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SOME PROPERTIES OF AN ALGEBRA OF ALL SETS OF NATURALS E-REDUCIBLE TO A FIXED SET

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In this paper we consider the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$, where A is an arbitrary fixed set of natural numbers, $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, $W_0, W_1, \ldots$ is the sequence of all computably enumerable sets, considered as e-operators, and Non is the predicate detecting non-emptiness. It is shown that for any set of natural numbers A the algebra $\mathfrak{N}^A$ has a least enumeration, admits equivalent representation with 3 operators and is finitely generated.

Keywords: Enumeration, enumeration degree, enumeration operator, degree of a structure, least degree of a structure, algebra.

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1. INTRODUCTION

In attempts to classify the family of all sets of naturals with respect to effective computability, different kinds of reducibilities have been introduced. In [8] Post first introduced the so-called "strong" reducibilities (m-,tt-,. . .) and later on in [9] – the Turing reducibility.

Every reducibility defines a pre-order. Thus in a natural way m-degrees, T-degrees, etc. have been introduced. Enumeration reducibility was introduced in 1959 by Friedberg and Rogers [5]. In [7] embedding of the semi-lattice of Turing degrees (T-degrees) into the semi-lattice of enumeration degrees (e-degrees) was found.This fact showed that two semi-lattices are closely related and any result or question about one of them triggered a question of validity for the other. In

1966 Sacks [12] and in 1967 Rogers [11] stated the basic question about T-degrees, namely whether there exist non-trivial automorphisms in the upper semi-lattice of T-degrees. In case that such non-trivial automorphisms do not exist, we say that the upper semi-lattice is rigid. The same question was stated for e-degrees, m-degrees, etc. This question is important because it is connected with definability in these semi-lattices. For m-degrees it was shown by Shore that there exist $2^{2^{\aleph_0}}$ automorphisms.

In 1977 Jockusch and Solovay [6] and in 1979 Richter [10] and Epstein [4] proved that for Turing degrees every automorphism is the identity on the cone above $0^{(3)}$. In 1986 Slaman and Woodin [13] improved the above result by showing that every automorphism is the identity on the cone above $0''$. Using the connections between both T- and e-jumps, Soskov and Ganchev [15] proved that for e-degrees every automorphism is the identity on the cone above $0^{(4)}$.

Since the upper semi-lattice of all e-degrees (e-degrees $\leq$ **a**) is defined by $\leq_e A$, in this paper for any fixed set of natural numbers A the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is considered. Here $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$ and $W_0, W_1, \ldots$ is the standard sequence of all computably enumerable (c.e.) sets, considered as e-operators and Non is the predicate for "non-emptiness". We would like to mention that the empty set plays a special role and we distinguish it from the other c.e. sets. We modify slightly the relation $\leq_e$ and show that the algebra $\mathfrak{N}^A$ has a least enumeration, admits equivalent representation with 3 operators ant is finitely generated. We use unary partial structures without equality [3, 2].

In Section 2 we give all necessary definitions, notions and propositions concerning normal and least enumerations of unary partial structures. Here we slightly modify the definitions of e-reducibility and e-operators, concerning the empty set. In Section 3 we prove our main result: The algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ admits a least enumeration. Then we prove that this algebra is recursively equivalent to an algebra with only 3 operators, and that the latter algebra is finitely generated. At the end we see that among all algebras with different enumeration of all e-operators the standard one has a least enumeration.

2. PRELIMINARIES

In this paper we denote by ω the set of all natural numbers. By $Dom(f)$, $Ran(f)$ and G_f we denote the domain, the range and the graph of a function f, respectively; $\langle f\rangle$ or $\langle G_f\rangle$ stands for the set $\{\langle x_1, \ldots, x_n, y\rangle | (x_1, \ldots, x_n, y) \in G_f\}$, where $\langle ., \ldots, .\rangle$ is some fixed coding function for all finite sequences of natural numbers. We shall use $f(x)\downarrow$ to denote that $x \in Dom(f)$; also we say that $f(x)$ *is conditionally equal to* $g(x)$, or that the conditional equality $f(x) \cong g(x)$ is true if and only if

$$(f(x)\downarrow \& g(x)\downarrow \& f(x) = g(x)) \vee (\neg(f(x)\downarrow) \& \neg(g(x)\downarrow)).$$

W_0, W_1,... denotes the standard enumeration of all computably enumerable (c.e.) sets; $\{E_v\}_{v\in\omega}$ is an effective coding of the family of all finite subsets of ω.

If W is c.e. set, then we write $W_{[n]} = \{x|\langle n, x\rangle \in W\}$.

If A is an arbitrary subset of ω, then by $W(A)$ we denote the set

$$W(A) = \{x|\exists v(\langle x, v\rangle \in W \& E_v \neq \emptyset \& E_v \subseteq A)\}.$$

Notice that there is a slight deviation from the usual definition of the term e-operator. It concerns $\emptyset$.

