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A METHOD FOR SOLVING THE SPECTRAL PROBLEM OF HAMILTONIAN MATRICES WITH APPLICATION TO THE ALGEBRAIC RICCATI EQUATION

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In this paper an effective iterative method for computing the eigenvalues and eigenvectors of a real Hamiltonian matrix is described and its applicability discussed. The method is an adaptation for Hamiltonian matrices of the methods for computing eigenvalues of real matrices due to Veselić and Voevodin. It uses symplectic similarity transformations and preserves the Hamiltonian structure of the matrix. Our method can be used for solving algebraic Riccati equation. The method is tested numerically and a comparison with the performance of other numerical algorithms is presented.

Keywords: Hamiltonian matrix, Jacobi-like methods, algebraic Riccati equation
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1. INTRODUCTION

Many applications lead to solving the real spectral problem

$$Hx = \lambda x,$$

where

$$H = H(A, B, D) = \begin{pmatrix} A & B \\ D & -A^T \end{pmatrix},$$

$A \in \mathbb{R}^{n\times n},\ B = B^T \in \mathbb{R}^{n\times n},\ D = D^T \in \mathbb{R}^{n\times n}$.

Recall that the real matrix H is called Hamiltonian or J-skewsymmetric if $J^T HJ = -H^T$, where $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, I_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix [7, 11, 12]. A matrix $U \in \mathbb{R}^{n\times n}$ is called a symplectic or J-orthogonal if $U^T JU = J$. It is well-known that if H is a Hamiltonian matrix and U is a symplectic matrix, then the matrix $U^{-1}HU$ is a Hamiltonian matrix.

Generalizations of the Jacobi process for arbitrary matrices, based on the fact that there exists a matrix P such that $\tilde{A} = P^{-1}AP$ is arbitrarily close to being normal, have been proposed [3, 15, 16]. In other words, the absolute value of every element of $\tilde{A}\tilde{A}^* - \tilde{A}^*\tilde{A}$ is arbitrarily small. Byers [2] has proposed a symplectic Jacobi-like algorithm for the computation of the Hamiltonian-Schur decomposition of Hamiltonian matrices. Byers' method is an adaptation of the non-symmetric Jacobi method proposed by Stewart [14].

In this paper an iterative method for solving the spectral problem for a Hamiltonian matrix is developed. It is a modification for Hamiltonian matrices of Veselić's and Voevodin's methods for computing eigenvalues of real matrices [15, 16]. The method uses similarity transformations with symplectic matrices. These transformations keep the block structure of a Hamiltonian matrix. This method can be used for solution of the algebraic Riccati equation.

The algebraic Riccati equation is of great practical importance due to its key role in control theory. There exist different procedures for solving this equation: a method solving a suitable matrix equation [6], a method solving a spectral problem for the Hamiltonian matrix [13]. Other methods are discussed in [8, 10, 17].

2. DESCRIPTION OF THE ALGORITHM

Now we describe the algorithm of our method. In this algorithm we construct the following sequence of Hamiltonian similar matrices:

$$\begin{aligned} H_1(A_1, B_1, D_1) &= H(A, B, D), \\ H_{k+1} = H(A_{k+1}, B_{k+1}, D_{k+1}) &= U_k^{-1} H_k U_k = (h_{rs}^{(k+1)}), \ldots, \end{aligned} \tag{2.1}$$

$k = 1, 2, 3$, where $U_k = U_{p_k q_k}(\varphi_k)$ is a suitable symplectic matrix. The matrix U_k depends on three parameters p_k, q_k and φ_k for each k. At each iteration step the parameters of U_k are chosen either to minimize

$$||U_k^{-1} H_k U_k||, \quad \text{where} \quad ||H_k|| = \sum_{rs} \left(h_{rs}^{(k)}\right)^2,$$

or to annihilate the off-diagonal elements of the symmetric matrix $H_{k+1} + H_{k+1}^T$.

