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ON THE STRUCTURE OF SOME ARCS RELATED TO CAPS AND THE NONEXISTENCE OF SOME OPTIMAL CODES

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In this paper we solve two instances of the main problem in coding theory for linear codes of dimension 5 over $\mathbb{F}_4$. We prove the nonexistence of $[395, 5, 295]_4$- and $[396, 5, 296]_4$-codes which implies the exact values $n_4(5, 295) = 396$ and $n_4(5, 296) = 397$. As a by-product, we characterize the arcs with parameters $(100, 26)$ in $\mathrm{PG}(3, 4)$.

Keywords: Linear codes, finite projecive geometries, arcs, extendable arcs, the Griesmer bound, Griesmer codes, Griesmer arcs.

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1. INTRODUCTION

In this paper we study two instances of the main problem in coding theory: the problem of determining the exact value of $n_q(k, d)$ defined as the minimal length of a k-dimensional linear code of minimum distance d over the field with q elements. This problem has been studied intensively in the past 30 years and has been solved completely for some small fields $\mathbb{F}_q$, and small dimensions k. The problem has a clear geometric relevance since every linear code of full length is known to be equivalent to an arc in the appropriate finite projective space and optimal codes correspond in the rule to nice geometric configurations.

A natural lower bound on $n_q(k, d)$ is the Griesmer bound [5]:

$$n_q(k, d) \geq g_q(k, d) \stackrel{def}{=} \sum_{i=1}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil .$$

Linear codes meeting this bound are called *Griesmer codes*. Arcs associated with Griesmer codes are called *Griesmer arcs*. Given an integer k and a prime power q, Griesmer codes are known to exist for all sufficiently large values of d. A standard approach to the problem of finding the exact value of $n_q(k,d)$ is to solve the problem for fixed k and q for all d. In this setting the main problem in coding theory is a finite one. This paper deals with linear codes over the field with four elements. The exact value of $n_4(k,d)$ was found for $k \le 4$ for all d [4,12]. For the next dimension $k = 5$ there exist 104 values of d for which $n_4(5,d)$ is unknown [12].

In this paper, we prove the nonexistence of the hypothetical quaternary Griesmer codes of dimension $k = 5$ with $d = 295, 296$, a fact which was hitherto unknown. The problem is studied purely geometrically due to the equivalence of linear $[n,k,d]_q$-codes and arcs with parameters $(n, n-d)$ in $\mathrm{PG}(k-1,q)$ [3,8,9,11]. Thus the existence of the codes in question that have parameters $[395,5,295]_4$ and $[396,5,296]_4$ is equivalent to the existence of $(395,100)$- and $(396,100)$-arcs in $\mathrm{PG}(4,4)$. The nonexistence proof relies on the classification of arcs with parameters $(100,26)$ in $\mathrm{PG}(3,4)$. These arcs are related to caps in $\mathrm{PG}(3,4)$ and can be obtained trivially from $(102,26)$-arcs by deleting two points. The latter are obtained as the sum of the maximal 17-cap in $\mathrm{PG}(3,4)$ and the whole space. Remarkably, there exists a $(100,26)$-arc which is not extendable to the unique $(102,26)$-arc.

This paper is organized as follows. In section 2 we present some basic facts on arcs in the geometries $\mathrm{PG}(r,q)$. We explain briefly the connections between linear codes over finite fields and arcs in finite projective geometries. Furthermore, we state without proof some results that are used in the paper. These include the so-called Hill–Lizak's Extension Theorem and H. N. Ward's Divisibility Theorem. Both theorems are formulated in their geometric form. Section 3 contains the geometric characterization of the arcs with parameters $(100,26)$ in $\mathrm{PG}(3,4)$. In section 4, we prove the nonexistence of arcs with parameters $(395,100)$, and $(396,100)$ in $\mathrm{PG}(4,4)$, which settles the problem of finding the exact value of $n_4(5,d)$ for $d = 295, 296$.

2. PRELIMINARIES

A *multiset in* $\mathrm{PG}(k-1,q)$ is a mapping $\mathcal{K}\colon \mathcal{P} \to \mathbb{N}_0$, where $\mathcal{P}$ denotes the pointset of $\mathrm{PG}(k-1,q)$. The integer $\mathcal{K}(\mathcal{P}) = \sum_{P\in\mathcal{P}} \mathcal{K}(P)$ is called the *cardinality* of the multiset $\mathcal{K}$. For a subset $\mathcal{Q}$ of $\mathcal{P}$, we set $\mathcal{K}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} \mathcal{K}(P)$. The integer $\mathcal{K}(\mathcal{Q})$ is called the *multiplicity* of the subset $\mathcal{Q}$. A point of multiplicity i is called an i-point; i-lines, i-planes, i-solids etc. are defined in a similar way. Given a set of points $S \subseteq \mathcal{P}$, we define the *characteristic function* χ_S of S by

$$\chi_S(P) = \begin{cases} 1 & \text{if } P \in S; \\ 0 & \text{if } P \notin S. \end{cases}$$

A multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an $(n,w,k-1,q)$-*arc*, or an (n,w)-arc for short, if

(a) $\mathcal{K}(\mathcal{P}) = n$;

(b) for each hyperplane H in $\mathrm{PG}(k-1,q)$, $\mathcal{K}(H) \le w$, and

(c) there is a hyperplane H with $\mathcal{K}(H) = w$.