We shall say that A is e-reducible to B ($A \leq_e B$) if there exists a c.e. set W such that $A = W(B)$; A is e-equivalent to B ($A \equiv_e B$) if $A \leq_e B \& B \leq_e A$; $\mathbf{d}_e(A) = \{B|A \equiv_e B\}$. Thus we obtain $\mathbf{0}_e$ — the family of all non-empty c.e. sets and $\mathbf{\text{-}1}_e = \{\emptyset\}$.

For two arbitrary sets A and B of naturals, set

$$A \oplus B := \{2x|x \in A\} \cup \{2x+1|x \in B\}.$$

If A_0, A_1,... is a sequence of sets of naturals, the notation $\oplus_{i\in\omega} A_i$ stands for the set $\{\langle i, x\rangle | x \in A_i\}$.

We recall some definitions from [14, 1].

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k\rangle$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1, \ldots, \theta_n$ are partial unary functions in B and $R_1, \ldots, R_k$ are unary partial predicates on B. We allow any of the sequences $\theta_1, \ldots, \theta_n$ and $R_1, \ldots, R_k$ to be infinite, as well. We call such structures unary. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false.

Let $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k\rangle$ be a partial structure over the set ω. By $\langle\mathfrak{B}\rangle$ we denote the set $\langle\varphi_1\rangle \oplus \cdots \oplus \langle\varphi_n\rangle \oplus \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_k\rangle$ (in the case the when the set of functions or predicates is infinite we shall use the corresponding infinite version of $\bigoplus$).

Definition 1. *An enumeration of a structure $\mathfrak{A}$ is any ordered pair $\langle\alpha, \mathfrak{B}\rangle$, where $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k\rangle$ is a partial unary structure on ω and α is a partial surjective mapping of ω onto B such that the following conditions hold:*

(i) $Dom(\alpha) \leq_e \langle\mathfrak{B}\rangle$;

(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ *for every* $x \in \omega$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ *for every* $x \in \omega$, $1 \leq j \leq k$.

An enumeration $\langle\alpha, \mathfrak{B}\rangle$ is said to be *total* if $Dom(\alpha) = \omega$.

Let $A \subseteq B$. The set A is called admissible in the enumeration $\langle\alpha, \mathfrak{B}\rangle$ if and only if there exists a set W of naturals such that $W \leq_e \langle\mathfrak{B}\rangle$ and for every $x \in \omega$, $x \in W \iff \alpha(x) \in A$.

A partial multiple-valued (p.m.v) function θ is called *admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a set $W \subseteq \omega^2$ such that $W \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$ and $t \in B$, the following equivalence is true:

$$t \in \theta(\alpha(x)) \iff \exists y((x,y) \in W \& \alpha(y) = t).$$

The above definition can be reformulated as follows: A p.m.v function θ is called *admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a p.m.v function φ in ω such that $\langle G_\varphi \rangle \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$, $\alpha(\varphi(x)) = \theta(\alpha(x))$.

A set A or p.m.v function θ is called $\forall$-*admissible* in $\mathfrak{A}$ if it is admissible in every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of the structure $\mathfrak{A}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of $\mathfrak{A}$ if for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n unary functional symbols $\mathbf{f_1}, \ldots, \mathbf{f_n}$ and k unary predicate symbols $\mathbf{T_1}, \ldots, \mathbf{T_k}$. We admit any of the sequences $\mathbf{f_1}, \ldots, \mathbf{f_n}$ and $\mathbf{T_1}, \ldots, \mathbf{T_k}$ to be infinite. Let us fix some denumerable set $X_1, X_2, \ldots$ of variables. We use capital letters X, Y, Z and the same letters indexed to denote variables.

We use the standard definition of a term in the language $\mathcal{L}$: Every variable is a term; if τ is a term, then $\mathbf{f_i}(\tau)$ is a term. If τ is a term in the language $\mathcal{L}$, then we write $\tau(Y_1, \ldots, Y_k)$ to denote that all variables which occur in the term τ are among $Y_1, \ldots, Y_k$.

Termal predicate in the language $\mathcal{L}$ is defined by the following inductive clauses:

1) If $\mathbf{T} \in \{\mathbf{T_0}, \ldots, \mathbf{T_k}\}$ and τ is a term, then $\mathbf{T}(\tau)$ and $\neg\mathbf{T}(\tau)$ are termal predicates.

2) If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Suppose that $\mathfrak{B}$ is a structure, $a_1, \ldots, a_k$ are elements of B and $\tau(Y_1, \ldots, Y_k)$ is a term. By $\tau_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_k/a_k)$ we denote the value of the term τ in $\mathfrak{A}$ over the elements $a_1, \ldots, a_k$, if it exists.

Let $\Pi(Y_1, \ldots, Y_m)$ be a termal predicate whose variables are among $Y_1, \ldots, Y_m$ and $a_1, \ldots, a_m$ be elements of B. The value $\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m)$ of Π over $a_1, \ldots, a_r$ in $\mathfrak{A}$ is defined as follows:

If $\Pi = \mathbf{T}_j(\tau)$, $0 \leq j \leq k$, then $\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong R_j(\tau_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m))$.

