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EXACT EFFECTIVE ENUMERATIONS OF TOTAL FUNCTIONAL STRUCTURES[1]

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In the present paper is considered an introduced by the author notion of exact effective enumeration. In some sense those enumerations are "the least one". It is proved that in case of total structure without predicates there exist exact effective enumerations, even infinitely many such enumerations.

Keywords: Structure, Total structure, Enumeration, Exact effective enumeration
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1. INTRODUCTION

In [1] Lacombe and in [2] Moschovakis have defined different kinds of computability in abstract structure. The first one uses enumerations of the structure and the second one, called search computability, uses only the functions and predicates in the structure. Mosckovakis [3] has proved that both computabilities are equivalent in the case when the equality is among the basic predicates. Soskov [4] has proved that both computabilities coincide in the general case.

Skordev has defined an "effective" version of Lacombe's computability as follows: It is said φ is effective in $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff φ has a partial recursive "associate". It is said φ is e-admissible iff φ is effective in all effective enumerations of the structure $\mathfrak{A}$. Skordev has stated a conjecture in the case when the structure has at most a denumerable domain, and it admits an effective enumeration. Skordev's conjecture

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is that e-admissibility coincides with search computability. Attempts were made to prove Skordev's conjecture [5, 6, 7, 8]. They were successful for some special cases, but not for the general one. As Manasse, Chisholm, Vencov [9, 10, 11] showed, the above mentioned conjecture wasn't true. Nevertheless, it is interesting to know for what kind of structures Skordev's conjecture is valid. The author puts the question: Which are the structures $\mathfrak{A}$ for which we could find effective enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ such that, for every function φ, φ is effective in $\langle\alpha_0, \mathfrak{A}_0\rangle$ iff φ is e-admissible. Such kind of enumeration we shall call an *exact effective enumeration*. In his master thesis Stoyan Atanasov showed that there exist exact effective enumerations for the structures with only unary total functions and no predicates. It is natural to expect that this kind of enumerations have to have some minimal(maximal) properties. In this paper we investigate exact enumerations for the structures with only total functions and no predicates. Different partial orders can be taken in the set of enumerations. Here we choose a partial order in the set of enumerations as the one in [12]. We prove that for total structures there exist exact enumerations. Furthermore, there exist infinitely many mutually incomparable exact enumerations. It is shown that above (in the considered partial order) every strongly effective enumeration there exists an exact enumeration.

2. PRELIMINARIES

In this paper we use ω to denote the set of all natural numbers. We shall recall some definitions from [4, 7].

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; \Sigma_1, \dots, \Sigma_k\rangle$ be a partial structure, where B is an arbitrary most denumerable set, $\theta_1, \dots, \theta_n$ are partial functions of many arguments on B, and $\Sigma_1, \dots, \Sigma_k$ are partial predicates of many arguments on B. The relational type of $\mathfrak{A}$ is the order pair $\langle\langle k_1, \dots, k_n\rangle, \langle l_1, \dots, l_k\rangle\rangle$, where each θ_i is k_i-ary and each Σ_j is l_j-ary. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false. We use also $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$ to denote the domain and the range of the function f respectively.

An *effective enumeration* of the structure $\mathfrak{A}$ is any ordered pair $\langle\alpha, \mathfrak{B}\rangle$ where $\mathfrak{B} = \langle\omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k\rangle$ is a partial structure of the same relational type as $\mathfrak{A}$, and α is a surjective mapping of ω onto B such that the following conditions hold:

a) $\varphi_1, \dots, \varphi_n$ are partial recursive (p.r.) and $\sigma_1, \dots, \sigma_k$ are recursively enumerable (r.e.);

b) $\alpha(\varphi_i(x_1, \dots, x_{k_i})) \cong \theta_i(\alpha(x_1), \dots, \alpha(x_{k_i}))$ for every natural numbers $x_1, \dots, x_{k_i}$, $1 \le i \le n$;

c) $\sigma_j(x_1, \dots, x_{l_j}) \cong \Sigma_j(\alpha(x_1), \dots, \alpha(x_{l_j}))$ for all natural numbers $x_1, \dots, x_{l_j}$, $1 \le j \le k$.

If θ is a partial function of m variables on B, then we say θ is effective in the enumeration $\langle\alpha, \mathfrak{B}\rangle$ iff $\mathrm{Dom}(\theta) \ne \emptyset$ and there exists such p.r. function f that for all natural numbers $i_1, \dots, i_m$,

$$\theta(\alpha(i_1), \dots, \alpha(i_m)) \cong \alpha(f(i_1, \dots, i_m)).$$

We exclude the trivial case of an empty function because it doesn't depend on any enumeration.