In a similar way, we define an $(n, w; k-1, q)$-*blocking set* (or just (n, w)-blocking set if the geometry is clear from the context) as a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ satisfying

(d) $\mathcal{K}(\mathcal{P}) = n$;

(e) for each hyperplane H in $\mathrm{PG}(k-1,q)$, $\mathcal{K}(H) \ge w$, and

(f) there is a hyperplane H with $\mathcal{K}(H) = w$.

Given a $(n, w; k-1, q)$-arc $\mathcal{K}$, we denote by $\gamma_i(\mathcal{K})$ the maximal multiplicity of an i-dimensional flat in $\mathrm{PG}(k-1,q)$, i.e. $\gamma_i(\mathcal{K}) = \max_\delta \mathcal{K}(\delta)$, $i = 0, \ldots, k-1$, where δ runs over all i-dimensional flats in $\mathrm{PG}(k-1,q)$. If $\mathcal{K}$ is clear from the context we shall write just γ_i. In what follows, we repeatedly use the following lemma which is proved by straightforward counting.

Lemma 1. *Let $\mathcal{K}$ be an $(n, w; k-1, q)$-arc, and let Π be an $(s-1)$-dimensional flat in $\mathrm{PG}(k-1,q)$, $2 \le s < k$, with $\mathcal{K}(\Pi) = u$. Then, for any $(s-2)$-dimensional flat Δ contained in Π, we have*

$$\mathcal{K}(\Delta) \le \gamma_{s-1}(\mathcal{K}) - \frac{n-u}{q^{k-s} + \ldots + q}.$$

For an $(n, w; k-1, q)$ arc $\mathcal{K}$, denote by a_i the number of hyperplanes H in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}(H) = i$, $i \ge 0$. Let further λ_j be the number of points P from $\mathcal{P}$ with $\mathcal{K}(P) = j$. The sequence $(a_0, a_1, \ldots)$ is called *the spectrum* of $\mathcal{K}$. Simple counting arguments yield the following identities, which are equivalent to the first three MacWilliams identities for linear codes:

$$\sum_{i=0}^{n-d} a_i = \frac{q^k - 1}{q-1}, \tag{2.1}$$

$$\sum_{i=1}^{n-d} i a_i = n \cdot \frac{q^{k-1} - 1}{q-1}, \tag{2.2}$$

$$\sum_{i=2}^{n-d} \binom{i}{2} a_i = \binom{n}{2} \frac{q^{k-2} - 1}{q-1} + q^{k-2} \cdot \sum_{i=2}^{\gamma_0} \binom{i}{2} \lambda_i. \tag{2.3}$$

Set $w = n-d$ and $v_i = (q^i - 1)/(q-1)$. The following identity is easily obtained from (2.1)–(2.3):

$$\sum_{i=0}^{w} \binom{w-i}{2} a_i = \binom{w}{2} v_k - n(w-1) v_{k-1} + \binom{n}{2} v_{k-2} + q^{k-2} \cdot \sum_{i=2}^{\gamma_0} \binom{i}{2} \lambda_i. \tag{2.4}$$

Note that the sum on the left-hand side can be written as $\sum_H \binom{w-\mathcal{K}(H)}{2}$, where H runs over all hyperplanes of $\mathrm{PG}(k-1,q)$. Let us fix a hyperplane H_0. Given a subspace δ of codimension 2 contained in H_0, denote by $H_1, H_2, \ldots, H_q$ the remaining hyperplanes through δ. Set

$$\eta_i = \max_{\delta:\, \mathcal{K}(\delta)=i} \sum_{j=1}^{q} \binom{w-\mathcal{K}(H_j)}{2}. \tag{2.5}$$

Here the maximum is taken over all hyperlines δ of multiplicity i contained in H_0. Assume the spectrum (b_i) of the restriction of $\mathcal{K}$ to H_0, is known. We have

$$\sum_H \binom{w-\mathcal{K}(H)}{2} \leq \sum_j b_j \eta_j + \binom{w-\mathcal{K}(H_0)}{2},$$

which by (2.4) implies

$$\sum_j b_j \eta_j + \binom{w-\mathcal{K}(H_0)}{2} \geq \binom{w}{2} v_k - n(w-1)v_{k-1} + \binom{n}{2} v_{k-2} + q^{k-2} \cdot \sum_{i=2}^{\gamma_0} \binom{i}{2} \lambda_i. \tag{2.6}$$

Clearly, (2.6) is a necessary condition for the existence of an (n,w)-arc in $\mathrm{PG}(k-1,q)$. It can also be used to rule out the existence of hyperplanes H for which $\mathcal{K}|_H$ has a given spectrum.

