le h} T_i \cap T_j$ of T. Holzapfel has shown in [5] that the blow-up A' of an abelian surface A at the singular locus $T^{\text{sing}} = \sum\limits_{1 \le i < j \le h} T_i \cap T_j$ of a multi-elliptic divisor $T = \sum\limits_{i=1}^{h} T_i$ is the toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a smooth ball quotient $\mathbb{B}/\Gamma$ if and only if $A = E \times E$ is the Cartesian square of an elliptic curve E and

$$\sum_{i=1}^{h} \text{card}(T_i \cap T^{\text{sing}}) = 4\text{card}(T^{\text{sing}}). \tag{1.1}$$

In order to describe the smooth irreducible elliptic curves T_i on A and their intersections, let us note that the inclusions $T_i \subset A = E \times E$ are morphisms of abelian varieties. Consequently, they lift to affine linear maps of the corresponding universal covers and

$$T_i = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, a_i u + b_i v + c_i \in \pi_1(E)\}$$

for some $a_i, b_i, c_i \in \mathbb{C}$. The fundamental group

$$\pi_1(T_i) = \{t \in \mathbb{C} \,|\, b_i t + \pi_1(E) = -a_i t + \pi_1(E) = \pi_1(E)\} = a_i^{-1}\pi_1(E) \cap b_i^{-1}\pi_1(E).$$

If Γ is an arithmetic lattice then the elliptic curve E has complex multiplication by an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$. As a result, Γ is commensurable with the full Picard modular group $SU_{2,1}(\mathcal{O}_{-d})$ over the integers ring $\mathcal{O}_{-d}$ of $\mathbb{Q}(\sqrt{-d})$. Such Γ are called Picard modular groups. Moreover, all T_i are defined over K. For simplicity, we assume that $\pi_1(E) = \mathcal{O}_{-d}$, in order to have maximal endomorphism ring $End(E) = \mathcal{O}_{-d}$. Since $K = \mathbb{Q}(\sqrt{-d})$ is the fraction field of $\mathcal{O}_{-d}$, one can choose $a_i, b_i \in \mathcal{O}_{-d}$. Thus, $\pi_1(T_i) \supseteq \mathcal{O}_{-d}$, $a_i\pi_1(E) + b_i\pi_1(E) \subseteq \mathcal{O}_{-d}$ and T_i has minimal fundamental group $\pi_1(T_i) = \mathcal{O}_{-d}$ exactly when $a_i\pi_1(E) + b_i\pi_1(E) = \pi_1(E) = \mathcal{O}_{-d}$. In particular, if K is of class number 1, then all the smooth elliptic curves $T_i \subset A = \mathbb{C}^2/(\mathcal{O}_{-d} \times \mathcal{O}_{-d})$, defined over $K = \mathbb{Q}(\sqrt{-d})$, have minimal fundamental groups $\pi_1(T_i) = \mathcal{O}_{-d}$. From now on, we do not restrict the class number of $K = \mathbb{Q}(\sqrt{-d})$, but confine only to smooth irreducible elliptic curves T_i with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$. If $b_i \ne 0$, then

$$T_i^{(1)} = \{(b_i t + \pi_1(E), -a_i t - b_i^{-1}c_i + \pi_1(E)) \,|\, t \in \mathbb{C}\} \subseteq T_i.$$

Moreover, the complete pre-image of $T_i^{(1)}$ in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A is $\pi_1(T_i)$-invariant family of complex lines. Therefore, $T_i^{(1)}$ is an elliptic curve and coincides with T_i.

The notations from the next lemma will be used throughout:

Lemma 1. *Let* $T_s = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s \in \mathcal{O}_d\}$ *and* $D_s = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s + \mu_s \in \mathcal{O}_{-d}\}$ *for* $1 \le s \le 3$ *be elliptic curves with minimal fundamental groups* $\pi_1(T_s) = \pi_1(D_s) = \mathcal{O}_{-d}$ *on* $A = (\mathbb{C}/\mathcal{O}_{-d}) \times (\mathbb{C}/\mathcal{O}_{-d})$ *and*

$$\Delta_{ij} := \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}, \quad \Delta := \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Then for any even permutation $\{i, j, l\}$ *of* $\{1, 2, 3\}$ *there hold the following:*

(i) *the intersection number is* $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$, *where* $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} : \mathbb{Q}(\sqrt{-d}) \to \mathbb{Q}$ *stands for the norm;*

(ii) $T_i \cap T_j \subset D_l$ *if and only if* $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ *and both* $\Delta_{ij}^{-1}\Delta_{jl}$ *and* $\Delta_{ij}^{-1}\Delta_{li}$ *belong to* $\operatorname{End}(E) = \mathcal{O}_{-d}$;

(iii) $T_1 \cap T_2 \cap T_3 = \emptyset$ *if and only if* $\Delta \notin \Delta_{12}\mathcal{O}_{-d} + \Delta_{23}\mathcal{O}_{-d} + \Delta_{31}\mathcal{O}_{-d}$.

Proof. (i) If $T_i \cap T_j = \emptyset$, then the liftings of T_i, T_j to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are discrete families of mutually parallel lines. In such a case, we say briefly that T_i and T_j are parallel. That allows to choose $a_j = a_i$, $b_j = b_i$ and to calculate $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(0) = 0 = T_i.T_j$. When $T_i \cap T_j \ne \emptyset$, one can move the origin $\breve{o}_A = (\breve{o}_E, \breve{o}_E) \in A$ in $T_i \cap T_j$ and represent

$$T_i = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, t \in \mathbb{C}\}, \quad T_j = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_j u + b_j v \in \mathcal{O}_{-d}\}.$$

Then the intersection is

$$T_i \cap T_j = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, \Delta_{ij} t \in \mathcal{O}_{-d} \subset \mathbb{C}\} \simeq$$

$$(\Delta_{ij}^{-1}\mathcal{O}_{-d})/(b_i^{-1}\mathcal{O}_{-d} \cap a_i^{-1}\mathcal{O}_{-d}) = (\Delta_{ij}^{-1}\mathcal{O}_{-d})/\mathcal{O}_{-d} \simeq \mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}.$$

For an arbitrary lattice $\Lambda \subset \mathbb{C}$, let us denote by $\mathcal{F}(\Lambda)$ a Λ-fundamental domain on $\mathbb{C}$. As far as $\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})$ is the $\mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}$-orbit of $\mathcal{F}(\mathcal{O}_{-d})$, the index equals

$$[\mathcal{O}_{-d} : \Delta_{ij}\mathcal{O}_{-d}] = \frac{\operatorname{vol}\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})}{\operatorname{vol}\mathcal{F}(\mathcal{O}_{-d})} = \frac{\operatorname{vol}\mathcal{F}(|\Delta_{ij}|\mathcal{O}_{-d})}{\operatorname{vol}\mathcal{F}(\mathcal{O}_{-d})} = |\Delta_{ij}|^2 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}).$$

(ii) The intersection $T_i \cap T_j$ consists of the $\pi_1(A)$-equivalence classes of the solutions $(u, v) \in \mathbb{C}^2$ of

$$\left| \begin{array}{l} a_i u + b_i v = \lambda_1 - c_i \\ a_j u + b_j v = \lambda_2 - c_j \end{array} \right.$$

for arbitrary $\lambda_1, \lambda_2 \in \pi_1(E) = \mathcal{O}_{-d}$. A point

$$(\Delta_{ij}^{-1}(b_i c_j - b_j c_i) + \Delta_{ij}^{-1}(b_j \lambda_1 - b_i \lambda_2), \; \Delta_{ij}^{-1}(a_j c_i - a_i c_j) + \Delta_{ij}^{-1}(a_i \lambda_2 - a_j \lambda_1))$$

belongs to the lifting of D_l if and only if

$$\begin{aligned}
&-\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}(c_i\Delta_{jl} + c_j\Delta_{li}) + c_l + \mu_l \\
&= -\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d}
\end{aligned}$$

for $\forall \lambda_1, \lambda_2 \in \pi_1(E)$. That, in turn, is equivalent to $\Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d}$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in End(E) = \mathcal{O}_{-d}$.