It is said that θ is *e-admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff θ is effective in every effective enumeration of the structure $\mathfrak{A}$.

We say that $\langle \alpha, \mathfrak{B}_0 \rangle$ is an *exact effective enumeration* if it is an effective enumeration, and for every partial function θ, θ is e-admissible iff θ is effective in $\langle \alpha, \mathfrak{B}_0 \rangle$.

It is well known that there are structures which don't have effective enumerations [13].The above definitions don't make good sense in all those cases. We'll consider those definitions only in case when the structure admits an effective enumeration. Actually, when the structure has only total functions and no predicates it admits an effective enumeration.

There isn't an established definition for partial order of the set of enumerations. There are different possibilities to define partial order, depending on different reducibilities in the set of all sets of natural numbers and aims of research. Here we shall take one of the possibilities, connected with m-reducibility.

Definition 1. *It is said that $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$ iff there exist partial recursive function f such that for all natural numbers n,*

$$\alpha_0(n) \cong \alpha(f(n)).$$

It is said that $\langle \alpha_0, \mathfrak{B}_0 \rangle$, $\langle \alpha, \mathfrak{B} \rangle$ are incomparable iff neither $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$ nor $\langle \alpha, \mathfrak{B} \rangle \leq \langle \alpha_0, \mathfrak{B}_0 \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n functional symbols $\mathbf{f_1}, \ldots, \mathbf{f_n}$ and k predicate symbols $\mathbf{T_1}, \ldots, \mathbf{T_k}$, where $\mathbf{f_i}$ is k_i-ary and $\mathbf{T_j}$ is l_j-ary. We add a new unary predicate symbol $\mathbf{T_0}$ which will represent the unary total predicate Σ_0, where $\Sigma_0(s) = 0$ for all $s \in B$.

Let us have a denumerable set of variables. We shall use capital letters X, Y, Z and the same letters by indexes to denote variables.

If τ is a term in the language $\mathcal{L}$, then we write $\tau(X_1, \ldots, X_l)$ to denote that all the variables in the term τ are among $X_1, \ldots, X_l$. If $s_1, \ldots, s_l$ are elements of B and $\tau(X_1, \ldots, X_l)$ is a term, then by $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_l/s_l)$ we denote the value of the term τ in the structure $\mathfrak{A}$ over the elements $s_1, \ldots, s_l$, if it exists.

We intend to show that all structures with total functions and no predicates have effective exact enumerations. Let from now on $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n \rangle$ be an arbitrary structure, where $\theta_1, \ldots, \theta_n$ are total functions and B is a denumerable set. The case when B is a finite set is analogous. As in [7] we shall construct a special kind of enumerations. Later this kind of enumerations is generalized and called normal enumerations [4].

Let $\langle p_1, \ldots, p_n \rangle$be some fixed coding function of all finite sequences of natural numbers.

Define $f_i(p_1, \ldots, p_{k_i}) = \langle i-1, p_1, \ldots, p_{k_i} \rangle$, $i = 1, \ldots, n$ and

$N_0 = \omega \setminus (\mathrm{Ran}(f_1) \cup \cdots \cup \mathrm{Ran}(f_{k_n}))$. It is obvious that N_0 is a recursive set. For every surjective mapping α^0 of N_0 onto B(called basis) we define a partial mapping of ω onto B by the following inductive clauses:

(i) If $p \in N_0$, then $\alpha(p) = \alpha^0(p)$;

(ii) If $p = f_i(q_1, \ldots, q_{k_i})$, $\alpha(q_1) = s_1, \ldots, \alpha(q_{k_i}) = s_{k_i}$ and $\theta_i(s_1, \ldots, s_{k_i}) = t$, then $\alpha(p) = t$.

It is well known that α is well defined and let $\mathfrak{B} = \langle \omega; f_1, \ldots, f_n \rangle$. We shall recall some obvious propositions for such kind $\langle \alpha, \mathfrak{B} \rangle$. The proofs are the same as in the case of normal enumerations [4].