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong \begin{cases} 1, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 0, \\ 0, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong \begin{cases} \Pi^2_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m), & \text{if} \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 0, \\ 1, & \text{if} \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Formulae of the kind $\exists Y_1' \dots \exists Y_l'(\Pi)$, where Π is a termal predicate, are called *conditions*. Every variable which occurs in Π and is different from $Y_1', \dots, Y_l'$ is called free in the condition $\exists Y_1' \dots \exists Y_l'(\Pi)$.

Let $\exists Y_1' \dots \exists Y_l'(\Pi)$ be a condition, let all free variables in C be among $Y_1, \dots, Y_m$, and $a_1, \dots, a_m$ be elements of B. The value $C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ is defined by the equivalence:

$$C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \iff$$
$$\exists t_1 \dots \exists t_l (\Pi_{\mathfrak{A}}(Y_1'/t_1, \dots, Y_l'/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

We assume that some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$ is fixed. We shall use superscripts to denote the corresponding codes.

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* (or just *definable*) in the structure $\mathfrak{A}$ if and only if there exists a recursive function γ of $r+1$ variables such that for all $n, x_1, \dots, x_r$, $C^{\gamma(n,x_1,\dots,x_r)}$ is a condition with free variables among $Z_1, \dots, Z_l, Y_1, \dots, Y_m$ and for some fixed elements $t_1, \dots, t_l$ of B the following equivalence is true:

$$(x_1, \dots, x_r, a_1, \dots, a_m) \in A \iff$$
$$\exists n \in \omega (C_{\mathfrak{A}}^{\gamma(n,x_1,\dots,x_r)}(Z_1/t_1, \dots, Z_l/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

If Π is a termal predicate and τ is a term, then $\exists Y_1' \dots \exists Y_l'(\Pi \supset \tau)$ is called a *conditional expression*.

Let $Q = \exists Y_1' \dots \exists Y_l'(\Pi \supset \tau)$ be a conditional expression with free variables among $X_1, \dots, X_a$, and $s_1, \dots, s_a \in B$. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \dots, X_a/s_a)$ of Q is the following subset of B:

$$\{\tau_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) | \Pi_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) \cong 0\}.$$

Let θ be a p.m.v. function in B. Then the function θ is called *definable in* $\mathfrak{A}$ if and only if for some c.e. set $\{Q^v\}_{v \in V}$ of conditional expressions with free variables among $X, Z_1, \dots, Z_r$ and for some fixed elements $t_1, \dots, t_r$ of B the following equivalence is true:

$$t \in \theta(s) \iff \exists v (v \in V \& t \in Q_{\mathfrak{A}}^v(Z_1/t_1, \dots, Z_r/t_r, X/s)).$$

In [14] Soskov has proved the following result.

Theorem 1. (Soskov [14]) *Let θ be a unary p.m.v. function in B. Then θ is $\forall$-admissible in $\mathfrak{A}$ if and only if θ is definable in $\mathfrak{A}$.*

Define $f_i(p) = \langle i-1, p \rangle$, $i = 1, \dots, n$ and $N_0 = \omega \setminus (Ran(f_1) \cup \dots \cup Ran(f_n))$. It is obvious that N_0 is an infinite recursive set and let $\{\mathbf{p_0}, \mathbf{p_1}, \dots\} = N_0$, where $\mathbf{p_i} < \mathbf{p_j}$ if $i < j$. In the case when the sequence f_i is infinite ($i \in \omega$) we can ensure N_0 to be infinite by taking for example $f_i(p) = \langle i-1, p, 0 \rangle$.

Next we recall the definition and some properties of normal enumerations [14] for the case of total enumerations. For every surjective mapping α^0 of N_0 onto B (called basis) we define a mapping α of ω onto B by the following inductive clauses:

(i) If $p \in N_0$, then $\alpha(p) = \alpha^0(p)$;

(ii) If $p = f_i(q)$, then $\alpha(q) = a$ and $\theta_i(a) = b$, then $\alpha(p) = b$.

Let $\sigma_1, \dots, \sigma_k$ be the partial predicates, defined by $\sigma_j(x) \cong R_j(\alpha(x))$, $j = 1, \dots, k$. Denote by $\mathfrak{B}$ the partial structure $\langle \omega; f_1, \dots, f_n; \sigma_1, \dots, \sigma_k \rangle$. It is well known [1, 14] that α is well defined and that the basis α^0 completely determines the normal enumeration $\langle \alpha, \mathfrak{B} \rangle$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal enumeration. We recall some obvious propositions for normal enumerations. Their proofs are the same as in [14].