(iii) For arbitrary $\lambda_1, \lambda_2, \lambda_3 \in \pi_1(E) = \mathcal{O}_{-d}$, the linear system

$$\left| \begin{aligned}
a_1 u + b_1 v &= \lambda_1 - c_1 \\
a_2 u + b_2 v &= \lambda_2 - c_2 \\
a_3 u + b_3 v &= \lambda_3 - c_3
\end{aligned} \right.$$

has no solutions exactly when

$$\det \begin{pmatrix} a_1 & b_1 & \lambda_1 - c_1 \\ a_2 & b_2 & \lambda_2 - c_2 \\ a_3 & b_3 & \lambda_3 - c_3 \end{pmatrix} = \Delta_{23}\lambda_1 + \Delta_{31}\lambda_2 + \Delta_{12}\lambda_3 - \Delta \neq 0.$$

Lemma 1 is proved. $\qquad\square$

The non-arithmetic lattices $\Gamma \subset SU_{2,1}$ correspond to abelian surfaces $A = E \times E$, whose elliptic factors E have minimal endomorphism rings $End(E) = \mathbb{Z}$. Then the liftings of the elliptic curves $T_i \subset A$ with $\pi_1(T_i) = \pi_1(E)$ to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are given by $a_i u + b_i v + c_i \in \pi_1(E)$ with $a_i, b_i \in \mathbb{Z}$. As a result, the intersection numbers $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$ are comparatively large and there are very few chances for construction of a multi-elliptic divisor $T = \sum\limits_{i=1}^{h} \subset A$, subject to (1.1). This is a sort of a motivation for restricting our attention to the arithmetic case.

The smooth irreducible elliptic curves $T_i' \subset A'$ contract to the Γ-orbits $\kappa_i = \Gamma(p) \in \partial_\Gamma \mathbb{B}/\Gamma$ of the Γ-rational boundary points $p \in \partial_\Gamma \mathbb{B}$. These κ_i are called cusps. The resulting Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma} = (\mathbb{B}/\Gamma) \cup (\partial_\Gamma \mathbb{B}/\Gamma)$ is a normal projective surface.

Definition 2. *Let Γ be a Picard modular group, $\gamma \in \Gamma$ and $Jac(\gamma) = \frac{\partial(\gamma_1, \gamma_2)}{\partial(z_1, z_2)}$ be the Jacobian matrix of $\gamma = (\gamma_1, \gamma_2) : \mathbb{B} \to \mathbb{B} \subset \mathbb{C}^2$. The global holomorphic functions $\delta : \mathbb{B} \to \mathbb{C}$ with transformation law*

$$\gamma^*(\delta)(z) = \delta\gamma(z) = [\det Jac(\gamma)]^{-n} \delta(z) \quad \text{for } \forall \gamma \in \Gamma, \; \forall z \in \mathbb{B}$$

are called Γ*-modular forms of weight* n.

The Γ*-modular forms of weight* n *constitute a* $\mathbb{C}$*-linear space, which is denoted by* $[\Gamma, n]$.

Definition 3. *A* Γ*-modular form* $\delta \in [\Gamma, n]$ *is cuspidal if* $\delta(\kappa_i) = 0$ *at all the cusps* $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$.

The cuspidal Γ-modular forms of weight n form the subspace $[\Gamma, n]_{\mathrm{cusp}}$ of $[\Gamma, n]$.

For any natural number n there is a $\mathbb{C}$-linear embedding

$$ j_n : H^0(\mathbb{B}, \mathcal{O}_\mathbb{B}) \longrightarrow H^0\left(\mathbb{B}, \left(\Omega_\mathbb{B}^2\right)^{\otimes n}\right) $$

$$ j_n(\delta)(z) = \delta(z)(dz_1 \wedge dz_2)^{\otimes n} $$

of the global holomorphic functions on the ball in the global holomorphic sections of the n-th pluri-canonical bundle $\left(\Omega_\mathbb{B}^2\right)^{\otimes n}$. It restricts to an isomorphism

$$ j_n : [\Gamma, n] \longrightarrow H^0\left(\mathbb{B}, \left(\Omega_\mathbb{B}^2\right)^{\otimes n}\right)^\Gamma $$

of the Γ-modular forms of weight n with the Γ-invariant holomorphic sections of $\left(\Omega_\mathbb{B}^2\right)^{\otimes n}$. Note that the subspace $H^0\left(\mathbb{B}, \left(\Omega_\mathbb{B}^2\right)^{\otimes n}\right)^\Gamma$ of $H^0\left(\mathbb{B}/\Gamma, \left(\Omega_{\mathbb{B}/\Gamma}^2\right)^{\otimes n}\right)$ acts on $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$, extending over the cusps $\partial_\Gamma \mathbb{B}/\Gamma$ of codimension 2 in $\widehat{A}$.

The tensor product $\Omega_{A'}^2(T') = \Omega_{A'}^2 \otimes_\mathbb{C} \mathcal{O}_{A'}(T')$ is called logarithmic canonical bundle of A', while $\Omega_{A'}^2(T')^{\otimes n}$ are referred to as logarithmic pluri-canonical bundles. Hemperly has observed in [3] that

$$ j_n[\Gamma, n] = H^0\left(\mathbb{B}, \left(\Omega_\mathbb{B}^2\right)^{\otimes n}\right)^\Gamma = H^0\left(A', \Omega_{A'}^2(T')^{\otimes n}\right) $$

as long as the holomorphic sections from these spaces have one and the same coordinate transformation law. A classical result of Baily-Borel establishes that $\Omega_{A'}^2(T')$ is sufficiently ample on $\widehat{A}$. The present article provides sufficient conditions for the ampleness of $\Omega_{A'}^2(T')$ on $\widehat{A}$.

Note that the canonical bundle

$$ K_{A'} = \xi^* K_A + \mathcal{O}_{A'}(L) = \xi^* \mathcal{O}_A + \mathcal{O}_{A'}(L) = \mathcal{O}_{A'}(L) $$

is associated with the exceptional divisor $L = \xi^{-1}(T^{\mathrm{sing}})$ of $\xi : A' \to A$. If s is a global meromorphic section of $\Omega_{A'}^2$ and t is a global meromorphic section of $\mathcal{O}_{A'}(T')$, then the tensoring

$$ (s \otimes_\mathbb{C} t)^{\otimes(-n)} : H^0\left(A', \Omega_{A'}^2(T')^{\otimes n}\right) \longrightarrow \mathcal{L}_{A'}(n(L + T')) $$

is a $\mathbb{C}$-linear isomorphism with

$$ \mathcal{L}_{A'}(n(L + T')) = \{ f \in \mathfrak{Mer}(A') \,|\, (f) + n(L + T') \geq 0 \}. $$

The isomorphism $\xi^* : \mathfrak{Mer}(A) \to \mathfrak{Mer}(A')$ of the meromorphic function fields induces a linear isomorphism

$$(\xi^*)^{-1} : \mathcal{L}_{A'}(n(L+T')) \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}}),$$

where $m_p : \mathrm{Div}(A) \to \mathbb{Z}$ stands for the multiplicity at a point $p \in A$ and

$$\mathcal{L}_A(nT, nT^{\mathrm{sing}}) = \{f \in \mathfrak{Mer}(A) \,|\, (f) + nT \geq 0,\, m_p(f) + n \geq 0 \text{ for } \forall p \in T^{\mathrm{sing}}\}.$$

The linear isomorphisms

$$\tau_n := (\xi^*)^{-1}(s \otimes_{\mathbb{C}} t)^{\otimes(-n)} : j_n[\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})$$

are called transfers of modular forms of weight n to abelian functions.

For any $\delta \in [\Gamma, 1]$, note that $\delta(\kappa_i) \neq 0$ if and only if $T_i \subset (\tau_1 j_1(\delta))_\infty$. Observe also that $\tau_1 j_1[\Gamma, 1]_{\mathrm{cusp}} = \{f \in \mathcal{L}_A(T, T^{\mathrm{sing}}) \,|\, (f)_\infty = \emptyset\} = \mathbb{C}$ and fix the cuspidal form $\eta_o = (\tau_1 j_1)^{-1}(1)$ of weight 1.