Proposition 1. *For every $1 \leq i \leq n$ and $p_1, \ldots, p_{k_i} \in \omega$,*

$$\alpha(f_i(p_1, \ldots, p_{k_i})) = \theta_i(\alpha(p_1), \ldots, \alpha(p_{k_i})).$$

Corollary 1. *Let $\tau(Y_1, \ldots, Y_m)$ be a term and $p_1, \ldots, p_m \in \omega$. Then*

$$\alpha(\tau_{\mathfrak{B}}(Y_1/p_1, \ldots, Y_m/p_m)) = \tau_{\mathfrak{A}}(\alpha(p_1), \ldots, \alpha(p_m)).$$

Corollary 2. *$\langle \alpha, \mathfrak{B} \rangle$ is an effective enumeration.*

Proposition 2. *There exists an effective way to define for every p of ω elements $q_1, \ldots, q_m \in N_0$ and term $\tau(Y_1, \ldots, Y_m)$ such that*

$$p = \tau_{\mathfrak{B}}(Y_1/p_1, \ldots, Y_m/p_m).$$

A term τ which we define by the above proposition from the natural number p we will denote by τ^p.

We can define the just mentioned enumerations also in the following way. Let $B = \{a_0, a_1, \ldots\}$, where $a_0, a_1, \ldots$ are different. Let $A_0, A_1, \ldots$ be a sequence of disjoint subsets of N_0 such that $\bigcup_{i \in \omega} A_i$. We define $[A_0], [A_1], \ldots$ as follows:

(a) If $p \in A_i$, then $p \in [A_i]$;

(b) $1 \leq i \leq n$ and $p_1 \in [A_{j_1}], \ldots, p_{k_i} \in [A_{j_{k_i}}]$ and $a_q = \theta_i(a_{j_1}, \ldots, a_{j_{k_i}})$, then $f_i(p_1, \ldots, p_{k_i}) \in [A_q]$.

Taking $\alpha^0(A_i) = a_i$ we have the basis and then we have $\alpha^{-1}(a_i) = [A_i]$. From now on if we define some sequence $[A_0], [A_1], \ldots$ of disjoint subsets of N_0 we shall have in mind the above mentioned enumeration.

Corollary 3. *Let $A_0, A_1, \ldots$ be a sequence of disjoint nonempty subsets of N_0. Then $\langle \alpha, \mathfrak{B} \rangle$ is an effective enumeration of the structure $\mathfrak{A}$.*

3. THE MAIN RESULTS

Theorem 1. *Let* $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n \rangle$ *be a structure, where* $\theta_1, \dots, \theta_n$ *are total functions. Then there exists an exact effective enumeration* $\langle \alpha, \mathfrak{B} \rangle$.

Proof. First we shall recall that in [7] it is shown that all e-admissible functions in the structure $\mathfrak{A}$, which are defined at least in one point, are exactly all search computable functions, which in this case are all superpositions of the functions $\theta_1, \dots, \theta_n$, projecting and constant functions of many variables.

We will build an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$ building a sequence $A_0, A_1, \dots$ by steps. In each step s we will define a sequence $A_{0,s}, A_{1,s}, \dots$ of N_0 such that:

(i) $A_{i,s}$ is a finite subset of N_0, $i, s \in \omega$;

(ii) $A_{i,s} \subseteq A_{i,s+1}$, $i, s \in \omega$.

At the end we will take $A_i = \cup_{s=0}^{+\infty} A_{i,s}$ and $\alpha([A_i]) = a_i$, $i \in \omega$. With the even steps we shall ensure that there is no subset of some Cartesian product of B different of that Cartesian product of B which is a domain of some e-admissible function. With the odd steps we shall ensure that the only e-admissible functions are all superpositions of the functions $\theta_1, \dots, \theta_n$.

Let $\varphi_0^{(k)}, \varphi_1^{(k)}, \dots$ be the standard enumeration of all partial recursive functions on k variables, $W_0^{(k)}, W_1^{(k)}, \dots$ be the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \dots\}$, where $a_0, a_1, \dots$ are different.

Step $s = 0$. Set $A_{i,s} = \emptyset$, $i \in \omega$.

Step $s = 2\langle e, k \rangle + 1$. We check if there exist different elements $p_1, \dots, p_k$, $p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ such that:

(i) $\varphi_e^{(k)}(p_1, \dots, p_k)$ and $\varphi_e^{(k)}(p'_1, \dots, p'_k)$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau^p_{\mathfrak{B}}(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
$\varphi_e^{(k)}(p'_1, \dots, p'_k) = q =$
$\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
where $q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p'_1, \dots, p'_k\})$ and $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$;

(iii) There exist $a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l} \in B$ such that
$\tau^p_{\mathfrak{A}}(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau^q_{\mathfrak{A}}(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

It it is so we set $A_{i_1,s} = A_{i_1,s-1} \cup \{p_1, p'_1\}, \dots, A_{i_k,s} = A_{i_k,s-1} \cup \{p_k, p'_k\}$, $A_{n_1,s} = A_{n_1,s-1} \cup \{q_1\}, \dots, A_{n_l,s} = A_{n_l,s-1} \cup \{q_l\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \notin \{i_1, \dots, i_k, n_1, \dots, n_l\}$. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

Step $s = 2\langle e, j \rangle + 2$. Let p be the least element of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{j,s} = A_{j,s-1} \cup \{p\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \neq j$, if such elements exist. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

We fix $A_i = \cup_{s=0}^{+\infty} A_{i,s}$, $\alpha([A_i]) = a_i$, $i \in \omega$ and construction is completed.