Proposition 1. *For every $1 \le i \le n$ and $y \in \omega$, $\alpha(f_i(y)) = \theta_i(\alpha(y))$.*

Corollary 1. *Let $\tau(Y)$ be a term and $y \in \omega$. Then*

$$\alpha(\tau_{\mathfrak{B}}(Y/y)) = \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Proposition 2. *There exists an effective way for every x of ω to find $y \in N_0$ and a term $\tau(Y)$, such that $x = \tau_{\mathfrak{B}}(Y/y)$.*

If $\langle \alpha, \mathfrak{B} \rangle$ is a normal enumeration, we denote the set $\cup_{j=1}^{k} \{\langle j, x, z \rangle | \sigma_j(x) = z\}$ by R_α. In the general case we have to add some additional members, but in our situation the functions f_i are totally defined and no additional terms are needed. It is clear that for every $W \subseteq \omega$, $W \le_e R_\alpha$ if and only if $W \le_e \langle \mathfrak{B} \rangle$.

Proposition 3. *There exists an effective way for every natural u to find elements $y_1, \dots, y_m \in N_0$ and a termal predicate $\Pi(Y_1, \dots, Y_m)$ such that for every normal enumeration $\langle \alpha, \mathfrak{B} \rangle$,*

$$u \in R_\alpha \iff \Pi_{\mathfrak{A}}(Y_1/\alpha(y_1), \dots, Y_m/\alpha(y_m)) \cong 0.$$

Proposition 4. *There exists an effective way for every code v of a finite set E_v to find elements $y_1^v, \dots, y_{m_v}^v \in N_0$ and a termal predicate $\Pi^v(Y_1, \dots, Y_{m_v})$ such that for every normal enumeration $\langle \alpha, \mathfrak{B} \rangle$,*

$$E_v \subseteq R_\alpha \iff \Pi^v_{\mathfrak{A}}(Y_1/\alpha(y_1^v), \dots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

To be precise, we have to mention that, for the sake of simplicity, in the above proposition we have used just Π^v instead of $\Pi^{\gamma(v)}$ with some recursive function γ.

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a unary partial structure. *Type of the sequence* $b_1, \dots, b_m$ of elements of B is called the set

$$\{v | \Pi^v_{\mathfrak{A}}(X_1/b_1, \dots, X_m/b_m) \cong 0 \,\&\, \Pi^v \text{ is a termal predicate with variables} \in \{X_i\}_{i=1}^m\}.$$

The type of the sequence $b_1, \dots, b_m$ is denoted by $[b_1, \dots, b_m]_{\mathfrak{A}}$. The type of an element a of B is the type of the sequence a.

A condition is called *simple* if it does not contain free variables and it is in the form $\exists X_1 \Pi$, where Π is a termal predicate. Let $V_0^{\mathfrak{A}} = \{v | C_{\mathfrak{A}}^v \cong 0 \ \& \ C^v$ be a simple condition$\}$.

Definition 2. *Let $\mathcal{A}$ be a family of subsets of ω. A set $U \subseteq \omega^2$ is said to be universal for the family $\mathcal{A}$, if the following conditions hold:*

a) For every fixed $e \in \omega$, $\{x_1 | (e, x_1) \in U\} \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, then there exists e such that $A = \{x_1 | (e, x_1) \in U\}$.

Theorem 2. ([3]) *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$ is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

3. THE MAIN RESULT

We shall consider the *standard structure* $\mathfrak{N} = \langle \mathcal{P}(\omega); W_0, W_1, \dots; Non \rangle$, where $\mathcal{P}(\omega)$ is the family of all subsets of ω, $W_0, W_1, \dots$ is a fixed sequence of all c.e. sets considered as functions (e-operators) and Non is the family of all non-empty sets of naturals. To be more precise, Non is a partial unary predicate defined as follows: $Non(A) = 0$, if $A \neq \emptyset$ and $Non(\emptyset) \uparrow$.

First we shall consider the structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \dots; Non \rangle$, where $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, which we call standard as well. Let us mention that the functions $W_0, W_1, \dots$ are totally defined as e-operators and we do not use the equality among the predicates. Let in addition $\mathbf{W}$ be the family of all c.e. sets considered as e-operators.

Let $\mathcal{L}^*$ be the first order language $\langle \mathbf{f_0}, \mathbf{f_1}, \dots; \mathbf{T} \rangle$, containing a countable set of unary functional symbols $\mathbf{f_0}, \mathbf{f_1}, \dots$ and a unary predicate symbol $\mathbf{T}$. We call $\overline{\mathfrak{A}}$ a *generalized structure* if $\overline{\mathfrak{A}} = \langle B; \mathbf{\Theta}; R \rangle$, where B is a denumerable set, $\mathbf{\Theta}$ – denumerable set of unary functions on B and R is a unary predicate on B. When we consider structures with finite functions and finite predicates, the considerations do not depend on the enumerations of the functions and the predicates. In the case when we consider denumerable set of functions the situation is different.

Enumeration of a family $\mathbf{\Theta}$ of functions is any sequence $\theta_0, \theta_1, \dots$ such that $\mathbf{\Theta} = \{\theta_{\mathbf{0}}, \theta_{\mathbf{1}}, \dots\}$. We do not require all members of the sequence $\theta_0, \theta_1, \dots$ to be different.