Towards the construction of abelian functions $f \in \mathcal{L}_A(T, T^{\mathrm{sing}})$, let us recall from [7] that any elliptic function $g : E \to \mathbb{P}^1$ can be represented as

$$g(z) = C_o \prod_{i=1}^{k} \frac{\sigma(z - \alpha_i)}{\sigma(z - \beta_i)}, \tag{1.2}$$

where

$$\sigma(z) = z \prod_{\lambda \in \pi_1(E) \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right)^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$$

is the Weierstrass σ-function, $\alpha_i, \beta_i, C_o \in \mathbb{C}$ and $\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i (\mathrm{mod}\, \pi_1(E))$. The points of $E = \mathbb{C}/\pi_1(E)$ are of the form $\overline{a} = a + \pi_1(E)$ for some $a \in \mathbb{C}$. The elliptic function (1.2) takes all the values from $\mathbb{P}^1$ with one and a same multiplicity k. Moreover, if $g^{-1}(x) = \{\overline{p_i(x)} \in E \,|\, 1 \leq i \leq k\}$ for some $x \in \mathbb{C} \subset \mathbb{P}^1$, then $\sum_{i=1}^{k} \overline{p_i(x)} = \sum_{i=1}^{k} \overline{\beta_i}$. Observe that $\sigma : \mathbb{C} \to \mathbb{C}$ is a non-periodic entire function, but its divisor $(\sigma)_{\mathbb{C}} = \pi_1(E)$ on $\mathbb{C}$ is $\pi_1(E)$-invariant. That enables to define the divisor $(\sigma)_E = \breve{o}_E$ of σ on E. In global holomorphic coordinates $(u, v) \in \mathbb{C}^2$, the divisor

$$(\sigma(a_i u + b_i v + c_i))_{\mathbb{C}^2} = \{(u, v) \in \mathbb{C}^2 \,|\, a_i u + b_i v + c_i \in \pi_1(E) = \mathcal{O}_{-d}\}$$

is the complete pre-image of T_i in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A. That allows to define the divisor

$$(\sigma(a_i u + b_i v + c_i)) = (\sigma(a_i u + b_i v + c_i))_A = T_i.$$

Let $f \in \mathcal{L}_A(T)$ be an abelian function with pole divisor $(f)_\infty = \sum\limits_{i=1}^{k} T_i$, after an eventual permutation of the irreducible components of T. Then

$$f_\infty := \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i) \quad \text{and} \quad f_0 := f f_\infty \qquad (1.3)$$

are (non-periodic) entire functions on $\mathbb{C}^2$. Let $\zeta = \frac{\sigma'}{\sigma}$ be Weierstrass' ζ-function, $\eta : \pi_1(E) \to \mathbb{C}$ be the $\mathbb{Z}$-linear homomorphism, satisfying $\zeta(z + \lambda) = \zeta(z) + \eta(\lambda)$ for all $z \in \mathbb{C}$, $\lambda \in \pi_1(E)$ and

$$\varepsilon(\lambda) = \begin{cases} 1 & \text{for } \lambda \in 2\pi_1(E), \\ -1 & \text{for } \lambda \in \pi_1(E) \setminus 2\pi_1(E). \end{cases}$$

Recall from [6] the $\pi_1(E)$-transformation law

$$\frac{\sigma(z + \lambda)}{\sigma(z)} = \varepsilon(\lambda) e^{\eta(\lambda)\left(z + \frac{\lambda}{2}\right)} \quad \text{for} \quad \forall \lambda \in \pi_1(E), \forall z \in \mathbb{C}.$$

Under the assumption (1.3), the $\pi_1(A)$-periodicity of f is equivalent to

$$\frac{f_0(u + \lambda, v)}{f_0(u, v)} = \frac{f_\infty(u + \lambda, v)}{f_\infty(u, v)} = \prod_{i=1}^{k} \varepsilon(a_i \lambda) e^{\eta(a_i \lambda)\left(a_i u + b_i v + c_i + \frac{a_i \lambda}{2}\right)}$$

and

$$\frac{f_0(u, v + \lambda)}{f_0(u, v)} = \frac{f_\infty(u, v + \lambda)}{f_\infty(u, v)} = \prod_{i=1}^{k} \varepsilon(b_i \lambda) e^{\eta(b_i \lambda)\left(a_i u + b_i v + c_i + \frac{b_i \lambda}{2}\right)}$$

for $\forall \lambda \in \pi_1(E) = \mathcal{O}_{-d}$, $\forall (u, v) \in \mathbb{C}^2$. We choose

$$f_0(u, v) = \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i + \mu_i)$$

and reduce the $\pi_1(A)$-periodicity of f to

$$1 = \frac{f(u + \lambda, v)}{f(u, v)} = e^{\sum\limits_{i=1}^{k} \eta(a_i \lambda) \mu_i}, \quad 1 = \frac{f(u, v + \lambda)}{f(u, v)} = e^{\sum\limits_{i=1}^{k} \eta(b_i \lambda) \mu_i} \quad \forall \lambda \in \mathcal{O}_{-d}, \forall (u, v) \in \mathbb{C}^2.$$

Let us mention that Holzapfel has studied $f \in \mathcal{L}_A(T)$ of the above form with at most three non-parallel irreducible components of $(f)_\infty$, intersecting pairwise in single points. The next lemma provides a sufficient (but not necessary) condition for $\pi_1(A)$-periodicity of a σ-quotient, whose pole divisor has an arbitrary number of irreducible components with arbitrary intersection numbers.

Lemma 4. *If*

$$\sum_{i=1}^{k} a_i \mu_i = \sum_{i=1}^{k} \overline{a_i} \mu_i = \sum_{i=1}^{k} b_i \mu_i = \sum_{i=1}^{k} \overline{b_i} \mu_i = 0 \,, \qquad (1.4)$$

then the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_i u + b_i v + c_i + \mu_i)}{\sigma(a_i u + b_i v + c_i)} \qquad (1.5)$$

is $\mathcal{O}_{-d} \times \mathcal{O}_{-d}$ -periodic.

Proof. Let us recall from [1] that the integers ring of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ is of the form $\mathcal{O}_{-d} = \mathbb{Z} + 2\omega\mathbb{Z}$ for

$$2\omega = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1 (\mathrm{mod}\, 4), \\ \frac{-1+\sqrt{-d}}{2} & \text{for } -d \equiv 1 (\mathrm{mod}\, 4). \end{cases}$$

Any $\nu \in \mathcal{O}_{-d}$ has unique representation $\nu = x + 2\omega y$ with

$$x = \frac{2\omega\overline{\nu} - 2\overline{\omega}\nu}{2\omega - 2\overline{\omega}} \in \mathbb{Z}, \quad y = \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} \in \mathbb{Z}.$$

Legendre's equality

$$\eta(2\omega) - 2\omega\eta(1) = 2\pi\sqrt{-1},$$

(cf.[6]) implies that

$$\eta(\nu) = \nu\eta(1) + \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} 2\pi\sqrt{-1} \quad \text{for } \forall \nu \in \mathcal{O}_{-d}.$$

As a result,

$$\sum_{i=1}^{k} \eta(a_i\lambda)\mu_i = \left(\sum_{i=1}^{k} a_i\mu_i\right)\lambda\eta(1) + \left(\sum_{i=1}^{k} a_i\mu_i\right)\frac{2\pi\sqrt{-1}\lambda}{2\omega - 2\overline{\omega}} - \left(\sum_{i=1}^{k} \overline{a_i}\mu_i\right)\frac{\overline{\lambda}2\pi\sqrt{-1}}{2\omega - 2\overline{\omega}}.$$

Lemma 4 is proved $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Mutually parallel smooth elliptic curves $T_1, \ldots, T_k$ admit liftings

$$T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_1 u + b_1 v + c_i \in \mathcal{O}_{-d}\}.$$

For arbitrary $\mu_j \in \mathbb{C}$ with $\sum_{i=1}^{k} \mu_i = 0$, the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_1 u + b_1 v + c_i + \mu_i)}{\sigma(a_1 u + b_1 v + c_i)} \qquad (1.6)$$

belongs to $\mathcal{L}_A(T, T^{\text{sing}})$ and has smooth pole divisor $(f)_\infty = \sum_{i=1}^{k} T_i$. Following [4], we say that (1.6) is a k-fold parallel σ-quotient. A σ-quotient (1.5) has smooth pole divisor if and only if it is k-fold parallel.

Definition 5. *A special σ-quotient of order k is a function of the form (1.5), which is subject to (1.4), has singular pole divisor $(f)_\infty$ and $\mu_i \notin \mathcal{O}_{-d}$ for all $1 \leq i \leq k$.*

Lemma 6. *If $f \in \mathcal{L}_A(T, T^{\text{sing}})$ is a special σ-quotient of order $k \geq 2$, then at any point $p \in (f)_\infty^{\text{sing}}$ the multiplicity $m_p(f)_\infty$ satisfies*

$$2 \leq m_p(f)_\infty \leq \left[\frac{k+1}{2}\right],$$

where $\left[\frac{k+1}{2}\right]$ is the greatest natural number, non-exceeding $\frac{k+1}{2}$.

In particular, $\mathcal{L}_A(T, T^{\text{isng}})$ does not contain a special σ-quotient of order 2.

Proof. The smoothness of the irreducible components $T_1, \ldots, T_k$ of $(f)_\infty$ results in $(f)_\infty^{\text{sing}} \subset \sum_{1 \leq i < j \leq k} (T_i \cap T_j)$ and implies that $m_p(f)_\infty \geq 2$ for all $p \in (f)_\infty^{\text{sing}}$. Suppose that $m_p(f)_\infty = m$ for some $2 \leq m \leq k$. After an eventual permutation of $T_1, \ldots, T_k$, one can assume that $p \in T_1 \cap \ldots \cap T_m$ and $p \notin T_{m+1} + \ldots + T_k$. Then

$$m_p(f) + 1 = m_p(f)_0 - m_p(f)_\infty + 1 = m_p(f)_0 - m + 1 \geq 0$$

requires the existence of $D_{m+1}, \ldots, D_{2m-1} \subset (f)_0 = \sum_{i=1}^{k} D_i$ with $p \in D_{m+1} \cap \ldots \cap D_{2m-1}$, after a further permutation of $D_{m+1}, \ldots, D_k$. Now $2m - 1 \leq k$ implies that $m_p(f)_\infty = m \leq \left[\frac{k+1}{2}\right]$.