Lemma 1. *For every natural number s, $A_{0,s-1} \cup A_{1,s-1} \cup \dots$ is finite.*

Proof. For every step s we add only finitely many numbers to

$$A_{0,s-1} \cup A_{1,s-1} \cup \dots .$$

□

Lemma 2. *Let e be such that $W_e^{(1)}$ is infinite. Then for every $j \in \omega$ there exists $p \in W_e^{(1)}$, such that $p \in A_j$.*

Proof. Let j be an arbitrary element of ω. Then on step $s = 2\langle e, j\rangle + 2$ we find $p \in W_e^{(1)}$ such that $p \in N_0$. Then we set $p \in A_{j,s} \subseteq A_j$. □

Corollary 4. *For every natural i, A_i is infinite and immune and $[A_i]$ is not recursively enumerable.*

Proof. Indeed, for every infinite r.e. subset $W_e^{(1)}$ of N_0 and every element $a_i \in B$ there exists an element $p \in W_e^{(1)}$ such that $p \in A_i$. Therefore, A_i is infinite and $W_e^{(1)} \cap (N_0 \setminus A_i) = W_e^{(1)} \cap (\cup_{j \neq i} A_j) \neq \emptyset$, i.e. A_i is immune and not recursively enumerable. □

Analogously one can prove the following

Corollary 5. *For every nonempty subset L of ω, $L \neq \omega$, $\cup_{i \in L} A_i$ is infinite and immune and $\cup_{i \in L}[A_i]$ is not recursively enumerable.*

Corollary 6. *For every natural $m \geq 1$ and every nonempty subset L of ω^m such that $L \neq \omega^m$ the set*

$$M = \cup\{(p_1, \dots, p_m) | \exists j_1 \dots \exists j_m [(j_1, \dots, j_m) \in L \& p_1 \in A_{j_1} \& \dots \& p_m \in A_{j_m}]\}$$

is not recursively enumerable.

Proof. First we claim: there exist coordinate i, $1 \leq i \leq m$, and $j_1, \dots, j_{i-1}$, $j_{i+1}, \dots, j_m$ such that the set $L' = \{j | (j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_m) \in L\}$, which is an i-th projection of L for the fixed $j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_m$, is nonempty and $L' \neq \omega$. For the sake of simplicity let $m = 2$. Since $L \neq \emptyset$, there exists (j'_1, j'_2) such that $(j'_1, j'_2) \in L$. Analogously, $L \neq \omega^2$ and there exists (i_1, i_2) such that $(i_1, i_2) \notin L$. If $(j'_1, i_2) \in L$, then fix $i = 1$ and $j_2 = i_2$ and the claim is true; otherwise $(j'_1, i_2) \notin L$, $(j'_1, j'_2) \in L$, fix $i = 2$ and $j_1 = j'_1$. Thus the claim is true again.

Let us assume that M is r.e. Then for some fixed i and $j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_m$ the set

$L' = \{j | (j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_m) \in L\} \neq \emptyset$ and $L' \neq \omega$. Let

$A' = \{a_k | (a_{j_1}, \dots, a_{j_{i-1}}, a_k, a_{j_{i+1}}, \dots, a_{j_m}) \in A\}$. Then

$M' = \{p | \exists j [p \in A_j \& a_j \in A']\} = \cup_{j \in L'} A_j$. According to the previous corollary, M' is not r.e. On the other hand, if we fix

$p_1 \in A_{j_1}, \dots, p_{i-1} \in A_{j_{i-1}}, p_{i+1} \in A_{j_{i+1}}, p_m \in A_{j_m}$, then

$M' = \{p | \exists j [p \in A_j \& (j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_m) \in L]\} =$

$\{p | \exists p_{j_1} \dots \exists p_{j_{i-1}} \exists p_{j_{i+1}} \dots \exists p_{j_m} [(p_{j_1}, \dots, p_{j_{i-1}}, p, p_{j_{i+1}}, \dots, p_{j_m}) \in M]\}$ is r.e. The obtained contradiction shows that the assumption is wrong. □

It is easy to check the following

Corollary 7. *For every natural $m \geq 1$ and every nonempty subset L of ω^m, such that $L \neq \omega^m$*

$$\cup\{(p_1, \dots, p_m) | \exists j_1 \dots \exists j_m [(j_1, \dots, j_m) \in L \& p_1 \in [A_{j_1}] \& \dots \& p_m \in [A_{j_m}]]\}$$

is not recursively enumerable.