The following argument will be used throughout the paper. Let $\mathcal{K}$ be an $(n, n-d; k-1, q)$-arc, i.e. an arc associated with an $[n,k,d]_q$-code. Fix an i-dimensional flat δ in $\mathrm{PG}(k-1,q)$, with $\mathcal{K}(\delta) = t$. Let further π be a j-dimensional flat in $\mathrm{PG}(k-1,q)$ of complementary dimension, i.e. $i+j = k-2$ and $\delta \cap \pi = \emptyset$. Define the projection $\varphi = \varphi_{\delta,\pi}$ from δ onto π by

$$\varphi\colon \begin{cases} \mathcal{P} \setminus \delta & \to \quad \pi \\ Q & \to \quad \pi \cap \langle \delta, Q \rangle. \end{cases} \tag{2.7}$$

Here $\mathcal{P}$ is the set of points of $\mathrm{PG}(k-1,q)$. Note that φ maps $(i+s)$-flats containing δ into $(s-1)$-flats in π. Given a set of points $\mathcal{F} \subset \pi$, define the induced arc $\mathcal{K}^{\varphi}$ by

$$\mathcal{K}^{\varphi}(\mathcal{F}) = \sum_{\varphi_{\delta,\pi}(P) \in \mathcal{F}} \mathcal{K}(P).$$

If $\mathcal{F}$ is a k'-dimensional flat in π then $\mathcal{K}^{\varphi}(\mathcal{F}) \leq \gamma_{k'+i+1} - t$.

In this paper, we consider arcs in $\mathrm{PG}(3,4)$ or $\mathrm{PG}(4,4)$ and always take δ to be a point (in the three-dimensional case) or a line (in the four-dimensional case); in both cases π will be a plane disjoint from δ. Every line L in π is then the image of a hyperplane (a plane or a solid) containing δ. If $P_0, \ldots, P_q$ are the points on L we call the $(q+1)$-tuple $(\mathcal{K}^\varphi(P_0), \ldots, \mathcal{K}^\varphi(P_q))$ the *type of* L.

It was mentioned already, that the existence of linear $[n, k, d]_q$ codes of full length is equivalent to that of $(n, n-d; k-1, q)$-arcs. Two linear codes with the same parameters are semilinearly isomorphic if and only if the corresponding arcs are projectively equivalent. H.N. Ward proved in [13] a remarkable theorem on the divisibility of codes meeting the Griesmer bound. Below we give Ward's result restated for arcs in $\mathrm{PG}(k-1, q)$ (cf. [9]).

Theorem 1. *Let $\mathcal{K}$ be a Griesmer (n, w)-arc in $\mathrm{PG}(k-1, p)$, p prime, with $w \equiv n \bmod p^e$, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \bmod p^e$ for every hyperplane H.*

For codes over $\mathbb{F}_4$ (resp. arcs in geometries over $\mathbb{F}_4$) we have the following weaker version of this result [13].

Theorem 2. *Let $\mathcal{K}$ be a Griesmer (n, w)-arc in $\mathrm{PG}(k-1, 4)$ with $w \equiv n \bmod 2^e$. Then $\mathcal{K}(H) \equiv n \bmod 2^{e-1}$ for every hyperplane H.*

An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ is called *extendable* if there exists an $(n+1, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}'(x) \geq \mathcal{K}(x)$ for every point of $\mathrm{PG}(k-1, q)$. The next extension result about arcs stated below follows directly from Hill-Lizak's extension theorem [6,7]:

Theorem 3. *Let $\mathcal{K}$ be an $(n, w; k-1, q)$-arc with $\gcd(n-w, q) = 1$. Assume that the multiplicities of all hyperplanes are congruent to n or w modulo q. Then $\mathcal{K}$ can be extended to an $(n+1, w)$-arc.*

The following theorem from [10] follows from a result by Beutelspacher [1] and can be viewed as a generalization of Hill-Lizak's extension theorem.

Theorem 4. *Let $\mathcal{K}$ be a (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q = p^s$, with spectrum $(a_i)_{i \geq 0}$. If $w \not\equiv n \bmod p$ and*

$$\sum_{i \not\equiv w \bmod q} a_i \leq q^{k-2} + q^{k-3} + \ldots + 1 + q^{k-3} \cdot r(q)$$

where $q + r(q) + 1$ is the minimal size of a non-trivial blocking set of $\mathrm{PG}(2, q)$, then there exists an $(n+1, w)$-arc.

As a corollary we can derive the following useful result [10]:

Corollary 1. *Let $\mathcal{K}$ be a nonextendable (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q = p^s$, with $w \equiv n + 1 \pmod q$ and with spectrum $(a_i)_{i \geq 0}$. Let θ denote the maximal number of hyperplanes of multiplicity $\not\equiv n+1 \pmod q$ incident with any subspace of codimension 2 of H, where H is a hyperplane of multiplicity $\mathcal{K}(H) \equiv w \pmod q$. Then $\sum_{i \not\equiv n, n+1 \pmod q} a_i > q^{k-3} \cdot r(q)/(\theta - 1)$, where $r(q)$ is as in Theorem 4. In particular, we have $\sum_{i \not\equiv n, n+1 \pmod q} a_i > q^{k-3} \cdot r(q)/(q-1)$.*

3. CLASSIFICATION OF THE $(100, 26)$-ARCS IN $\mathrm{PG}(3, 4)$

In this section we classify the arcs with parameters $(100, 26)$ in $\mathrm{PG}(3, 4)$. It is known that a $(102, 26)$-arc in $\mathrm{PG}(3, 4)$ is the sum of a 17-cap plus the whole space, and hence is unique. By Hill-Lizak's extension theorem every $(101, 26)$-arc is extendable to a $(102, 26)$-arc. One obvious way to construct $(100, 26)$-arcs in $\mathrm{PG}(3, 4)$ is to delete a point from a $(101, 26)$-arc, or equivalently, to delete two points from a $(102, 26)$-arc. It turns out however that there exist $(100, 26)$-arcs that cannot be obtained in this way.