In particular, for $k = 2$ the inequality $2 \leq m_p(f)_\infty \leq \left[\frac{3}{2}\right]$ cannot be satisfied. $\square$

Proposition 7. *If*

$$f(u, v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i + \mu_i)}{\sigma(a_i u + b_i v + c_i)} \tag{1.7}$$

is a special σ-quotient from $\mathcal{L}_A(T, T^{\text{sing}})$, then $T_1 \cap T_2 \cap T_3 = \emptyset$ and the intersection numbers $T_1.T_2 = T_2.T_3 = T_3.T_1 \in \mathbb{N}$ are equal.

Proof. By Lemma 6 there follows $m_p(f)_\infty = 2$ for $\forall p \in (f)_\infty^{\text{sing}}$. In particular, $(f)_\infty = T_1 + T_2 + T_3$ has no triple point and $T_1 \cap T_2 \cap T_3 = \emptyset$. Further, for any $p \in T_i \cap T_j$ the condition $m_p(f) + 1 \geq 0$ requires that $p \in D_l$, therefore $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}$, according to Lemma 1 (ii). A cyclic change of the even permutation $\{i, j, l\}$ by $\{j, l, i\}$ and $\{l, i, j\}$ results in $\Delta_{jl}^{-1}\Delta_{li}, \Delta_{jl}^{-1}\Delta_{ij} \in \mathcal{O}_{-d}$ and, respectively, $\Delta_{li}^{-1}\Delta_{ij}, \Delta_{li}^{-1}\Delta_{jl} \in \mathcal{O}_{-d}$. Consequently, $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}^*$, whereas $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{jl}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{li})$. Now, by Lemma 1 (i) it follows that $T_i.T_j = T_j.T_l = T_l.T_i$. $\square$

Definition 8. *The divisor $T_1 + T_2 + T_3$ with three smooth elliptic irreducible components is called a triangle if $T_1 \cap T_2 \cap T_3 = \emptyset$ and $T_1.T_2 = T_2.T_3 = T_3.T_1 = 1$.*

Examples of special σ-quotients with triangular pole divisors are constructed by Holzapfel in [4]. We show that any triangular divisor can be realized as a pole divisor of a special σ-quotient $f \in \mathcal{L}_A(T, T^{\mathrm{sing}})$ and provide a general formula for such f.

Proposition 9. *Let $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_i'u + b_i'v + c_i' \in \mathcal{O}_{-d}\}$ with $1 \le i \le 3$ be the smooth irreducible elliptic components of a triangle $T_1 + T_2 + T_3$ and $a_i = \Delta_{jl}'a_i'$, $b_i = \Delta_{jl}'b_i'$, $c_i = \Delta_{jl}'c_i'$. Then $a_1 + a_2 + a_3 = 0$, $b_1 + b_2 + b_3 = 0$, $\Delta_{12}^{-1}\Delta \notin \mathcal{O}_{-d}$ and for any $\nu \in \mathcal{O}_{-d}$ the function*

$$f(u,v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i - \Delta_{12}^{-1}\Delta + \nu)}{\sigma(a_i u + b_i v + c_i)} \tag{1.8}$$

is a special σ-quotient from $\mathcal{L}_A(T, T^{\mathrm{sing}})$ with pole divisor $(f)_\infty = T_1 + T_2 + T_3$.

Proof. Let $v_i' = \begin{pmatrix} a_i' \\ b_i' \end{pmatrix}$ for $1 \le i \le 3$. Expanding along the third row, one obtains

$$0 = \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ a_1' & a_2' & a_3' \end{vmatrix} = \Delta_{23}'a_1' + \Delta_{31}'a_2' + \Delta_{12}'a_3' = 0,$$

$$0 = \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ b_1' & b_2' & b_3' \end{vmatrix} = \Delta_{23}'b_1' + \Delta_{31}'b_2' + \Delta_{12}'b_3' = 0,$$

and concludes that

$$v_1 + v_2 + v_3 = \Delta_{23}'v_1' + \Delta_{31}'v_2' + \Delta_{12}'v_3' = 0_{2 \times 1}, \quad \Delta_{12} = \Delta_{23} = \Delta_{31}. \tag{1.9}$$

Now, according to Lemma 1 (iii), $T_1 \cap T_2 \cap T_3 = \emptyset$ is equivalent to $\Delta \notin \Delta_{12}\mathcal{O}_{-d}$. Then the condition $m_p(f)_0 \ge m_p(f)_\infty - 1$ for $\forall p \in (f)_\infty^{\mathrm{sing}}$ reduces to $T_i \cap T_j \subset D_l$ for any even permutation $\{i, j, l\}$ of $\{1, 2, 3\}$. Making use of Lemma 1 (ii), one can choose $\mu_1 = \mu_2 = \mu_3 = \nu - \Delta_{12}^{-1}\Delta \notin \mathcal{O}_{-d}$. Then (1.9) implies (1.4) from Lemma 4 and reveals that (1.8) is a special σ-quotient from $\mathcal{L}_A(T, T^{\mathrm{sing}})$. $\square$

Definition 10. *The special σ-quotients (1.8) from $\mathcal{L}_A(T, T^{\mathrm{sing}})$ with triangular pole divisors $(f)_\infty = T_1 + T_2 + T_3$ are called triangular.*

For elliptic curves $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_i u + b_i v + c_i \in \mathcal{O}_{-d}\}$, $1 \le i \le 2$ with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$ and intersection number $T_1.T_2 = 1$, Lemma 1 (i) implies that

$$M = \begin{pmatrix} a_2 & b_2 \\ a_1 & b_1 \end{pmatrix} \in GL_2(\mathcal{O}_{-d}).$$

As a result, there arises an automorphism

$$\varphi : A \longrightarrow A,$$

$$\varphi(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) = \left[M \left(\begin{array}{c} \overline{u} \\ \overline{v} \end{array} \right) + \left(\begin{array}{c} \overline{c_2} \\ \overline{c_1} \end{array} \right) \right]^t$$

with $\varphi(T_1) = E \times \check{o}_E$, $\varphi(T_2) = \check{o}_E \times E$. Making use of $\sigma(\alpha z) = \alpha \sigma(z)$ for $\forall \alpha \in \mathcal{O}^*_{-d}$, $\forall z \in \mathbb{C}$, one observes that any triangular σ-quotient can be reduced by an automorphism of A to the form

$$f_{012}(u, v) = \frac{\sigma(u + a_0^{-1}c_0)\sigma(v + b_0^{-1}c_0)\sigma(a_0 u + b_0 v)}{\sigma(u)\sigma(v)\sigma(a_0 u + b_0 v + c_0)} \tag{1.10}$$

with $a_0, b_0 \in \mathcal{O}^*_{-d}$, $c_0 \notin \mathcal{O}_{-d}$.

We are going to describe the complete divisor of a triangular σ-quotient.

Definition 11. *The divisor $D = \sum\limits_{i=0}^{2} D_i - \sum\limits_{i=0}^{2} T_i$ with smooth elliptic irreducible components D_i, T_j is called a tetrahedron (cf. Figure 1) if:*
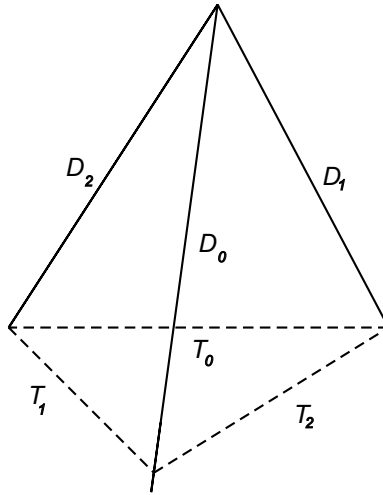

Figure 1: Tetrahedron

(i) $\sum\limits_{i=0}^{2} T_i$ *is a triangle;*

(ii) D_i *are parallel to T_i for all $0 \le i \le 2$;*

(iii) $D_0 \cap D_1 \cap D_2 = D_0 \cap D_1 = D_1 \cap D_2 = D_2 \cap D_0 = \{p_0\}$ *for some point $p_0 \in A$;*

(iv) $\left(\sum\limits_{i=0}^{2} D_i\right) \cap \left(\sum\limits_{i=0}^{2} T_i\right) = \left(\sum\limits_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum\limits_{i=0}^{2} D_i\right)^{\text{smooth}}.$