Corollary 8. *For every natural $m \geq 1$ and every nonempty subset A of B^m such that $A \neq B^m$*

$$\cup\{(p_1, \dots, p_m) | \exists j_1 \dots \exists j_m [(a_{j_1}, \dots, a_{j_m}) \in A \& p_1 \in A_{j_1} \dots \& p_m \in A_{j_m}]\}$$

is not recursively enumerable.

Corollary 9. *For every function θ such that $\mathrm{Dom}(\theta) \subseteq B^m$ and θ is effective in the enumeration $\langle \alpha, \mathfrak{B} \rangle$, the equality $\mathrm{Dom}(\theta) = B^m$ holds.*

Proof. It is an immediate corollary of the previous one. □

Lemma 3. $N_0 \subseteq \mathrm{Dom}(\alpha)$.

Proof. Let us assume that $N_0 \setminus \mathrm{Dom}(\alpha) \neq \emptyset$ and p_0 is the least element of $N_0 \setminus \mathrm{Dom}(\alpha)$. Then there exists a step $s = 2\langle e, j \rangle + 2$ such that p_0 is the least element of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ and $W_e^{(1)} = \omega$. At that step s we have to put p_0 in some $A_j \subseteq \mathrm{Dom}(\alpha)$. The contradiction obtained shows that $N_0 \subseteq \mathrm{Dom}(\alpha)$. □

Now it is obvious that

Corollary 10. $\mathrm{Dom}(\alpha) = \omega$.

Let θ be effective in $\langle \alpha, \mathfrak{B} \rangle$ and $\mathrm{Dom}(\theta) \subseteq B^k$ for some natural $k \geq 1$. Then $\mathrm{Dom}(\theta) = B^k$ and there exists p.r. function f such that for all natural numbers $i_1, \dots, i_k$,

$$\theta(\alpha(i_1), \dots, \alpha(i_k)) \cong \alpha(f(i_1, \dots, i_k)).$$

Therefore f is a total function and $f = \varphi_e^{(k)}$ for some natural e. Let us consider step $s = 2\langle e, k\rangle + 1$.

First assume there are different elements $p_1, \dots, p_k, p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ satisfying the conditions (i)–(iii) at step $s = 2\langle e, k\rangle + 1$ and fix such elements $p_1, \dots, p_k, p'_1, \dots, p'_k$. Then according to Corollary 1,
$\alpha(f(p_1, \dots, p_k)) = \alpha(\varphi_e^{(k)}(p_1, \dots, p_k)) =$
$\alpha(\tau^p_{\mathfrak{B}}(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)) =$
$\tau^p_{\mathfrak{A}}(X_1/\alpha(p_1), \dots, X_k/\alpha(p_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/\alpha(r_1), \dots, Z_m/\alpha(r_m))$
$= \tau^p_{\mathfrak{A}}(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau^q_{\mathfrak{A}}(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^q_{\mathfrak{A}}(X'_1/\alpha(p'_1), \dots, X'_k/\alpha(p'_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/\alpha(r_1), \dots, Z_m/\alpha(r_m))$
$= \alpha(\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)) =$
$\alpha(\varphi_e^{(k)}(p'_1, \dots, p'_k)) = \alpha(f(p'_1, \dots, p'_k))$.

On the other hand, $\alpha(f(p_1, \dots, p_k)) = \theta(\alpha(p_1), \dots, \alpha(p_k)) = \theta(a_{i_1}, \dots, a_{i_k}) = \theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \alpha(f(p'_1, \dots, p'_k))$. That contradiction shows this case isn't possible. Therefore, there aren't different elements $p_1, \dots, p_k, p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ satisfying the conditions (i) – (iii)