Let $\mathcal{K}$ be a $(100, 26)$-arc. By Lemma 1

$$\gamma_0(\mathcal{K}) = 2, \gamma_1(\mathcal{K}) = 7, \gamma_2(\mathcal{K}) = 26.$$

From now on we assume that $\mathcal{K}$ is a non-extendable $(100, 26)$-arc in $\mathrm{PG}(3, 4)$. The restriction of $\mathcal{K}$ to a maximal hyperplane is a $(26, 7)$-arc. The characterization of such arcs is given by the following lemma.

Lemma 2. *A* $(26, 7)$*-arc in* $\mathrm{PG}(2, 4)$ *is one of the following:*

(1) two copies of the plane minus three non-concurrent lines minus a point (type (A));

(2) the sum of the plane plus a hyperoval minus a point (type (B));

(3) two 7-lines through a common 0-point; all points outside these two 7-lines are 1-points (type (C)).

The arcs of the first two types are extendable while an arc of the third type is not. This result is easily obtained from the known results on arcs and blocking sets in $\mathrm{PG}(2, 4)$ and we omit the proof. Below we present the spectra of these arcs, as well as the possible line types after a projection from a 0-point. For the second spectrum of type (B) there are no 0-points.

Type	a_7	a_6	a_5	a_4	a_3	a_2	λ_2	λ_1	λ_0	Line types
(A)	14	4	0	0	2	1	9	8	4	77732 77633 66662
	13	5	0	0	3	0	8	10	3	77633
(B)	12	3	4	2	0	0	6	14	1	66644
	10	5	6	0	0	0	5	16	0	-
(C)	11	6	1	3	0	0	6	14	1	77444

It is important to note that a $(26, 7)$-arc does not have 0- or 1-lines, as well as 5-lines with a 0-point. It is also worth noting that a $(26, 7)$-arc cannot contain a 3- and a 4-line simultaneously.

Lemma 3. *Let $\mathcal{K}$ be a $(25,7)$-arc in $\mathrm{PG}(2,4)$ having a 6-line L with three 2- and two 0-points. Then $\mathcal{K}$ has a also a 7-line incident with a 0-point.*

Proof. Denote by λ_i, $i = 0, 1, 2$ the number of i-points in $\mathcal{K}$. Obviously $\lambda_2 - \lambda_0 = 4$, and since $\lambda_0 \geq 2$ and $\lambda_2 \leq 9$, we are left with four cases: $\lambda_2 = 6 + i$, $\lambda_0 = 2 + i$, where $i = 0, 1, 2, 3$.

Assume for a contradiction that there is no 7-line with three 2-points and one 1-point. We consider the case $\lambda_0 = 2$. Let the three 2-points outside L be collinear. Then the line defined by them meets the 6-line L in a 0-point. This line should have another 0-point, because of our assumption, which gives $\lambda_0 \geq 3$, a contradiction. If the three 2-points outside L form a triangle, at least one of the lines defined by the vertices of this triangle meets L in a 2-point and hence there must be another 0-point, again a contradiction.

The cases $\lambda_0 = 3, 4, 5$ are dealt with in a similar way. □

Lemma 4. *Let $\mathcal{K}$ be a $(100,26)$-arc in $\mathrm{PG}(3,4)$. Then for every plane π in $\mathrm{PG}(3,4)$ $\mathcal{K}(\pi) \geq 12$.*

Proof. Let us note that by Lemma 1 $\mathcal{K}(\pi) \neq 7, 10, 11, 23$. Without loss of generality we consider the case when $\mathcal{K}$ is a non-extendable arc. If $\mathcal{K}$ is extendable the possible plane multiplicities are $26, 25, 24, 22, 21, 20$, and the lemma holds trivially.

Planes π of multiplicity ≤ 5 are ruled out by the nonexistence of 0- or 1-lines in $(26,7)$-arcs. It is easily seen that for planes of multiplicity at most 5, there is always a 0-point P in π which is incident only with 0- or 1-lines. Since P lies in at least one 26-plane π' ($\mathcal{K}$ was assumed to be non-extendable) the line $\pi \cap \pi'$ is a 0- or 1-line, a contradiction.

Assume there exists a 6-plane π_0. Consider a 2-line L in π_0. The line L is incident with at least two 26-planes, π_1 and π_2 say. Clearly π_1 and π_2 are of type (A). There exists a 0-point on L such that after a projection from that point the images of π_1 and π_2 are $(7,7,7,3,2)$. Now in the projection plane there is a line of type $(3,3,2/0,x,y)$ for some integers x, y. Now $x, y \leq 4$ since a 26-plane does not have a 5-line with a 0-point. This is a contradiction since $\mathcal{K}$ cannot have a 6-plane and a plane of multiplicity < 14 simultaneously.