Definition 12. *An inscribed (ordered) pair of triangles (cf. Figure 2) is a divisor* $D = \sum\limits_{i=0}^{2} D_i - \sum\limits_{i=0}^{2} T_i$, *such that:*

(i) $\sum\limits_{i=0}^{2} D_i$ *and* $\sum\limits_{i=0}^{2} T_i$ *are triangles;*

(ii) D_i *are parallel to* T_i *for all* $0 \le i \le 2$;

(iii) $\left(\sum\limits_{i=0}^{2} D_i\right) \cap \left(\sum\limits_{i=0}^{2} T_i\right) = \left(\sum\limits_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum\limits_{i=0}^{2} D_i\right)^{\text{smooth}}.$
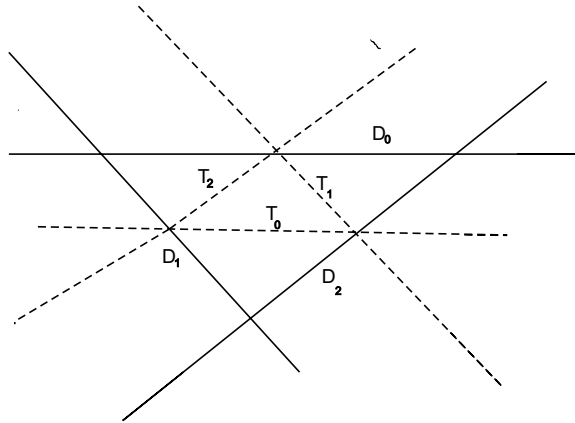

Figure 2: Inscribed (ordered) pair of triangles

An explicit calculation of the singular points of the complete divisor yields the following

Corollary 13. *Let (1.10) with* $a_0, b_0 \in \mathcal{O}_{-d}^*$, $c_0 \notin \mathcal{O}_{-d}$ *be a triangular* σ-*quotient with complete divisor* $(f_{012}) = \sum\limits_{i=0}^{2} D_i - \sum\limits_{i=0}^{2} T_i$. *Then:*

(i) $c_0 + \mathcal{O}_{-d} \in E_{2-\text{tor}}$ *is a 2-torsion point if and only if* (f_{012}) *is a tetrahedron;*

(ii) $c_0 + \mathcal{O}_{-d} \notin E_{2-\text{tor}}$ *exactly when* (f_{012}) *is an inscribed pair of triangles.*

In either case, the multiplicity $m_p(f_{012}) = -1$ *at all* $p \in (f_{012})_\infty \cap T^{\text{sing}}$.

In [4] Holzapfel introduces the idea for detecting the linear independence of co-abelian modular forms by the poles of the corresponding transfers to abelian functions. Instead of his strongly descending divisor condition, we use a natural complete decreasing flag on $[\Gamma, 1]$. That enables to supply a criterion for some modular forms to constitute a basis of $[\Gamma, 1]$ and to show that $[\Gamma, 1]$ has always a basis of the considered form.

Observe that the subspaces

$$V_i = j_1[\Gamma, 1]_i := \{\omega \in j_1[\Gamma, 1] \,|\, \omega(\kappa_1) = \ldots = \omega(\kappa_{i-1}) = 0\}$$

of $V_1 = j_1[\Gamma, 1]$ form a non-increasing flag

$$j_1[\Gamma, 1] = V_1 \supseteq V_2 \supseteq \ldots \supseteq V_{m-1} \supseteq V_m \supseteq \ldots \supseteq V_h \supseteq V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}.$$

For any $\omega, \omega' \in V_i$ one has $\omega'(\kappa_i)\omega - \omega(\kappa_i)\omega' \in V_{i+1}$, so that $0 \leq \dim_{\mathbb{C}}(V_i/V_{i+1}) \leq 1$ for all $1 \leq i \leq h$. We prove that there is a permutation of the cusps $\kappa_1, \ldots, \kappa_h$, so that $V_i/V_{i+1} \simeq \mathbb{C}$ for $1 \leq i \leq m$ and $V_{m+1} = V_{m+2} = \ldots = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}} \simeq \mathbb{C}$. If so, then $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1$.

Proposition 14. *If the pole divisors of $f_i \in \mathcal{L}_A(T, T^{\mathrm{sing}})$ are subject to*

$$T_i \subset (f_i)_\infty \subseteq T_i + T_{i+1} + \ldots + T_h \quad \text{for all} \quad 1 \leq i \leq m,$$

then $\omega_i = \tau_1^{-1}(f_i) \in j_1[\Gamma, 1]$ with $1 \leq i \leq m$ form a basis of a complement of $V_{m+1} = j_1[\Gamma, 1]_{m+1}$.

In particular, if $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}$, then $j_1(\eta_o), \omega_1, \ldots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$.

Proof. It suffices to show that for arbitrary $b_1, \ldots, b_m \in \mathbb{C}$ the linear system

$$\sum_{i=1}^m \omega_i(\kappa_j)t_i = b_j, \quad 1 \leq j \leq m \tag{1.11}$$

has a unique solution $(t_1, \ldots, t_m)$. On one hand, that implies the linear independence of $\omega_1, \ldots, \omega_m$ over $\mathbb{C}$. On the other hand, for any $\omega \in j_1[\Gamma, 1]$ there is uniquely determined $\sum_{i=1}^m c_i\omega_i$ with $\omega_0 = \omega - \sum_{i=1}^m c_i\omega_i \in j_1[\Gamma, 1]_{m+1} = V_{m+1}$. In other words, $j_1[\Gamma, 1] = \mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m) \oplus V_{m+1}$, so that $\omega_1, \ldots, \omega_m$ is a basis of the complement $\mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m)$ of V_{m+1}.

Towards the existence of a unique solution of (1.11), note that the requirement $T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \ldots + T_h$ is equivalent to $\omega_i(\kappa_i) \neq 0$ and $\omega_i(\kappa_1) = \omega_i(\kappa_2) = \ldots = \omega_i(\kappa_{i-1}) = 0$. Thus, (1.11) is of the form

$$\begin{pmatrix} \omega_1(\kappa_1) & \ldots & 0 & \ldots & 0 \\ \omega_1(\kappa_i) & \ldots & \omega_i(\kappa_i) & \ldots & 0 \\ \omega_1(\kappa_m) & \ldots & \omega_i(\kappa_m) & \ldots & \omega_m(\kappa_m) \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_i \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix},$$

with non-degenerate, lower-triangular coefficient matrix and has unique solution for all $b_1, \ldots, b_m \in \mathbb{C}$.

In the case of $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}$, note that $j_1[\Gamma, 1]_{\text{cusp}} = \mathbb{C}j_1(\eta_o)$ with $\tau_1 j_1(\eta_o) = 1 \in \mathcal{L}_A(T, T^{\text{sing}})$, so that $j_1(\eta_o), \omega_1, \ldots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$. $\quad\square$

The next proposition establishes that $j_1[\Gamma, 1]$ has always a $\mathbb{C}$-basis of the considered form.

Proposition 15. *Let $\Gamma \subset SU_{2,1}$ be a freely acting, co-abelian Picard modular group and $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1$. Then there is a permutation $\{\kappa_1, \ldots, \kappa_m, \kappa_{m+1}, \ldots, \kappa_h\}$ of the Γ-cusps, such that*

$$V_1/V_2 \simeq V_2/V_3 \simeq \cdots \simeq V_m/V_{m+1} \simeq \mathbb{C}, \quad V_{m+1} = V_{m+2} = \cdots = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}.$$

Any $\omega_i \in V_i \setminus V_{i+1}$ transfers to $\tau_1(\omega_i) \in \mathcal{L}_A(T, T^{\text{sing}})$ with

$$T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \cdots + T_h \quad for \quad 1 \leq i \leq m$$

and $j_1(\eta_o), \omega_1, \ldots, \omega_m$ is a $\mathbb{C}$-basis of $V_1 = j_1[\Gamma, 1]$.

In particular, if $T_{h-1}.T_h = 1$ then $V_{h-1} = j_1[\Gamma, 1]_{\text{cusp}}$ and $\dim[\Gamma, 1] \leq h - 1$.