Let us fix some enumeration $\theta_0^0, \theta_1^0, \dots$ of the family $\mathbf{\Theta}$ and consider the structure $\mathfrak{A}_0 = \langle B; \theta_0^0, \theta_1^0, \dots; R \rangle$.

We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of the generalized structure $\overline{\mathfrak{A}}$ if for every enumeration $\theta_0, \theta_1, \ldots$ of $\mathbf{\Theta}$ and every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A} = \langle B; \theta_0, \theta_1, \ldots; R \rangle$ the inequality $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$ holds.

Let us consider the structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non \rangle$ for the language $\mathcal{L}^*$ and define the m.v.f. $\Phi^A : \mathcal{P}(\omega)^A \setminus \{\emptyset\} \to \mathcal{P}(\omega)^A \setminus \{\emptyset\}$ as follows: $\Phi^A(B) = \{C | C \leq_e B \& C \neq \emptyset\}$ for nonempty B.

Proposition 5. *The m.v.f. Φ^A is definable in the structure $\mathfrak{N}^A$.*

Proof. Let Q^n be the conditional expression $\mathbf{T}(X)\&\mathbf{T}(\mathbf{f_n}(X)) \supset \mathbf{f_n}(X)$. Notice that the sequence $\{Q^n\}_{n\in\omega}$ is c.e. and

$$C \in Q^n_{\mathfrak{N}^A}(X/B) \iff Non(B)\&Non(W_n(B))\&C = W_n(B).$$

Then

$$\begin{aligned} &C \in \Phi^A(B) \iff C \leq_e B\&C \neq \emptyset\&B \neq \emptyset \iff \\ &\exists n(W_n(B) = C\&C \neq \emptyset\&B \neq \emptyset) \iff \exists n(C \in Q^n_{\mathfrak{N}^A}(X/B)). \end{aligned}$$

Proposition 5 is proved. □

Let $L_A = \{\langle n, x \rangle | x \in W_n(A)\}$. The following lemma is well-known, its proof is a simple application of the S^m_n-theorem.

Lemma 1. *There exists a recursive function δ of two variables such that for all naturals m, n and a set C of naturals the following equality is true:*

$$W_m(W_n(C)) = W_{\delta(m,n)}(C).$$

Let us fix a function δ in Lemma 1 and define the pair $\langle \alpha_0, \mathfrak{B}_0 \rangle$ as follows:

$\alpha_0(n) = W_n(A)$, $\mathfrak{B}_0 = \langle \omega; \varphi^0_0, \varphi^0_1, \ldots; \sigma^0 \rangle$, where $\varphi^0_i(x) = \delta(i, x)$, $i, x \in \omega$, $\sigma^0(x) \cong 0 \iff W_x(A) \neq \emptyset$ and $\sigma^0(x) \uparrow$ if $W_x(A) = \emptyset$.

Lemma 2. *The pair $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is an enumeration of the structure $\mathfrak{N}^A$.*

Proof. $W_i(\alpha_0(x)) = W_i(W_x(A)) = W_{\delta(i,x)}(A) = \alpha_0(\delta(i, x)) = \alpha_0(\varphi^0_i(x))$.
$Non(\alpha_0(x)) \cong 0 \iff W_x(A) \neq \emptyset \iff \sigma^0(x) \cong 0$. □

Let $W_A = \{n | \exists x(\langle n, x \rangle \in L_A)\} = \{n | W_n(A) \neq \emptyset\} = \{n | \sigma^0(n) \cong 0\}$.

Proposition 6. $W_A \equiv_e A$.

Proof. Let n_0 be a fixed element of ω and define the set B by the following equivalence: $\langle \langle n, x \rangle, m \rangle \in B \iff \langle n, x \rangle \in L_A \& m = n_0$. Obviously, $B \leq_e L_A \equiv_e A$.

Therefore, using the S_n^m-theorem we obtain

$$\begin{aligned}
\langle\langle n,x\rangle,m\rangle\in B \quad &\iff \exists v(\langle\langle\langle n,x\rangle,m\rangle,v\rangle\in W_a \& \emptyset\neq E_v\subseteq A)\\
&\qquad\qquad\qquad \text{(for some fixed natural } a)\\
&\iff \exists v(\langle\langle m,v\rangle,\langle n,x\rangle\rangle\in W_b \& \emptyset\neq E_v\subseteq A)\\
&\qquad\qquad\qquad \text{(for some fixed natural } b)\\
&\iff \exists v(\langle m,v\rangle\in W_{\gamma(\langle n,x\rangle)} \& \emptyset\neq E_v\subseteq A)\\
&\qquad\qquad\qquad \text{(for some fixed recursive function } \gamma)\\
&\iff m\in W_{\gamma(\langle n,x\rangle)}(A).
\end{aligned}$$

We will show that $L_A\leq_m W_A$ by recursive function γ.

Let us assume $\langle n,x\rangle\in L_A$. Then $\langle\langle n,x\rangle,n_0\rangle\in B$, thus $n_0\in W_{\gamma(\langle n,x\rangle)}(A)$, i.e. $W_{\gamma(\langle n,x\rangle)}(A)\neq\emptyset$, hence $\gamma(\langle n,x\rangle)\in W_A$.