Assume there exists a plane π_0 of multiplicity 8. Consider a projection from a 0-point in a 3-line L in π_0. The images of the other four planes through L are of type $(7,7,7-\epsilon,3,2+\epsilon)$ with $\epsilon \in \{0,1\}$. Now the projection plane necessarily contains a line of type $(7,7,6,6,0)$ which is an impossible type by Lemma 2. Planes of multiplicity 9 are ruled out similarly. In this case, we can even select the point on the 9-plane in such way that its image is of type $(3,3,3,0,0)$, which simplifies the proof. □

Lemma 5. *Let $\mathcal{K}$ be a $(100,26)$-arc in $\mathrm{PG}(3,4)$. Then there is no plane π in $\mathrm{PG}(3,4)$ with $12 \leq \mathcal{K}(\pi) \leq 15$.*

Proof. First, we shall rule out planes of multiplicity 15. Assume π_0 is a plane with $\mathcal{K} = 15$. The restriction of $\mathcal{K}$ to π_0 is a plane minus a line L and minus a point Q which lies off L. Assume there is a 0-point P outside π_0. A 26-plane π_1 through P (it exists since $\mathcal{K}$ is not extendable) has at least two 0-points (P and one on π_0). Hence this plane contains an arc of type (A) and therefore contains also Q. Now consider a 7-line L' in π_1 through P. It is incident with at least two further 26-planes that have at least two and hence at least three 0-points. On the other hand they meet π_0 in a 4-line which contradicts Lemma 4 (a $(26, 7)$-arc of type (A) does not have a 4-line). We have proved so far that there are no 0-points outside π_0. Now Q should be incident with a 26-plane that meets π_0 in a 3-line and hence has two 0-points, which is impossible.

In the same way we can rule out the existence of 14-planes (the complement of a line and two points or the complement of a Baer subplane), and of 16-planes in which the 0-points are collinear.

Now we are going to prove that planes of multiplicity 13 do not exist. The proof of the nonexistence of 12-planes is similar and more simple.

Assume there exists a 13-plane π_0. Fix a 4-line L in π_0 and denote the other four planes through L by π_i, $i = 1, \ldots, 4$. Without loss of generality π_1, π_2, π_3 are 26-planes and π_4 is a 25-plane. Consider a projection from P which we denote by φ and set $L_i = \varphi(\pi_i)$. Let us note that L_4 does not contain a 7-point. This follows from the fact that this point must be incident with three 26-lines and the types of L_1, L_2, L_3 are $(7, 7, 4, 4, 4)$ or $(6, 6, 6, 4, 4)$ and L_0 is forced to be of type $(4, 4, 4, 4, 0)$, a contradiction. Hence the type of L_4 is one of $(4, 6, 6, 6, 3)$, $(4, 6, 6, 5, 4)$ or $(4, 6, 5, 5, 5)$.

Now a 13-plane is the complement of a $(8, 1)$-blocking set, and hence one of the following: (a) the complement of a line and three points or (b) the complement of a Baer subplane and a point.

In case (a) there exists a point P such that the projection of π_0 from that point is of type $(4, 3, 3, 3, 0)$. Now none of the lines L_1, L_2, L_3 is of type $(7, 7, 4, 4, 4)$ since a 26-line through a 7-point should have two points of multiplicity at most 3. Consequently, the line L_4 should have two 3-points which is impossible. Therefore L_1, L_2, L_3 are of type $(6, 6, 6, 4, 4)$. Now with all three possibilities for L_4 we get a contradiction. For instance, if L_4 is of type $(4, 6, 6, 6, 3)$, the set of points

$$\mathcal{F} = \{X \in L_1 \cup L_2 \cup L_3 \cup L_4 \mid \mathcal{K}^{\varphi}(X) = 6\} \cup \{Y \in L_1 \cup L_2 \cup L_3 \cup L_4 \mid \mathcal{K}^{\varphi}(Y) = 3\}$$

must be a $(15, 4)$-arc and there is a line of type $(4, 4, 4, 3, 0)$. But we have already ruled out the existence of 15-planes. The other two possibilities for L_4 are dealt with in a similar fashion.

(b) As in the nonexistence proof for 15-planes we can show that there are no 0-points outside π_0. Now denote by P the extra 0-point on π_0 which is not from the removed Baer subplane. The lines in π_0 through P have multiplicities 3,3,3,3,1. hence a 26-plane through P (which necessarily exists) has two 0-points, a contradiction. □

Lemma 6. *There exists a unique* $(100, 26)$*-arc in* $\mathrm{PG}(3, 4)$ *with the following property:* $\mathrm{PG}(3, 4)$ *has a 24-plane with a 7-line consisting of three 2-points, one 1-point, and one 0-point.*

Proof. Denote by π_0 the 24-plane from the condition of the lemma. Let L be a 7-line in π_0 and let P be the unique 0-point on L. The remaining four planes through L, denoted by $\pi_1, \ldots, \pi_4$, are 26-planes. We consider a projection φ from the point P. Set $Q = \varphi(L)$, and $L_i = \varphi(\pi_i)$, $i = 0, \ldots, 4$. Clearly, $\mathcal{K}|_{\pi_i}$ $i = 1, \ldots, 4$, are of type (A) or (C) (cf. Lemma 4). Hence the possible types of the lines $L_1, \ldots, L_4$ are $(7, 7, 7 - \varepsilon, 3, 2 + \varepsilon)$, $\varepsilon = 0$ or 1, or $(7, 7, 4, 4, 4)$. Now we have the following possibilities for the four 26-planes through L:

(i) AAAA, (ii) AAAC, (iii) AACC, (iv) ACCC, (v) CCCC.