Proof. If $V_1 = V_{h+1}$, then there is nothing to be proved. From now on, we assume that $\dim V_1/V_{h+1} = m \in \mathbb{N}$. By induction on $1 \leq i \leq m$, we establish the existence of $\omega_j \in V_j \setminus V_{j+1}$ for all $1 \leq j \leq i$. First of all, for any $\omega_1 \in V_1 \setminus V_{h+1}$ there exists a cusp κ_1 with $\omega_1(\kappa_1) \neq 0$. Then for an arbitrary permutation of the remaining cusps, one has $\omega_1 \in V_1 \setminus V_2$. If we have chosen $\omega_j \in V_j \setminus V_{j+1}$ for $1 \leq j \leq i - 1$ and $V_i \supsetneq V_{h+1}$, then for an arbitrary $\omega_i \in V_i \setminus V_{h+1}$ there exists a permutation of $\{\kappa_i, \kappa_{i+1}, \ldots, \kappa_h\}$, such that $\omega_i(\kappa_i) \neq 0$. Clearly, $\omega_i \in V_i \setminus V_{i+1}$ and we have obtained a basis $j_1(\eta_o), \omega_1, \ldots, \omega_m$ of $V_1 = j_1[\Gamma, 1]$. The conditions $\omega_i \in V_i \setminus V_{i+1}$ amount to $T_i \subset (\tau_1(\omega_i))_\infty$ and $T_j \nsubseteq (\tau_1(\omega_i))_\infty$ for all $1 \leq j \leq i - 1$.

If $T_{h-1}.T_h = 1$, then up to an automorphism of A, one can assume that $T_{h-1} = E \times \check{o}_E$ and $T_h = \check{o}_E \times E$. We claim that $\mathcal{L}_A((E \times \check{o}_E) + (\check{o}_E \times E)) = \mathbb{C}$, so that $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1 \leq h - 1$. Indeed, for an arbitrary $Q \in E \setminus \check{o}_E$ the restriction $f|_{E \times Q}$ is an elliptic function of order 1. Therefore $f|_{E \times Q} \equiv C(Q) \in \mathbb{C}$ is a constant. Similarly, $f|_{P \times E} \equiv C'(P) \in \mathbb{C}$ for any $P \in E \setminus \check{o}_E$. As a result, $C'(P) = f(P, Q) = C(Q)$ for all $Q \in E$ and $f|_A$ is constant. $\quad\square$

Proposition 16. *(Holzapfel [5]) Let us fix the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{i}{2}$, $\omega_3 = \omega_1 + \omega_2$ of the lattice $\pi_1(E) = \mathcal{O}_{-1} = \mathbb{Z} + i\mathbb{Z}$, the 2-torsion points $Q_0 := 0 (\text{mod } \mathbb{Z} + i\mathbb{Z}) \in E$, $Q_j := \omega_j (\text{mod } \mathbb{Z} + i\mathbb{Z}) \in E$ for $1 \leq j \leq 3$ and $Q_{ij} := (Q_i, Q_j) \in A$. Consider the elliptic curves*

$$T_k = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - i^k v \in \pi_1(E)\} \quad for \quad 1 \leq k \leq 4,$$

$$T_{4+k} = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - \omega_k \in \pi_1(E)\} \quad for \quad 1 \leq k \leq 2,$$

$$T_{6+k} = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, v - \omega_k \in \pi_1(E)\} \quad \text{for} \quad 1 \le k \le 2.$$

Then the blow-up of A at the singular points

$$S_1 = Q_{00}, \quad S_2 = Q_{33}, \quad S_3 = Q_{11}, \quad S_4 = Q_{12}, \quad S_5 = Q_{21}, \quad S_6 = Q_{22}$$

of $T^{(6,8)}_{\sqrt{-1}} = \sum\limits_{k=1}^{8} T_k$ is the toroidal compactification $(\mathbb{B}/\Gamma_1)'$ of a ball quotient $\mathbb{B}/\Gamma_1$ by a freely acting Picard modular group Γ_1 over the Gaussian integers $\mathbb{Z}[i]$.

The self-intersection matrix $M(6,8) \in \mathbb{Z}_{6 \times 8}$ of $T^{(6,8)}_{\sqrt{-1}}$ is defined to have entries $M(6,8)_{ij} = 1$ for $S_i \in T_j$ and $M(6,8)_{ij} = 0$ for $S_i \notin T_j$. Straightforwardly,

$$M(6,8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$
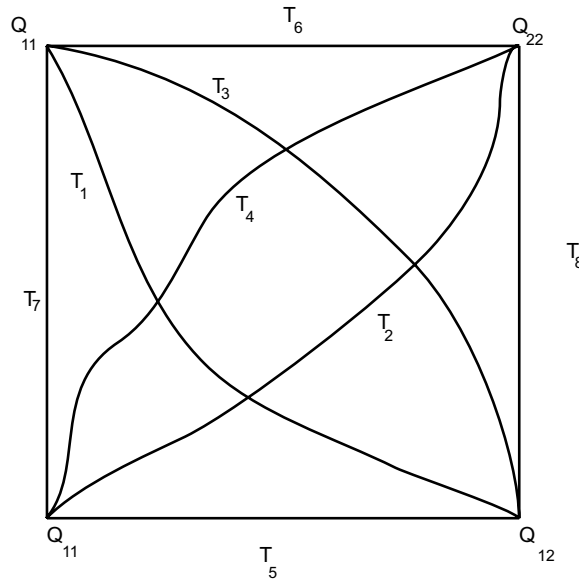

Figure 3: The incidence relations of $T^{(6,8)}_{\sqrt{-1}}$ and $\sum\limits_{i=1}^{2} \sum\limits_{j=1}^{2} Q_{ij} \subset \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\text{sing}}$.

According to $Q_{00}, Q_{33} \in T_k$ or $\forall 1 \le k \le 4$, there are no triangles $T_i + T_j + T_k \subset T^{(6,8)}_{\sqrt{-1}}$ with $1 \le i < j \le 4$, $1 \le i < j < k \le 8$. Bearing in mind that

$\left(T_{\sqrt{-1}}^{(6,8)}\right)^{\mathrm{sing}} \cap \left(\sum\limits_{k=5}^{8} T_k\right) = \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} Q_{ij}$, one makes use of Figure 3 and recognizes the triangles $T_{2k-1} + T_{4+m} + T_{6+m}$, $T_{2k} + T_{4+m} + T_{9-m}$ with $1 \leq k, m \leq 2$. An immediate application of Proposition 9 with $\nu = 2\omega_m$ and, respectively, $\nu = \omega_3 + \omega_m + (-1)^{k+1}\omega_{3-m}$, yields the following

Corollary 17. *The space* $\mathcal{L}_A\left(T_{\sqrt{-1}}^{(6,8)}, \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\mathrm{sing}}\right)$ *contains the binary parallel*

$$f_{56}(u, v) = \frac{\sigma(u - \omega_1 - \mu_1)\sigma(u - \omega_2 + \mu_1)}{\sigma(u - \omega_1)\sigma(u - \omega_2)},$$

$$f_{78}(u, v) = \frac{\sigma(v - \omega_1 - \mu_2)\sigma(v - \omega_2 + \mu_2)}{\sigma(v - \omega_1)\sigma(v - \omega_2)}$$

and the triangular σ-quotients

$$f_{2k-1,4+m,6+m}(u, v)$$
$$= \frac{\sigma(u + (-1)^k iv + \omega_3)\sigma(-u + \omega_m + \omega_3)\sigma((-1)^{k+1} iv + (-1)^k i\omega_m + \omega_3)}{\sigma(u + (-1)^k iv)\sigma(-u + \omega_m)\sigma((-1)^{k+1} iv + (-1)^k i\omega_m)}$$

$$f_{2k,4+m,9-m}(u, v)$$
$$= \frac{\sigma(u + (-1)^{k+1} v + \omega_3)\sigma(-u + \omega_m + \omega_3)\sigma((-1)^k v + (-1)^{k+1}\omega_{3-m} + \omega_3)}{\sigma(u + (-1)^{k+1} v)\sigma(-u + \omega_m)\sigma((-1)^k v + (-1)^{k+1}\omega_{3-m})}$$

with arbitrary $1 \leq k, m \leq 2$.

Proposition 14 provides the following

Corollary 18. *If f_{pq} and f_{ijk} are the binary parallel and triangular σ-quotients from the space* $\mathcal{L}_A\left(T_{\sqrt{-1}}^{(6,8)}, \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\mathrm{sing}}\right)$ *and $\omega_{pq} = \tau_1^{-1}(f_{pq})$, $\omega_{ijk} = \tau_1^{-1}(f_{ijk})$, then*

$$\omega_{157}, \quad \omega_{258}, \quad \omega_{368}, \quad \omega_{467}, \quad \omega_{56}, \quad \omega_{78}, \quad j_1(\eta_o)$$

is a $\mathbb{C}$-basis of $j_1[\Gamma_1, 1]$.

In particular, $\dim_{\mathbb{C}}[\Gamma, 1] = 7$.

2. SUFFICIENT CONDITIONS FOR THE NORMAL GENERATION OF A SPACE OF LOGARITHMIC CANONICAL SECTIONS

Definition 19. *A holomorphic line bundle $\mathcal{E}$ on an algebraic variety X is sufficiently ample if the holomorphic sections of a sufficiently large tensor power $\mathcal{E}^{\otimes m}$ provide a projective embedding of X.*

Definition 20. *A holomorphic line bundle $\mathcal{E}$ over an algebraic variety X is globally generated if the global holomorphic sections of $\mathcal{E}$ determine a regular projective morphism.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is globally generated if some (and therefore any) basis of V provides a regular projective morphism $X \to \mathbb{P}(V)$.