Let us fix different $p_1, \dots, p_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$. Then
$f(p_1, \dots, p_k) = \varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau^p_{\mathfrak{B}}(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$, where
$q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$, $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$. Furthermore, for every different $p'_1, \dots, p'_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$, $f(p'_1, \dots, p'_k) = \varphi_e^{(k)}(p'_1, \dots, p'_k) = q =$
$\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y'_1/q'_1, \dots, Y'_{l'}/q'_{l'}, Z_1/r_1, \dots, Z_m/r_m)$, where
$q'_1, \dots, q'_{l'} \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p'_1, \dots, p'_k\})$,
$r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$ and for every
$a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l}, a_{n'_1}, \dots, a_{n'_{l'}} \in B$
$\tau^p_{\mathfrak{A}}(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^q_{\mathfrak{A}}(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y'_1/a_{n'_1}, \dots, Y'_{l'}/a_{n'_{l'}}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

Therefore, $\theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \alpha(f(p'_1, \dots, p'_k)) =$
$\alpha(\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y'_1/q'_1, \dots, Y'_{l'}/q'_{l'}, Z_1/r_1, \dots, Z_m/r_m)) =$
$\tau^q_{\mathfrak{A}}(X'_1/\alpha(p'_1), \dots, X'_k/\alpha(p'_k), Y'_1/\alpha(q'_1), \dots, Y'_{l'}/\alpha(q'_{l'}), Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^p_{\mathfrak{A}}(X_1/\alpha(p'_1), \dots, X_k/\alpha(p'_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

Let $q_1 \in A_{n_1}, \dots, q_l \in A_{n_l}$ and θ' be the function
$\theta'(b_1, \dots, b_k) = \tau^p_{\mathfrak{A}}(X_1/b_1, \dots, X_k/b_k, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$ for fixed $a_{n_1}, \dots, a_{n_l}, a_{j_1}, \dots, a_{j_m}$. We'll prove that $\theta = \theta'$. Let $(b_1, \dots, b_k)$ be an arbitrary k-tuple of B^k. Take $p'_1, \dots, p'_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$ such that $\alpha(p'_1) = b_1, \dots, \alpha(p'_k) = b_k$. It is possible because every element of B has infinitely many numbers. Then $\theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \tau^p_{\mathfrak{A}}(X_1/\alpha(p'_1), \dots, X_k/\alpha(p'_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) = \theta(b_1, \dots, b_k)$ and θ' is a superposition of the function $\theta_1, \dots, \theta_n,$, projecting and constant functions of many variables. □

Theorem 2. *Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n \rangle$ be a structure, where $\theta_1, \dots, \theta_n$ are total functions. If $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is an effective enumeration in $\mathfrak{A}$, then there exists an exact effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ such that $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$.*

Proof. We shall give only the construction of the effective enumeration. The proof that it has the required properties is analogous to the previous one.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an effective enumeration in $\mathfrak{A}$, N_0 the same as in the proof of the previous Theorem and $N_0 = N_0' \cup N_0''$, where N_0', N_0'' are infinite recursive sets. Take recursive $f(i) = p_i''$, where $N_0'' = \{p_0'', p_1'', \dots\}$, $p_0'' < p_1'', < \dots$. As in the proof of the previous theorem we will build an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$ building a sequence $A_0, A_1, \dots$ by steps. In each step s we will define a sequence $A_{0,s}, A_{1,s}, \dots$ of N_0 such that:

(i) $A_{i,s} \cap N_0'$ is a finite subset of N_0', $i, s \in \omega$;

(ii) $A_{i,s} \subseteq A_{i,s+1}$, $i, s \in \omega$.

At the end we will take $A_i = \cup_{s=0}^{+\infty} A_{i,s}$ and $\alpha([A_i]) = a_i$, $i \in \omega$.

Let $\varphi_0^{(k)}, \varphi_1^{(k)}, \dots$ be the standard enumeration of all partial recursive functions of k variables, $W_0^{(k)}, W_1^{(k)}, \dots$ be the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \dots\}$ where $a_0, a_1, \dots$ are different.

Step $s = 0$. Set $A_{i,s} = \{p | p \in N_0'' \& \exists q[f(q) = p \& \alpha_0(q) = a_i]\}$, $i \in \omega$.