(i) The lines $L_1, \ldots, L_4$ are all of type $(7, 7, 7-\varepsilon, 3, 2+\varepsilon)$. Assume the pointset $\mathcal{X} = \{X \mid \mathcal{K}^{\varphi}(X) \geq 6\}$ in the projection plane has four collinear points and denote by M the line incident with them. Let Z be the fifth point on M. It has multiplicity at most 2. Now every line through Z, different from L_0 or M has at least two points from $\mathcal{X}$, which is impossible. Hence $\mathcal{X}$ is a $(9, 3)$-arc. Moreover, there is no external line to $\mathcal{X}$ since it would be of multiplicity ≤ 15. Now for every point $R \neq Q$ on L_0 we have $\mathcal{K}^{\varphi}(R) \leq 4$. This implies $\mathcal{K}^{\varphi}(L_0) \leq 7 + 4 \cdot 4 = 23 < 24$, a contradiction.

(ii) Let L_4 be the line of type $(7, 7, 4, 4, 4)$. In this case there exists a 26-line through a 7-point on L_1 (different from P) which is of type $(7, 4, *, *, *)$ and hence is forced to be of type $(7, 7, 4, 4, 4)$. This is clearly impossible since only L_0 and L_4 can have points of multiplicity 4.

(iii) The proof is similar to that of (ii).

(iv) Let L_1 be of type $(7, 7, 7 - \varepsilon, 3, 2 + \varepsilon)$, and let L_2, L_3, L_4 be of type $(7, 7, 4, 4, 4)$. Now L_0 is forced to be of type $(7, 5, 4, 4, 4)$. Two of the 7-points on L_1 plus the 7-points on L_2, L_3, and L_4 form an oval which is extendable to a hyperoval by adding a point on L_0. Now through the point of multiplicity $7 - \varepsilon$ on L_1 we have a secant to the hyperoval (different from L_1) which is of type $(7, 7, 7-\varepsilon, 4, 4)$, which is impossible.

(v) The pointset $\{X \mid \mathcal{K}^{\varphi}(P) = 7\}$ is an oval. Denote the nucleus of the oval by N. Clearly, $\mathcal{K}^{\varphi}(N) \geq 5$ since there is a line of type $(7, \mathcal{K}^{\varphi}(N), 4, 4, 4)$. If L_0 is of type $(7, \mathcal{K}^{\varphi}(N), x_1, x_2, x_3)$ then $2 \leq x_i \leq 4$. Hence the structure of $\mathcal{K}$ can be represented as

$$\mathcal{K} = \mathcal{F} - \mathcal{B}.$$

Here $\mathcal{F}$ is the sum of the whole space plus a cone with an hyperoval as a base curve minus twice the vertex of the cone P; $\mathcal{B}$ is a set of 8 points blocking once every plane that does not contain P. Clearly $\mathcal{B} \cup \{P\}$ is a $(9, 1)$-blocking set with a 4-line. □

Lemma 7. *There exists no* $(100, 26)$*-arc in* $\mathrm{PG}(3, 4)$ *with the following property: every 7-line with a 0-point is contained in two 25-planes.*

Proof. With L and P as in the previous theorem, let π_0, π_1, and π_2 be the 26-planes and π_3, π_4 – the 25-planes through L. We have the following possibilities for the three 26-planes through L:

(i) AAA, (ii) AAC, (iii) ACC, (iv) CCC.

In all cases we consider a projection φ from P. We set $L_i := \varphi(\pi_i)$, $Q = \varphi(L)$.

(i) Here we shall deal with the case when L_0, L_1, L_2 are all of type $(7,7,7,3,2)$. The case when some (or all) of these lines are of type $(7,7,6,3,3)$ is ruled out in the same way. Assume three of the 7-points different from Q are collinear. Then the line defined by them meets one of L_3 or L_4 (L_3 say) in a point of multiplicity at most 2. Hence there is a line through this point which is of type $(7,3,3,\leq 2,x)$ or $(3,3,3,\leq 2,x)$. In the first case we get a line of multiplicity at most 22, a contradiction. In the second we get $x \leq 4$, which gives a line of multiplicity at most 15 and is again impossible.

Now the six points different from Q on L_0, L_1, L_2 that are of multiplicity 7 form a hyperoval. Through the 2-point on L_0, there exist two external lines to the hyperoval and one of them has to be of type $(2,2,3,*,*)$ (or $(2,2,2,*,*)$). Now it is an easy check that this line should be of multiplicity less than 16, which is impossible.

(ii) Let L_0 and L_1 be of type $(7,7,7-\varepsilon,3,2+\varepsilon)$, $\varepsilon \in \{0,1\}$, and let L_2 be of type $(7,7,4,4,4)$. First observe that L_3 or L_4 do not have a point of multiplicity 7. In such case there is a line of type $(7,4,3/2,3/2,*)$, which is impossible since a 25- and a 24-plane do not meet in a 7-line. Assume one of L_0 and L_1, L_0 say, is of type $(7,7,7,3,2)$. Now through the 2-point on L_0 there exist two lines of type $(2,3,4,4,4)$ or $(2,2,4,4,4)$, and hence each of L_3, L_4 has two points of multiplicity 4. Since type $(7,5,5,4,4)$ is impossible for L_3 or L_4 (by the nonexistence of 26-planes with a 5-line which contains a 0-point), both lines are of type $(7,6,4,4,4)$. This implies that L_1 is also of type $(7,7,7,3,2)$ and the set

$$\{X \mid \mathcal{K}^{\varphi}(X) = 7, X \neq P\} \cup \{Y \mid Y \in L_3, \mathcal{K}^{\varphi}(Y) = 6\}$$

is not a hyperoval (since it has tangents). Hence there is a line of type $(7,7,7,4,4)$ or $(7,7,6,4,2)$. The former is clearly impossible and the latter is ruled out by Lemma 2.

