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FACTORIZATIONS OF SOME SIMPLE LINEAR GROUPS

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In this paper we have considered finite simple groups G which can be represented as a product $G = AB$ of two of their proper non-Abelian simple subgroups A and B. Any such representation is called a (simple) factorization of G. Supposing that G belongs to the infinite series of linear groups with some restrictions to the dimension of the natural vector space onto which G acts we have determined all the factorizations of G.

Keywords: Finite simple groups, groups of Lie type, factorizations of groups

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1. INTRODUCTION

Let G be a finite (simple) group. We are interested in the factorizations of G into the product of two simple subgroups. In the present work we suppose that G is the simple linear group $L_n(q)$ and start our investigation of this series of groups in case that n is at most 7. The results obtained are included in the following

Theorem. *Let $G = L_n(q)$ with $2 \le n \le 7$. Suppose $G = AB$ where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $n = 2,\ q = 9$ *and* $A \cong B \cong A_5$*;*

(2) $n = 4,\ q = 2$ *and* $A \cong L_3(2)$*,* $B \cong A_6$ *or* A_7*;*

(3) $n = 4,\ q > 2, q \not\equiv 1 \pmod 3$ *and* $A \cong L_3(q), B \cong PSp_4(q)$*;*

(4) $\quad n = 6,\ q \not\equiv 1 \pmod 5$ *and* $A \cong L_5(q), B \cong PSp_6(q)$*;*

(5) $\quad n = 6,\ q = 2^s > 2, s \not\equiv 0 \pmod 4$ *and* $A \cong L_5(q), B \cong G_2(q)$*.*

The factorizations of the groups $L_2(q)$ have been determined in [7]. This gives (1) in the theorem. The groups $L_3(q)$ have no factorizations, see [2]. The factorizations of the groups $L_4(2) \cong A_8$ and $L_4(q)$ (q odd, $q \not\equiv 1 \pmod 8$) have been determined in [10]. This leads to the (2) and (3) (q odd, $q \not\equiv 1 \pmod 8$) in the theorem. Some isolated linear groups as $L_4(4), L_5(2)$ and $L_6(2)$ have been treated in [1]and [3]. It has been proved that the groups $L_4(4)$ and $L_5(2)$ have no factorizations whilst the group $L_6(2)$ has one factorization listed in (4) (with $q = 2$) in the theorem.

The factorizations of all the classical simple groups into the product of two maximal subgroups (so called maximal factorizations) have been determined in [9]. Particularly, an explicit list of the maximal factorizations of the groups $L_n(q)$ have also been given in [9]. We shall make use of this result here.

Note that,using the result of the above theorem especially for $n = 4$ ($G = L_4(q)$) and the results in previously published papers [4], [5] and [6], we have finished determination of the factorizations (with two proper simple subgroups) of all the finite simple groups of Lie type of Lie rank three. Indeed, only the groups $PSU_7(q)$ and $P\Omega_8^-(q)$ of Lie rank three are not covered from the mentioned results; but according to [9] these groups have no maximal factorizations and so it follows they have no factorizations with any two proper factors A and B as well.

In our considerations we shall freely use the notation and basic information on the finite (simple) classical groups given in [8]. Let V be the n-dimensional vector space over the finite field $GF(q)$ on which $G = L_n(q)$ acts naturally, and let P_k be the stabilizer in G of a k-dimensional subspace of V. From Proposition 4.1.17 in [8] we can obtain the structure of P_k. In particular, it follows that $P_1 \cong P_{n-1} \cong \{[q^{n-1}] : GL_{n-1}(q)\}/Z_{(n,q-1)}$. From this it follows immediately that P_1 ($\cong P_{n-1}$) contains a subgroup isomorphic to $L_{n-1}(q)$ if and only if $(n-1, q-1) = 1$.

If a, b are positive integers and $(a, b) = 1$, then $Ord_a(b)$ denotes the multiplicative order of b modulo a (i.e. the least positive integer n with $b^n \equiv 1 \pmod a$).

The following lemma is needed in the proof of the theorem.

Lemma 1.1 (see [9]). *Let q be a prime power and n a positive integer. Then there exists a prime r such that $Ord_r(q)$ unless $n = 6$ and $q = 2$ or $n = 2$ and q a Mersenne prime.*

Such a prime r is called a *primitive prime divisor* of $q^n - 1$.

2. PROOF OF THE THEOREM

Let $G = L_n(q)$, where $q = p^s$ and p is a prime. In our assumptions here $2 \leq n \leq 7$, and $G = AB$ where A, B are proper non-Abelian simple subgroups of G. According to the information about known factorizations of G provided above, it remains to treat G in the cases $n = 4$, 5, 6 and 7. If $G = L_4(q)$ we only suppose $q > 2$ and no other restrictions on q will be applied, as for $G = L_5(q)$ or $L_6(q)$ we assume that $q > 2$ as well. The list of maximal factorizations of G is given in [9]. In case that $G = L_n(q)$ with $n = 5$ or $n = 7$ only one maximal factorization appears with one factor a (maximal) subgroup of G isomorphic to $\{Z_{q^n-1/q-1}.n\}/Z_{(n,q-1)}$. Obviously, there is no choice for one of the groups A and B to be a non-Abelian simple subgroup of G. Now we proceed with the group $G = L_n(q)$ where $n = 4$ or $n = 6$ and choose (by Lemma 1.1) a primitive prime divisor of $p^{sn} - 1$ (recall that if $n = 6$ then $q > 2$) to be a divisor of $|B|$. Using the list of maximal factorizations in [9], by order considerations, we come to the following possibilities:

1) $n = 4$ or $n = 6$ and $A \cong L_{n-1}(q)$ (in P_1), $B \cong PSp_n(q)$ with $(n-1, q-1) = 1$;

2) $n = 6$ and $A \cong L_5(q)$ (in P_1), $B \cong G_2(q)$ with $q = 2^s > 2, s \not\equiv 0 \pmod 4$;

3) $n = 6$ and $A \cong L_5(q)$ (in P_1), $B \cong L_3(q^2)$ with $(5, q-1) = 1$.

We consider these possibilities case by case.

Case 1. These are the factorizations in (3) and (4) of the theorem. It remains to show that these factorizations actually exist. From Proposition 3.3 in [10] we have

$$SL_n(q) = SL_{n-1}(q).Sp_n(q)$$

with natural embeddings of $SL_{n-1}(q)$ and $Sp_n(q)$ in $SL_n(q)$. Moreover, the intersection of these naturally embedded subgroups $SL_{n-1}(q)$ and $Sp_n(q)$ is a subgroup isomorphic to $Sp_{n-2}(q)$ with natural embedding in $SL_n(q)$, too. Factoring out by $Z(SL_n(q))$, we obtain the factorizations in (3) and (4), as $SL_{n-1}(q) \equiv L_{n-1}(q)$ (by the condition $(n-1, q-1) = 1$).

Case 2. Here $q = 2^s > 2$, $s \not\equiv 0 \pmod 4$, and from the previous case it follows that $G = A.B_1$ where $A \cong L_5(q)$, $B_1 \cong PSp_6(q)$, and $A \cap B_1 \cong PSp_4(q)$. In [5] we have proved that $B_1 = (A \cap B_1).B$ where $B \cong G_2(q)$ with an explicit construction in B_1; also $(A \cap B_1) \cap B$ $(= A \cap B) \cong L_2(q)$. This leads, by order considerations, to the factorization $G = A.B$ in (5) of the theorem.

Case 3. This case is similar to one of those considered in [4]. Denote $D = A \cap B$; then $|D| = q(q^4 - 1).(6, q-1)/(3, q^2-1)$ (recall $(5, q-1) = 1$). By the known subgroup structure of $L_3(q^2)$, it follows that D is contained in a subgroup of B isomorphic to

$$H = \left\{ \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a, b, c \in GF(q^2);\ A \in GL_2(q^2),\ a. \det A = 1 \right\} \Big/ \langle \omega E \rangle$$

where ω is an element of order $(3, q^2 - 1)$ in $GF(q^2)$. Further, $H = FK$ and $F \triangleleft H, F \cap K = 1$ where

$$F = \left\{ \left(\begin{array}{c|cc} 1 & b & c \\ \hline 0 & & \\ & & E \\ 0 & & \end{array} \right) \mid b, c \in GF(q^2) \right\} \cong E_{q^4},$$

$$K = \left\{ \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a \in GF(q^2);\ A \in GL_2(q^2),\ a. \det A = 1 \right\} \Big/ \langle \omega E \rangle$$

$\cong GL_2(q^2)/Z_{(3,q^2-1)}$.

Suppose that $T = D \cap F \neq 1$. Then $T \triangleleft D$ and $T \cong E_{p^k}$ where $p \leq p^k \leq q$. The centralizer of any non-identity p-element in $L_3(q^2)$ has order dividing $q^6(q^2 - 1)$. Hence $|C_D(T)|$ divides $q(q^2 - 1).(6, q - 1)/(3, q^2 - 1)$. Then $|D/C_D(T)|$ is divisible by $q^2 + 1$. However, $D/C_D(T)$ is a subgroup of $Aut(T) \cong GL_k(p)$, so

$$|GL_k(p)| = p^{k(k-1)/2}.(p - 1) \cdots (p^k - 1)$$

must be divisible by $q^2 + 1$ which (in view of $p^k \leq q$) contradicts Lemma 1.1. Indeed,using this lemma we can choose a primitive prime divisor of $p^{4s} - 1$ dividing $q^2 + 1$ but not dividing the order of $GL_k(p)$, which is impossible.

Thus $D \cap F = 1$ and hence D is isomorphic to a subgroup of $H/F \cong K$. Of course, K contains a subgroup $L \cong SL_2(q^2)$ of index $(q^2 - 1)/(3, q^2 - 1)$ and then $D \cap L$ is a proper subgroup of L of order divisible by $q(q^2 + 1).(6, q - 1)$. It follows that $L_2(q^2)$ has a proper subgroup of order divisible by $q(q^2 + 1)$ which (for $q \geq 3$) contradicts the structure of $L_2(q^2)$.

This completes the proof.

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