Step $s = 2\langle e, k \rangle + 1$. We check if there exist different elements $p_1, \dots, p_k$, $p_1', \dots, p_k'$ of $N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ such that:

(i) $\varphi_e^{(k)}(p_1, \dots, p_k)$ and $\varphi_e^{(k)}(p_1', \dots, p_k')$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau_{\mathfrak{B}}^p(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
$\varphi_e^{(k)}(p_1', \dots, p_k') = q =$
$\tau_{\mathfrak{B}}^q(X_1'/p_1', \dots, X_k'/p_k', Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
where $q_1, \dots, q_l \in N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p_1', \dots, p_k'\})$ and $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$;

(iii) There exist $a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l} \in B$ such that
$\tau_{\mathfrak{A}}^p(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau_{\mathfrak{A}}^q(X_1'/a_{i_1}, \dots, X_k'/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

If it is so we set $A_{i_1,s} = A_{i_1,s-1} \cup \{p_1, p_1'\}, \dots, A_{i_k,s} = A_{i_k,s-1} \cup \{p_k, p_k'\}$, $A_{n_1,s} = A_{n_1,s-1} \cup \{q_1\}, \dots, A_{n_l,s} = A_{n_l,s-1} \cup \{q_l\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \notin \{i_1, \dots, i_k, n_1, \dots, n_l\}$. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

Step $s = 2\langle e, j \rangle + 2$. Let p be the least element of $N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{j,s} = A_{j,s-1} \cup \{p\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \neq j$, if such elements exist. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

We fix $A_i = \cup_{s=0}^{+\infty} A_{i,s}$, $\alpha([A_i]) = a_i$, $i \in \omega$ and construction is completed. □

Theorem 3. *There exist infinitely many mutually incomparable exact effective enumerations.*

Proof. We will build effective enumerations $\langle \alpha_j, \mathfrak{B}_j \rangle$, $j \in \omega$ of the structure

$\mathfrak{A}$ building a sequence $A_0^j, A_1^j, \dots,\ j \in \omega$, by steps. In each step s we will define a sequence $A_{0,s}^j, A_{1,s}^j, \dots$, of subsets of N_0, $j \in \omega$, such that:

(i) $A_{i,s}^j$ is a finite subset of N_0, $i, j, s \in \omega$;

(ii) $A_{i,s}^j \subseteq A_{i,s+1}^j$, $i, j, s \in \omega$.

With the steps of the kind $3k+1$ we shall ensure for every j that there isn't a subset of some Cartesian product of B different of that Cartesian product of B which is a domain of some e-admissible function for the enumeration $\langle \alpha_j, \mathfrak{B}_j \rangle$. With the steps of the kind $3k+2$ we shall ensure that the only e-admissible functions for the enumeration $\langle \alpha_j, \mathfrak{B}_j \rangle$ are all superpositions of the functions $\theta_1, \dots, \theta_n$. With the steps of the kind $3k+3$ we shall ensure that $\langle \alpha_j, \mathfrak{B}_j \rangle \not\leq \langle \alpha_k, \mathfrak{B}_k \rangle$ for $j \neq k$, $j, k \in \omega$.

As above, $\varphi_0^{(k)}, \varphi_1^{(k)}, \dots$ is the standard enumeration of all partial recursive functions on k variables, $W_0^{(k)}, W_1^{(k)}, \dots$ is the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \dots\}$, where $a_0, a_1, \dots$ are different.

Step $s = 0$. Set $A_{i,s}^j = \emptyset$, $i, j \in \omega$.

Step $s = 3\langle e, k, j \rangle + 1$. We check if there exist different elements
$p_1, \dots, p_k, p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \dots)$
such that:

(i) $\varphi_e^{(k)}(p_1, \dots, p_k)$ and $\varphi_e^{(k)}(p'_1, \dots, p'_k)$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau_{\mathfrak{B}}^p(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,

$\varphi_e^{(k)}(p'_1, \dots, p'_k) = q =$
$\tau_{\mathfrak{B}}^q(X'_1/p'_1, \dots, X'_k/p'_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
where $q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \dots \cup \{p_1, \dots, p_k, p'_1, \dots, p'_k\})$ and $r_1, \in A_{j_1,s-1}^j, \dots, r_m \in A_{j_m,s-1}^j$;

(iii) There exist $a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l} \in B$ such that
$\tau_{\mathfrak{A}}^p(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau_{\mathfrak{A}}^q(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

If it is so we set $A_{i_1,s}^j = A_{i_1,s-1}^j \cup \{p_1, p'_1\}, \dots, A_{i_k,s}^j = A_{i_k,s-1}^j \cup \{p_k, p'_k\}$, $A_{n_1,s}^j = A_{n_1,s-1}^j \cup \{q_1\}, \dots, A_{n_l,s}^j = A_{n_l,s-1}^j \cup \{q_l\}$ and $A_{i,s}^j = A_{i,s-1}^j$ for all $i \notin \{i_1, \dots, i_k, n_1, \dots, n_l\}$. Otherwise set $A_{i,s}^j = A_{i,s-1}^j$ for all $i \in \omega$.