Definition 21. *A holomorphic line bundle $\mathcal{E}$ over an algebraic manifold X is normally generated if $\mathcal{E}$ is globally generated and $H^0(X, \mathcal{E})$ defines a projective immersion of X with normal image.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is normally generated if it is globally generated and the morphism $X \to \mathbb{P}(V)$ is a projective immersion with normal image.

The normal generation of a sufficiently ample line bundle is a classical topic under study. Various works provide normally generated and non-normally generated line bundles over curves and abelian varieties. According to [2], if $\mathcal{E}$ is a sufficiently ample line bundle on an abelian variety of dimension n, then $\mathcal{E}^{\otimes(n-1)}$ is normally generated. In particular, any sufficiently ample line bundle on an abelian surface is normally generated.

Our aim is to provide sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ over the Baily-Borel compactification $\widehat{A}$. That cannot be derived from the normal generation of a subspace $W \subseteq H^0(A, \mathcal{E})$ of holomorphic sections of a line bundle $\mathcal{E} \to A$. Namely, $\xi^* W$ cannot be a normally generated space of global holomorphic sections of $\xi^* \mathcal{E}$, as far as the morphism, associated with $\xi^* W$ is not immersive on the exceptional divisor $L = \xi^{-1}(T^{\mathrm{sing}})$ of $\xi : A' \to A$.

Corollary 22. *Let X be an irreducible normal projective variety X and $f : X \to Y$ be a finite, regular, generically injective morphism onto Y. Then $f : X \to Y$ is a regular immersion with normal image Y.*

Proof. If $f : X \to Y$ is a regular morphism of degree $d \in \mathbb{N}$, then the generic fiber of f consists of d points, while the exceptional ones are constituted by $\leq d$ points. In particular, for $d = 1$, any regular, generically injective morphism is bijective onto its image. As a result, $f : X \to Y$ is a regular immersion with normal image. $\square$

Our specific considerations will be based on the following immediate consequence of Corollary 22

Corollary 23. *Let X be an irreducible normal projective variety, $\mathcal{E} \to X$ be e holomorphic line bundle over X and $V \subseteq H^0(X, \mathcal{E})$ be a space of global holomorphic sections of $\mathcal{E}$. If $f : X \to \mathbb{P}(V)$ is a finite, regular, generically injective morphism then V is normally generated.*

Lemma 24. *A subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, is globally generated over $\widehat{A}$ if and only if it satisfies simultaneously the following two geometric conditions:*

(i) *for any irreducible component T_i of T there is $\omega_i \in V$ with $(\tau_1(\omega_i))_\infty \supset T_i$;*

(ii) *for any $p \in T^{\text{sing}}$ there exists $\omega_p \in V$ with $m_p(\tau_1(\omega_p)) = -1$.*

Proof. The space V is globally generated over $\widehat{A}$ exactly when for any point $q \in \widehat{A}$ there is $v_q \in V$ with $v_q(q) \neq 0$. If $q \in \left(\widehat{\mathbb{B}/\Gamma}\right) \setminus \left(L \cup \sum_{i=1}^{h} \kappa_i\right)$, then $j_1(\eta_o)(q) \neq 0$. A modular form $\omega_i \in V$ does not vanish on the cusp κ_i if and only if $T_i \subset (\tau_1(\omega_i))_\infty$. A modular form $\omega_p \in V$ takes non-zero values on the rational (-1)-curve $\xi^{-1}(p)$ exactly when the multiplicity $m_p(\tau_1(\omega_p)) = -1$. $\qquad\square$

From now on, we say briefly that a modular form $\omega \in H^0(A', \Omega^2_{A'}(T'))$ is binary parallel or triangular if its transfer $\tau_1(\omega) \in \mathcal{L}_A(T, T^{\text{sing}})$ is binary parallel or, respectively, triangular.

Proposition 25. *Let us suppose that the subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ contains the cuspidal form $j_1(\eta_o)$, two binary parallel forms ω_{13}, ω_{24}, a triangular ω_{012} with $T_0 \cap T_3 \cap T_4 = \emptyset$ and satisfies the following three conditions:*

(i) *for any $i \notin \{0, 1, \ldots, 4\}$ there exists $\omega_i \in V$ with $(\tau_1(\omega_i))_\infty \supset T_i$;*

(ii) *for any $p \in T^{\text{sing}} \setminus \left(\sum_{j=0}^{4} T_j\right)$ there exists $\omega_p \in V$ with $m_p(\tau_1(\omega_p)) = -1$;*

(iii) *for any $1 \leq i < j \leq h$ there is $\omega_{ij} \in V$, such that $(\tau_1(\omega_{ij}))_\infty$ contains exactly one of T_i or T_j.*

Then V is normally generated.

Proof. In the presence of Corollary 23, it suffices to establish that the projective morphism $f : \widehat{A} \to \mathbb{P}(V)$, associated with V is regular, finite and generically injective. Assumption (i) from the present proposition and $(\tau_1(\omega_{ij}))_\infty = T_i + T_j$, $(\tau_1(\omega_{012}))_\infty = T_0 + T_1 + T_2$ imply assumption (i) from Lemma 24. Further, no-one $p \in T^{\text{sing}} \cap (T_1 + T_3)$ belongs to $(\tau_1(\omega_{13}))_0 = D_1 + D_3$, as far as T_1, T_3, D_1 and D_3 are mutually parallel and distinct. Therefore, $m_p(\tau_1(\omega_{13})) = -1$. Similarly, $m_p(\tau_1(\omega_{24})) = -1$ for $p \in T^{\text{sing}} \cap (T_2 + T_4)$. By Corollary 13, $m_p(\tau_1(\omega_{012})) = -1$ for all $p \in T^{\text{sing}} \cap \left(\sum_{i=0}^{2} T_i\right)$. Combining with assumption (ii) from the present proposition, one obtains (ii) from Lemma 24. That allows to conclude that $f : \widehat{A} \to \mathbb{P}(V)$ is regular.

The assumption (iii) guarantees that $f : \widehat{A} \to f(\widehat{A}) \subset \mathbb{P}(V)$ is finite. First of all, on $\widehat{A} \setminus [L + (\partial_\Gamma \mathbb{B}/\Gamma)] = (\mathbb{B}/\Gamma) \setminus L = A \setminus T$, the morphism

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13} \circ \xi = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24} \circ \xi = f_{24}\right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^2$$

is of degree 4. More precisely, if

$$f_{13}(u,v) = \frac{\sigma(u-\mu_1)\sigma(u-c_3+\mu_1)}{\sigma(u)\sigma(u-c_3)}, \; f_{24}(u,v) = \frac{\sigma(v-\mu_2)\sigma(v-c_4+\mu_2)}{\sigma(v)\sigma(v-c_4)}, \quad (2.1)$$

then for any $x,y \in \mathbb{P}^1$ the fiber is

$$(f_{13}, f_{24})^{-1}(x,y) = \{(P_i(x), Q_j(y)) \,|\, 1 \le i,j \le 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_1(y) + Q_2(y) = \overline{c_4}.$$

The condition (iii) provides the injectiveness of $f : \partial_\Gamma \mathbb{B}/\Gamma \to f(\partial_\Gamma \mathbb{B}/\Gamma)$, which suffices for $f : L \to f(L)$ to be discrete and, therefore, finite. Otherwise, f contracts some irreducible component $\xi^{-1}(p)$, $p \in T^{\mathrm{sing}}$ of L. If $p \in T_i \cap T_j$ then $\kappa_i, \kappa_j \in \xi^{-1}(p)$, whereas $f(\kappa_i) = f(\kappa_j)$. Thus, $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ and, therefore, $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

The generic injectiveness of the projective morphism $f : \widehat{A} \to f(\widehat{A})$ follows from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

This, in turn, is equivalent to the generic injectiveness of the rational surjective morphism

$$F = (f_{13}, f_{24}, f_{012}) : A \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Let us consider also the rational surjection $F_1 = (f_{13}, f_{24}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ and its factorization

$$
\begin{array}{ccc}
A & \xrightarrow{\;F\;} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
{\scriptstyle F_1}\big\downarrow & \swarrow {\scriptstyle \mathrm{pr}_{12}} & \\
\mathbb{P}^1 \times \mathbb{P}^1 & &
\end{array}
$$

through F and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ onto the first two factors. The irreducible components T_1 and T_2 of the triangle $T_0 + T_1 + T_2$ have intersection number $T_1.T_2 = 1$. That allows to assume that $T_1 = \breve{o}_E \times E$, $T_2 = E \times \breve{o}_E$ and (1.10).