Let us suppose that $\gamma(\langle n,x\rangle)\in W_A$. Then $\exists m(m\in W_{\gamma(\langle n,x\rangle)}(A))$, thus $n_0\in W_{\gamma(\langle n,x\rangle)}(A)$. Therefore $\langle\langle n,x\rangle,n_0\rangle\in B$ and $\langle n,x\rangle\in L_A$.

We proved the equivalence $\langle n,x\rangle\in L_A \iff \gamma(\langle n,x\rangle)\in W_A$, i.e. $L_A\leq_m W_A$. Therefore, $L_A\leq_e W_A$.

Conversely,

$$\begin{aligned}
n\in W_A \quad &\iff \exists x(\langle n,x\rangle\in L_A) \iff \exists x(x\in W_n(A))\\
&\iff \exists x\exists v(\langle x,v\rangle\in W_n \& \emptyset\neq E_v\subseteq A)\\
&\iff \exists v(\exists x(\langle n,v\rangle\in W_{\gamma_1(x)}) \& \emptyset\neq E_v\subseteq A)\\
&\iff \exists v(\langle n,v\rangle\in W_a) \& \emptyset\neq E_v\subseteq A) \iff n\in W_a(A)
\end{aligned}$$

for some fixed recursive function γ_1 and a fixed natural a. Hence, $W_A\leq_e A$. □

Lemma 3. *Let τ^v be the term with a code v. There exists a recursive function γ_0 such that for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v the equality $\tau^v_{\mathfrak{N}^A}(X/A)=W_{\gamma_0(v)}(A)$ holds.*

Proof. Decode $\tau^v(X)$ as a sequence of $f_{i_1},f_{i_2},\ldots,f_{i_p}$ and variable X. Then consider the composition of the operators $W_{i_1},W_{i_2},\ldots,W_{i_p}$ over A and use the recursive function δ. Thus there exists an effective way for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v to find a natural number n such that $\tau^v_{\mathfrak{N}^A}(X/A)=W_n(A)$. □

Lemma 4. $[A]_{\mathfrak{N}^A}\equiv_m W_A$.

Proof. Recall that $[A]_{\mathfrak{N}^A}=\{v|\tau^v_{\mathfrak{N}^A}(X/A)\neq\emptyset\}$. Let γ_0 be the recursive function from the previous lemma, then $v\in[A]_{\mathfrak{N}^A}\iff\tau^v_{\mathfrak{N}^A}(X/A)\neq\emptyset\iff W_{\gamma_0(v)}(A)\neq\emptyset\iff\sigma^0(\gamma_0(v))\cong 0\iff\gamma_0(v)\in W_A$. Thus, $[A]_{\mathfrak{N}^A}\leq_m W_A$.

Conversely, $n\in W_A\iff W_n(A)\neq\emptyset\iff$ the term $\mathbf{f}_n(X)$ with code $v(n)$ satisfies $(\mathbf{f}_n(X))^{v(n)}_{\mathfrak{N}^A}(X/A)\neq\emptyset$, i.e. $W_A\leq_m[A]_{\mathfrak{N}^A}$. □

Theorem 3. *The enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is the least enumeration of the structure $\mathfrak{N}^A$.*

Proof. According to Theorem 2, having in mind $W_A = V_0^{\mathfrak{N}^A}$, we need to show that all types of elements B such that B is a set of naturals and $B \leq_e A$ satisfy the condition $[B]_{\mathfrak{N}^A} \leq_e [A]_{\mathfrak{N}^A}$ and that there exists a universal set with e-degree $deg_e(A)$ for all types $[B]_{\mathfrak{N}^A}$.

Let $B \leq_e A$. Then there exists an e-operator W_n such that $W_n(A) = B$. Therefore, $v \in [B]_{\mathfrak{N}^A} \iff$ the code v_1 of the term $\mathbf{f_n}(\tau^v)$ belongs to $[A]_{\mathfrak{N}^A}$, thus $[B]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{N}^A}$. Further, using the type $[A]_{\mathfrak{N}^A}$, we define the set U^A by the equivalence: $(n, v) \in U^A \iff \exists v_1(\tau^{v_1} = \mathbf{f_n}(\tau^v) \& v_1 \in [A]_{\mathfrak{N}^A})$. Actually, we could define U^A by the equivalence: $(n, v) \in U^A \iff \langle n, v \rangle \in L_A$, as well. It is obvious that U^A is universal for the family of all types of the structure $\mathfrak{N}^A$. □

Let us consider the structure $\mathfrak{D}^A = \langle \mathcal{P}(\omega)^A; \Phi^A \rangle$. The following definition is natural, although it is not used because normally we do not consider structures with p.m.v. functions.