Step $s = 3\langle e, k, j \rangle + 2$. Let p be the least element of $N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \dots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{k,s}^j = A_{k,s-1}^j \cup \{p\}$ and $A_{i,s}^j = A_{i,s-1}^j$ for all $i \neq k$, if such elements exist. Otherwise set $A_{i,s}^j = A_{i,s-1}^j$ for all $i \in \omega$.

Step $s = 3\langle e, k, j \rangle + 3$.

Let first $k \neq j$, $\varphi_e^{(1)}$ be a total function. Let p be the least element of $N_0 \setminus (A_{0,s-1}^k \cup A_{1,s-1}^k \cup \dots)$,

$\varphi_e^{(1)}(p) = q = \tau_{\mathfrak{B}}^q(X/p, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$, where
$q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \dots \cup \{p\})$ and $r_1 \in A_{j_1,s-1}^j, \dots, r_m \in A_{j_m,s-1}^j$.

Fix $A_{j,s}^j = A_{j,s-1}^j \cup \{p, q_1, \dots, q_l\}$ and $A_{i,s}^j = A_{i,s-1}^j$, for $i \neq j$, $i \in \omega$.

We check if $\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \dots, Y_l/a_j, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) = a_j$. If so, fix $A^k_{k,s} = A^k_{k,s-1} \cup \{p\}$ and $A^k_{i,s} = A^k_{i,s-1}$ for $i \neq k$, $i \in \omega$; otherwise fix $A^k_{k-1,s} = A^k_{k-1,s-1} \cup \{p\}$ and $A^k_{i,s} = A^k_{i,s-1}$ for $i \neq k-1$, $i \in \omega$.

Fix $A^{i'}_{i,s} = A^{i'}_{i,s-1}$ for $i, i' \in \omega$, $i' \notin \{j,k\}$.

In case either $k = j$ or $\varphi^{(1)}_e$ is not a total function, fix $A^{i'}_{i,s} = A^{i'}_{i,s-1}$ for $i, i' \in \omega$, $i' \notin \{j,k\}$.

At the end we fix $A^j_i = \cup^{+\infty}_{s=0} A^j_{i,s}$, $\alpha_j^{-1}([A^j_i]) = a_i$, $i, j \in \omega$, and construction is completed.

The proof that $\langle \alpha_j, \mathfrak{B}_j \rangle$, $j \in \omega$, is an exact effective enumeration is analogous to the previous ones. We'll concentrate on the proof that $\langle \alpha_j, \mathfrak{B}_j \rangle$ and $\langle \alpha_k, \mathfrak{B}_k \rangle$ are incomparable.

Let us assume that $\langle \alpha_k, \mathfrak{B}_k \rangle \leq \langle \alpha_j, \mathfrak{B}_j \rangle$. Then there exists a total recursive function f such that for all natural p, $\alpha_k(p) = \alpha(f(p))$. Let $f = \varphi^{(1)}_e$, consider the step $s = 3\langle e, k, j\rangle + 3$ and let p be the element belonging to $N_0 \setminus (A^k_{0,s-1} \cup A^k_{1,s-1} \cup \dots)$ chosen on that step.

Then $\varphi^{(1)}_e(p) = q = \tau^q_{\mathfrak{B}}(X/p, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$, where $q_1, \dots, q_l \in N_0 \setminus (A^j_{0,s-1} \cup A^j_{1,s-1} \cup \dots \cup \{p\})$, $r_1 \in A^j_{j_1,s-1}, \dots, r_m \in A^j_{j_m,s-1}$ and $A^j_{j,s} = A^j_{j,s-1} \cup \{p, q_1, \dots, q_l\}$.

We have to consider two cases. We'll consider only the first one:
$\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \dots, Y_l/a_j, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) = a_j$.
Then $A^k_{k,s} = A^k_{k,s-1} \cup \{p\}$, $\alpha_k(p) = a_k \neq a_j =$
$\alpha_j(\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \dots, Y_l/a_j, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})) =$
$\tau^q_{\mathfrak{A}}(X/\alpha_j(p), Y_1/\alpha_j(q_1), \dots, Y_l/\alpha_j(q_l), Z_1/\alpha_j(r_1), \dots, Z_m/\alpha_j(r_m)) =$
$\alpha_j(\tau^q_{\mathfrak{B}}(X/p, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)) = \alpha_j(q) = \alpha_j(\varphi^{(1)}_e(p)) =$
$\alpha_j(f(p))$.
The contradiction obtained shows the assumption is not true. □

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