Suppose that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is not generically injective. By $F_1 = \mathrm{pr}_{12} \circ F$ and $\deg F_1 = 4$, the generic fiber of F on $F_1^{-1}(x,y)$ consists of 2 or 4 points. In either case, for any $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds at least one of the following pairs of relations:

Case (i): $\quad f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y)),$
$\qquad\qquad\; f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_2(y));$

Case (ii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_2(y)),$

$$f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_1(y));$$

Case (iii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_1(x), Q_1(y)),$

$$f_{012}(P_2(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y)).$$

We claim that the relations from at least one case are satisfied identically on $\mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, the locus of either case is a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ and their union is also a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$. The contradiction implies that for any $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds identically at least one of the Cases (i), (ii) or (iii). Note that (ii) and (iii) are equivalent under the transposition of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and, respectively, of $A = E \times E$.

Without loss of generality, one can suppose that $P_1(\infty) = \check{o}_E$ and $P_2(\infty) = \overline{c_3}$. In Case (i), up to a relabeling of $Q_1(y)$, $Q_2(y)$, one has $Q_1(\infty) = \check{o}_E$, $Q_2(\infty) = \overline{c_4}$. Then

$$\infty = f_{012}(\check{o}_E, \check{o}_E) = f_{012}(P_1(\infty), Q_1(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

However, $\overline{c_3} \neq \check{o}_E$, $\overline{c_4} \neq \check{o}_E$ and $T_3 \cap T_4 = \{(\overline{c_3}, \overline{c_4})\} \nsubseteq T_0$ reveal that $f_{012}(\overline{c_3}, \overline{c_4}) \neq \infty$, so that Case (i) does not hold identically on A. Similarly, in Case (ii), there follows

$$\infty = f_{012}(\check{o}_E, \overline{c_4}) = f_{012}(P_1(\infty), Q_2(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

The contradiction implies that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. $\qquad \square$

Here is another sufficient condition for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated.

Proposition 26. *Let V be a subspace of $H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, a binary parallel ω_{13}, triangular ω_{012}, ω_{234} with $T_0 \cap T_1 \cap T_4 = \emptyset$ and satisfying the following three conditions:*

(i) *for any $i \notin \{0, 1, \dots 4\}$ there exists $\omega_i \in V$ with $(\tau_1(\omega_i))_\infty \supset T_i$;*

(ii) *for any $p \in T^{\mathrm{sing}} \setminus \left(\sum\limits_{j=0}^{4} T_j \right)$ there exists $\omega_p \in V$ with $m_p(\tau_1(\omega_p)) = -1$;*

(iii) *for any $1 \leq i < j \leq h$ there is $\omega_{ij} \in V$, such that $(\tau_1(\omega_{ij}))_\infty$ contains exactly one of T_i or T_j.*

Then V is normally generated.

Proof. As in Proposition 25, first we establish the regularity of the projective morphism $f : \widehat{A} \to f(\widehat{A})$.

Further, $f : \widehat{A} \to f(\widehat{A})$ is finite, as far as the fibers of its restriction on $(\mathbb{B}/\Gamma) \setminus L = A \setminus T$ are contained in the fibers of

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : A \setminus T \longrightarrow \mathbb{C}^2.$$

Let $f_{012}(u, v)$ be of the form (1.10) and f_{13} be as in (2.1). Then for any $x, y \in \mathbb{P}^1$ the fiber

$$(f_{13}, f_{012})^{-1}(x, y) = \{(P_i(x), Q_{ij}(x, y)) \,|\, 1 \leq i, j \leq 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_{i1}(x, y) + Q_{i2}(x, y) = -a_0 b_0^{-1} P_i(x) - b_0^{-1}\overline{c_0}$$

consists of at most four points. The reason is that for any fixed $P_i(x) \in E$ the elliptic function $f_{012}(P_i(x),\ )$ is of order 2. Thus, $(f_{13}, f_{012}) : A \setminus T \to \mathbb{C}^2$ is finite. The assumption (iii) implies that $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ is finite, so that $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

We derive the generic injectiveness of $f : \widehat{A} \to f(\widehat{A})$ from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012}, \frac{\omega_{234}}{j_1(\eta_o)} = f_{234} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

To this end, let us factor the rational surjection $F_1 = (f_{13}, f_{012}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ through the rational surjection $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, along the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\ F\ } & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
{\scriptstyle F_1}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_{12}} & \\
\mathbb{P}^1 \times \mathbb{P}^1 & &
\end{array} \ .$$

If F is not generically injective, then at least one of the following three cases holds identically on $\mathbb{P}^1 \times \mathbb{P}^1$:

Case (i): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y)),$

$\qquad\quad f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y));$

Case (ii): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y)),$

$\qquad\quad f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y));$

Case (iii): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_1(x), Q_{11}(x, y)),$

$\qquad\quad f_{234}(P_2(x), Q_{22}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y)).$

In either case, denote by $P_1(\infty) = \breve{o}_E$ and $P_2(\infty) = \overline{c_3}$ the poles of the elliptic function f_{13} and note that $T_1 = P_1(\infty) \times E$, $T_3 = P_2(\infty) \times E$. Further, let

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$Q_{i1}(\infty, \infty) = \check{o}_E$, so that $T_2 = E \times Q_{11}(\infty, \infty) = E \times Q_{21}(\infty, \infty)$. Finally, let $Q_{i2}(\infty, \infty) = -a_0 b_0^{-1} P_i(\infty) - b_0^{-1} \overline{c_0}$, in order to have

$$\{q_{10}\} = T_1 \cap T_0 = \{(P_1(\infty), Q_{12}(\infty, \infty))\},$$

$$\{q_{30}\} = T_3 \cap T_0 = \{(P_2(\infty), Q_{22}(\infty, \infty))\}.$$

Denote also

$$\{q_{12}\} = T_1 \cap T_2 = \{(P_1(\infty), Q_{11}(\infty, \infty))\},$$

$$\{q_{32}\} = T_3 \cap T_2 = \{(P_2(\infty), Q_{21}(\infty, \infty))\}.$$

Bearing in mind that $(f_{234})_\infty = T_2 + T_3 + T_4$, note that $f_{234}(q_{ij}) = \infty$ whenever $\{i, j\} \cap \{2, 3, 4\} \neq \emptyset$. In the Case (i) one has $f_{234}(q_{10}) = f_{234}(q_{32}) = \infty$. If $q_{10} \in T_2$, then $q_{10} \in T_0 \cap T_1 \cap T_2$, contrary to the assumption that $T_0 + T_1 + T_2$ is a triangle. On the other hand, $T_3 \cap T_1 = \emptyset$ guarantees that $q_{10} \notin T_3$. Therefore $q_{10} \in T_4$ and $q_{10} \in T_0 \cap T_1 \cap T_4 = \emptyset$. The contradiction rejects the Case (i). If the first relation of Case (ii) is identical on $\mathbb{P}^1 \times \mathbb{P}^1$, then $f_{234}(q_{10}) = f_{234}(q_{30}) = \infty$. As in the Case (i), that leads to an absurd. Finally, $f_{234}(q_{10}) = f_{234}(q_{12}) = \infty$ contradicts the hypotheses and establishes that $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. $\qquad \square$

An immediate application of Proposition 26 to the example from Proposition 16 yields the following

Corollary 27. *In the terms of Proposition 16, the subspace*

$$V_1 = \mathrm{Span}_{\mathbb{C}}(j_1(\eta_o), \omega_{56}, \omega_{157}, \omega_{267}, \omega_{368}, \omega_{458}) \subset H^0(A_1', \Omega_{A_1'}^2(T'))$$

is normally generated, i.e., determines a regular projective immersion

$$f : \widehat{\mathbb{B}/\Gamma_1} \to \mathbb{P}(V_1) = \mathbb{P}^5$$

with normal image.

If one applies Proposition 25 to the cuspidal form $j_1(\eta_o)$, the binary parallel ω_{56}, ω_{78} and triangular ω_{157}, then one needs to adjoin the triangular $\omega_{2,4+k,9-k}$, $\omega_{3,4+l,6+l}$, $\omega_{4,4+m,9-m}$ for some $k, l, m \in \{1, 2\}$. The span of these modular forms is 7-dimensional and depletes the entire $[\Gamma_1, 1]$. It is clear that the normal generation of V_1 implies the normal generation of $H^0(A_1', \Omega_{A_1'}^2(T')) = j_1[\Gamma_1, 1]$.

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Boris Kotzev
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 blvd. J. Bourchier, BG-1164 Sofia
BULGARIA
e-mail: bkotzev@fmi.uni-sofia.bg

Azniv Kasparian
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 blvd. J. Bourchier, BG-1164 Sofia
BULGARIA
e-mail: kasparia@fmi.uni-sofia.bg