Definition 3. *Enumeration of the structure $\mathfrak{D}^A$ is called the pair $\langle \alpha, \mathfrak{B} \rangle$, where $\alpha : \omega \to \mathcal{P}(\omega)^A$, $\mathfrak{B} = \langle \omega; \varphi \rangle$ and φ is a partial m.v.f. in ω, such that for all natural n the equality $\alpha(\varphi(n)) = \Phi^A(\alpha(n))$ holds (here, we mean equality between sets).*

Proposition 7. *There exists an enumeration $\langle \alpha_0, \mathfrak{B}' \rangle$ of the structure $\mathfrak{D}^A$ such that $\langle \mathfrak{B}' \rangle \equiv_e A$.*

Proof. Let us recall that $\alpha_0(n) = W_n(A)$ and define the partial m.v.f. φ^0 as follows: $m \in \varphi^0(n) \iff \exists k(\sigma^0(m) \cong 0 \& \sigma^0(n) \cong 0 \& \delta(k, n) = m)$. It is clear that $\langle G_\varphi \rangle \leq_e A$. Then

$$\begin{aligned}
&C \in \alpha_0(\varphi^0(n)) \iff \exists m(m \in \varphi(n) \& \alpha_0(m) = C) \iff \\
&\exists m(\exists k(\sigma^0(m) \cong 0 \& \sigma^0(n) \cong 0 \& \delta(k, n) = m) \& W_m(A) = C) \iff \\
&\exists m \exists k(W_m(A) = W_k(W_n(A)) \& C = W_m(A) \neq \emptyset \& W_n(A) \neq \emptyset) \iff \\
&\exists m(W_m(A) \leq_e W_n(A) \& C = W_m(A) \neq \emptyset \& W_n(A) \neq \emptyset) \iff \\
&\exists m(C = W_m(A) \in \Phi^A(W_n(A)) \iff C \in \Phi^A(\alpha_0(n)).
\end{aligned}$$

Therefore $\langle \alpha_0, \mathfrak{B}' \rangle$ is an enumeration of $\mathfrak{D}^A$.

Further, let us fix some a such that $\alpha_0(a) = A$. Then $W_n(A) = W_n(W_a(A)) = W_{\delta(n,a)}(A)$ and hence

$$\begin{aligned}
W_A &= \{n | W_n(A) \neq \emptyset\} \equiv_e \{\delta(n, a) | W_{\delta(n,a)}(A) \neq \emptyset\} \\
&= \{\delta(n, a) | \sigma^0(\delta(n, a)) \cong 0\} \equiv_e \{\delta(n, a) | \delta(n, a) \in \varphi^0(a)\} \leq_e \langle G_\varphi \rangle \equiv_e \langle \mathfrak{B}' \rangle .
\end{aligned}$$

Proposition 7 is proved. □

Lemma 5. *There exist c.e. sets $V^{[n]}, n \in N, V', V^{[S]}$ such that the effective sequence of compositions $\{V^{[0]}(V^{[S]})^n V'\}_{n \in \omega}$ is recursively isomorphic to the sequence $\{W_n\}_{n \in \omega}$.*

Proof. Let us notice first that $V^{[0]}(V^{[S]})^n V'$ means the following:

$$V^{[0]}(V^{[S]})^0 V' = V^{[0]}V'; \quad V^{[0]}(V^{[S]})^{n+1}V' = ((V^{[0]}(V^{[S]})^n)V^{[S]})V'.$$

Let us denote

$$V^{[n]}=\{\langle x,v\rangle | x \in \omega \& E_v=\{\langle n,x\rangle\}\}, \ \ V^{[S]}=\{\langle\langle n,x\rangle,v\rangle | n,x \in \omega \& E_v=\{\langle n+1,x\rangle\}\}.$$

Further, let $V = \{\langle n,x\rangle | x \in W_n\}$ and $V' = \{\langle\langle k,x\rangle,v\rangle | \langle k,\langle x,v\rangle\rangle \in V\}$. Then

$$\begin{aligned} x \in V^{[n]}V'(X) &\iff \exists v_1(\langle x,v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n,x\rangle\} \subseteq V'(X)) \\ &\iff \exists v_1(\langle x,v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n,x\rangle\} \& \langle n,x\rangle \in V'(X)) \\ &\iff \exists v(\langle\langle n,x\rangle,v\rangle \in V' \& \emptyset \neq E_v \subseteq X) \\ &\iff \exists v(\langle x,v\rangle \in V_{[n]} \& \emptyset \neq E_v \subseteq X) \iff x \in V_{[n]}(X)\,, \end{aligned}$$

$$\begin{aligned} x \in V^{[n]}V^{[S]}(X) &\iff \exists v_1(\langle x,v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n,x\rangle\} \subseteq V^{[S]}(X)) \\ &\iff \exists v_1(\langle x,v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n,x\rangle\} \& \langle n,x\rangle \in V^{[S]}(X)) \\ &\iff \exists v(\langle\langle n,x\rangle,v\rangle \in V^{[S]} \& E_v = \{\langle n+1,x\rangle\} \subseteq X) \\ &\iff \exists v(\langle x,v\rangle \in V^{[n+1]} \& E_v = \{\langle n+1,x\rangle\} \subseteq X) \\ &\iff x \in V^{[n+1]}(X). \end{aligned}$$

We shall prove by induction the equivalence

$$x \in V^{[0]}(V^{[S]})^n V'(X) \iff x \in V_{[n]}(X). \tag{$*$}$$

Indeed, $x \in V^{[0]}(V^{[S]})^0 V'(X) \iff x \in V^{[0]}V'(X) \iff x \in V_{[0]}(X)$. Let us assume the equivalence $(*)$ is true. Then
$x \in V^{[0]}(V^{[S]})^{n+1}V'(X) \iff x \in V^{[n+1]}V'(X) \iff x \in V_{[n+1]}(X)$. □

The next two corollaries are obvious.