Now we are left with the case where the lines L_0 and L_1 are both of type $(7,7,6,3,3)$. The three 7-points different from P on L_0, L_1, L_2 are obviously not collinear. Now there exists a 26-line of type $(7,4,6/3,*,*)$ which is again ruled out by Lemma 2.

(iii) The proof is similar to that of (ii).

(iv) In this case the lines L_0, L_1, L_2 are all of type $(7,7,4,4,4)$ and the three 7-points different from Q are not collinear. Then L_3 and L_4 have three points of multiplicity at most 4, whence they are of type $(7,6,4,4,4)$. Now $\{X \mid \mathcal{K} \geq 6\}$ is a hyperoval. The arc $\mathcal{K}$ can be represented as $\mathcal{K} = \mathcal{F} - \mathcal{B}$, where $\mathcal{F}$ and $\mathcal{B}$ are as in Lemma 6(v). Again $\mathcal{B} \cup \{P\}$ is a $(9,1)$-blocking set with two 3-lines meeting in a point and four coplanar points in a general position. A blocking set with this structure does not exist. □

Summing up the results from Lemma 6 and Lemma 7 we get the following theorem.

Theorem 5. *Let* $\mathcal{K}$ *be a* $(100, 26)$*-arc in* $\mathrm{PG}(3, 4)$*. Then* $\mathcal{K}$ *is one of the following:*

(1) the sum of a cap and the whole space minus two points;

(2) the arc from the cone construction (case (v) in Lemma 6).

4. THE NONEXISTENCE OF $(395, 100)$- AND $(396, 100)$-ARCS IN $\mathrm{PG}(4, 4)$

In this section we prove the nonexistence of arcs with parameters $(395, 100)$ and $(396, 100)$ in $\mathrm{PG}(4, 4)$. Equivalently, there exist no $[395, 5, 295]_4$- and $[396, 5, 296]_4$-codes. This resolves two open cases in Maruta's tables for optimal linear codes with $k = 5$, $q = 4$, namely $n_4(5, 295) = 396$ and $n_4(5, 296) = 397$.

As already noted, we will tackle the problem geometrically and will prove the nonexistence of arcs in $\mathrm{PG}(4, 4)$ with parameters $(395, 100)$ and $(396, 100)$. The proof is based on the knowledge of the structure of the maximal planes which was completed in the previous section.

Theorem 6. *There exist no* $(396, 100)$*-arcs in* $\mathrm{PG}(4, 4)$*.*

Proof. Assume that $\mathcal{K}$ is a $(396, 100)$-arc in $\mathrm{PG}(4, 4)$. From the geometric version of Ward's divisibility theorem, as well as by easy counting we have that the admissible hyperplane multiplicities with respect to $\mathcal{K}$ are the following: $100, 96, 92, 86, 84, 82, 80, 78,$ and 76. Smaller hyperplanes are impossible since a $(100, 26)$-arc in $\mathrm{PG}(3, 4)$ does not have planes of multiplicity less than 20.

Since the number of 2-points in $\mathcal{K}$ is at least 55 and the maximal size of a cap in $\mathrm{PG}(4, 4)$ is 41 [2], there exist three collinear 2-points. A line incident with three 2-points is either a 6- or a 7-line.

First assume there is a 7-line L with three 2-points and consider a projection φ from L. This line is necessarily contained in a 26-plane, π say, and hence in a 100-solid. The five solids through π, denoted by Δ_i, $i = 0, \dots, 4$, are 100-solids of type (2). Hence $\mathcal{K}^\varphi$ has five 17- and sixteen 19-points. Moreover, the 17-points should form a blocking set and hence are collinear. Therefore there is a 92-solid through L.

Now consider another projection, denoted by ψ from the 0-point P of L. The image of a 100-solid has five 7-points, one 5-point and fifteen 4-points with the 7- and 5-points forming a hyperoval. Clearly, $\mathcal{K}^\psi$ has seventeen 7-points that form a cap. Each 7-point is incident with a unique tangent plane to the cap. This plane is forced to contain all the 5-points (since it is the image of the 92-solid above). This observation is true for every 7-point, which gives a contradiction.

Now consider a 6-line L consisting of three 2- and two 0-points, and a projection φ from L. Since a 100-solid does not have such a line, we have that L is contained in solids of multiplicity at most 96. Since every point is contained in five solids through L, counting the multiplicity of $\mathcal{K} - \chi_L$ we get

$$390 = |\mathcal{K} - \chi_L| \leq \frac{21 \cdot 90}{5} = 378,$$

a contradiction. □

Now we are going to prove the nonexistence of $(395, 100)$-arcs in $\mathrm{PG}(4, 4)$ by demonstrating that if such an arc exists it is extendable to the nonexisting $(396, 100)$-arc.

Theorem 7. *There exist no* $(395, 100)$*-arcs in* $\mathrm{PG}(4, 4)$.

Proof. Assume $\mathcal{K}$ is a $(395, 100)$-arc in $\mathrm{PG}(4, 4)$. As in the proof of theorem 6, there exist three collinear 2-points. We consider two cases: (a) the line L defined by these points is a 7-line, and (b) the line L defined by these points is a 6-line.

(a) The line L is necessarily contained in a 100-solid Δ_0, which is forced to be nonextedable. Since a $(100, 26)$-arc in $\mathrm{PG}(3, 4)$ has just planes of multiplicity 26, 24, 22, and 20, the possible multiplicities for solids with respect to $\mathcal{K}$ are: $100, 99, 92, 91, 86, \ldots, 83$, and $78, \ldots, 75$.

First we rule out the existence of 78- and 77-solids. By easy counting, solids of this multiplicity have to be projective. Hence such solids are either the complement of a line and two (resp. three points, or the complement of a Baer subplane (resp. Baer subplane and a point). Denote such a solid by Δ_1. Note that Δ_1 must meet Δ_0 in a 20-plane since the latter has no planes of smaller multiplicity. Consider a projection φ from a 4-line K in the plane $\Delta_0 \cap \Delta_1$. Now $\varphi(\Delta_0)$ is of type $(22, 22, 20, 16, 16)$. The possible types of the line $\varphi(\Delta_1)$ are the following:

if Δ_1 is a 77-plane: $(16, 16, 15, 15, 11)$, $(16, 16, 16, 14, 11)$, $(16, 16, 16, 15, 10)$, $(16, 16, 16, 16, 9)$;

if Δ_1 is a 78-plane: $(16, 16, 16, 14, 12)$, $(16, 16, 16, 15, 11)$, $(16, 16, 16, 16, 10)$.

We shall deal with the case when Δ_1 is a 78-solid (the case $\mathcal{K}(\Delta_1) = 77$ is treated analogously). The other three solids Δ_i, $i = 2, 3, 4$, through $\Delta_0 \cap \Delta_1$ are forced to be of multiplicity 99. Since a 26-plane in a 99-solid is contained in four 100-solids the image $\varphi(\Delta_i)$, $i = 2, 3, 4$, does not have a 22-point. Hence the lines $\varphi(\Delta_i)$, $i = 2, 3, 4$, are of type $(20, 20, 20, 19, 16)$. A 22-point in the projection plane is incident with four 96-lines (images of 100-solids) and one 95-line (the image of 99-solid). Therefore the 95-lines through each 22-point contain the 19-points on the lines $\varphi(\Delta_i)$, $i = 2, 3, 4$. This is obviously impossible.

Next we rule out the existence of 86-solids. A 22-plane in a 100-solid has one 2-point and twenty 1-points. Hence a 86-solid has one 2-point and eighty-four 1-points. Such a solid has just 21- and 22-planes and therefore the arc $\mathcal{K}$ has no further solids of multiplicity 85 and 86. This implies that $\mathcal{K}$ is extendable to the nonexistent $(396, 100)$-arc by Corollary 1.

Finally, an 85-solid Δ_1 must have a 2-point, otherwise all points are 1-points and all planes are 21-planes, which is impossible. If Δ_1 is an 85-solid then every 6-line is incident with one 21-plane, and four 22-planes; consequently, a 6-line has exactly one 2-point. This implies that Δ_1 consist either of one 2-point, one 0-point and all the rest 1-points, or of two 2-points, two 0-points (all these collinear) and all the rest 1-points. Now this is the only 85-solid since two such solids meet in a plane of multiplicity at most 18. Again $\mathcal{K}$ is extendable by Corollary 1 and we arrive at a contradiction.

(b) Now we assume that every three collinear 2-points determine a 6-line. Note that every 100-solid is an extendable $(100, 26)$-arc and consequently 26-planes cannot have three collinear 2-points. Consider such a 6-line, L say. Note that L is not contained in a 100-solid. Assume that L is contained in a 99-solid Δ. There exists a 25-plane π with $L \subset \pi \subset \Delta$. Denote by P, Q the two 0-points on L. If there exists a 7-line in π through P then counting the multiplicities of the planes through this line we get

$$99 \leq \mathcal{K}(\Delta) \leq 5 \cdot 25 - 4 \cdot 7 = 97,$$

a contradiction. This implies that $\mathcal{K}|_\pi$ is extendable to a $(26, 7)$-arc by turning P into 1-point. But this implies that Q is incident with a 7-line, again a contradiction.

We have proved that the multiplicities of the solids through L do not exceed 97. Consider a projection φ from L. We have $|\mathcal{K}^\varphi| = 389$ and by the above argument $\mathcal{K}^\varphi(M) \leq 91$ for every line M in the projection plane. Now counting the multiplicities of all lines in the plane of projection, we get

$$389 = |\mathcal{K}^\varphi| \leq \frac{21 \cdot 91}{5} = \frac{1911}{5} < 383,$$

a contradiction. □

Finally, we state Theorems 6 and 7 in coding-theoretic terms.

Corollary 2. *There exist no linear codes with parameters* $[395, 5, 295]_4$*, and* $[396, 5, 296]_4$*. Consequently,* $n_4(5, 295) = 396$*, and* $n_4(5, 296) = 397$*.*

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