Corollary 2. *The structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is equivalent to the structure $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$, where $V^{[0]}, V^{[S]}, V'$ is the c.e. sets from the previous lemma.*

Corollary 3. *For any set A of naturals the set $\mathcal{P}(\omega)^A$ is finitely generated in the structure $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$ by the single element A.*

Proposition 8. *For any enumeration $\{V_0, V_1, \ldots\}$ of the family $\boldsymbol{W}$ the structure $\mathfrak{M}^A = \langle \mathcal{P}(\omega)^A; V_0, V_1, \ldots; Non\rangle$ admits a least enumeration $\langle \alpha, \mathfrak{B}\rangle$ such that $A \leq_e \langle \mathfrak{B}\rangle$.*

Proof. Let $\alpha^0 : N_0 \to \mathcal{P}(\omega)^A$ be defined as follows: $\alpha^0(\mathbf{p_n}) = V_n(A)$. Take α^0 as a basis of a normal enumeration $\langle \alpha, \mathfrak{B}\rangle$, where $\mathfrak{B} = \langle \omega; \varphi_0, \varphi_1, \ldots; \sigma\rangle$ and $\varphi_i(x)$ is a computable function of both variables i, x. According to Proposition 2, there exists an effective way for any x to find $y = \mathbf{p_n} \in N_0$ and a term τ such that $x = \tau_{\mathfrak{B}}(Y/y)$; thus $\alpha(x) = \tau_{\mathfrak{A}}(Y/\alpha(y)) = \tau_{\mathfrak{A}}(Y/\alpha^0(\mathbf{p_n})) = \tau_{\mathfrak{A}}(Y/V_n(A)) = \tau'_{\mathfrak{A}}(Y/A)$, where $\tau' = \tau(\mathbf{f}_n(Y))$.

Let us denote $V_A = \{n|\sigma(n) \cong 0\}$. Then, using the term τ' obtained above, $x \in V_A \iff \sigma(x) \cong 0 \iff \alpha(x) \neq \emptyset \iff \tau'_{\mathfrak{A}}(Y/A) \neq \emptyset \iff v' \in [A]_{\mathfrak{M}^A}$ for the code v' of the term τ'. Thus, having in mind that we can find v' effectively from x, we have proved that $V_A \leq_m [A]_{\mathfrak{M}^A}$.

Analogously, let $v' \in [A]_{\mathfrak{M}^A}$, $\tau^{v'} = \tau^{v'}(Y)$ and n be a fixed natural, such that $\alpha^0(\mathbf{p}_n) = V_n(A) = A$, where $y = \mathbf{p}_n \in N_0$. Then $\tau^{v'}_{\mathfrak{A}}(Y/A) = \alpha(\tau^{v'}_{\mathfrak{B}}(Y/y)) \neq \emptyset$ and let $x = \tau^{v'}_{\mathfrak{B}}(Y/y)$. Then $\sigma(x) \cong 0$ and $x \in V_A$. Therefore, $[A]_{\mathfrak{M}^A} \leq_m V_A$.

Hence, $[A]_{\mathfrak{M}^A} \equiv_m V_A$ and $\langle \mathfrak{B} \rangle \equiv_e [A]_{\mathfrak{M}^A} \equiv_e V_A$. □

Corollary 4. $W_A \leq_e V_A$.

Proof. Let $V_{i_0} = V^{[0]}$, $V_{i_1} = V^{[S]}$ and $V_{i_2} = V'$ and consider the sequence of terms $\tau^{v(n)}$, where $\tau^{v(n)} = \mathbf{f}_{i_0} \circ \mathbf{f}^n_{i_1} \circ \mathbf{f}_{i_2}(X)$. Here, $\mathbf{f}^n_{i_1}$ means n times the term $\mathbf{f}_{i_1}$. Then it is easy to check that $n \in [A]_{\mathfrak{N}^A} \iff v(n) \in [A]_{\mathfrak{M}^A}$. Thus we have proved that $[A]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{M}^A}$, hence $W_A \leq_e V_A$. □

Corollary 5. *The enumeration* $\langle \alpha_0, \mathfrak{B}_0 \rangle$ *is the least for the generalized structure* $\overline{\mathfrak{N}^A} = \langle \mathcal{P}(\omega)^A; \boldsymbol{W}; Non \rangle$.

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