To give an idea for the iterative process (2.1), we shall explain only the k-th iteration step of the algorithm. We introduce the notation

$$\begin{aligned} H_k &= H(A_k, B_k, D_k) = (h_{rs}^{(k)}), \\ A_k = (a_{\beta\gamma}^{(k)}),\ B &= (b_{\beta\gamma}^{(k)}),\ D_k = (d_{\beta\gamma}^{(k)}). \end{aligned}$$

For the matrices $H_k + H_k^T$ and C_k we have

$$\begin{aligned} H_k + H_k^T &= H(A_k + A_k^T, B_k + D_k, D_k + B_k), \\ C_k = C(H_k) &= (c_{rs}^{(k)}) = H_k H_k^T - H_k^T H_k = H(F_k, E_k, E_k), \end{aligned}$$

where

$$\begin{aligned} F_k &= F_k^T = A_k A_k^T + B_k B_k - A_k^T A_k - D_k D_k = (f_{\beta\gamma}^{(k)}), \\ E_k &= E_k^T = A_k D_k - B_k A_k - A_k^T B_k + D_k A_k^T = (e_{\beta\gamma}^{(k)}). \end{aligned}$$

The strategy, determining U_k from (2.1) and parameters p_k, q_k, φ_k, is the following. At the k-th iteration step we find the numbers

$$c^{(k)} = \max_{r \neq s} |c_{rs}^{(k)}|^{\frac{1}{2}} \text{ and } h^{(k)} = \max_{r \neq s} |h_{rs}^{(k)} + h_{sr}^{(k)}|$$

for the matrices C_k and H_k.

Then there are six possible cases to be considered successively:

A.1. $|f_{pq}^{(k)}|^{1/2} = c^{(k)} \geq h^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi = \varphi_k$.

In this case we choose the matrix $U_k = U_{pq}(\varphi)$ of the form

$$U = U_{pq}(\varphi) = \begin{pmatrix} S_{pq}(\varphi) & 0 \\ 0 & S_{pq}^{-T}(\varphi) \end{pmatrix}, \tag{2.2}$$

where $S_{pq}(\varphi) \in \mathbb{R}^{n \times n}$ is the matrix

$$S_{pq}(\varphi) = (s_{ij}) = \begin{cases} s_{qp} = \varphi, \\ s_{ij} = \delta_{ij}, \quad (i,j) \notin \{(q,p)\}. \end{cases}$$

Note that $S_{pq}^{-1}(\varphi) = S_{pq}(-\varphi)$.

The parameter φ is computed by the formula

$$\varphi = \frac{2 f_{pq}^{(k)}}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)}, \tag{2.3}$$

where

$$\begin{aligned} M_{qp}^{(k)} &= 2\sum_{j \neq q} \left(\left(a_{jq}^{(k)}\right)^2 + \left(b_{pj}^{(k)}\right) \right)^2 + 2\sum_{j \neq p} \left(\left(a_{pj}^{(k)}\right)^2 + \left(d_{qj}^{(k)}\right)^2 \right) \\ &+ 2\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right)^2 + 4\left(b_{pq}^{(k)}\right)^2 + 4\left(d_{pq}^{(k)}\right)^2 \\ &+ \tau^2 \left(\left(b_{pp}^{(k)}\right)^2 + \left(b_{qq}^{(k)}\right)^2 + \left(d_{pp}^{(k)}\right)^2 + \left(d_{qq}^{(k)}\right)^2 + 2\left(a_{pq}^{(k)}\right)^2 + \left(a_{qp}^{(k)}\right)^2 \right). \end{aligned}$$

A.2. $|e_{pq}^{(k)}|^{1/2} = c^{(k)} \geq h^{(k)}$, $1 \leq p = p_k < q = q_k \leq n$, $\varphi_k = \varphi$.

Then the matrix $U = U_{pq}(\varphi)$ is of the form

$$U = \begin{pmatrix} I_n & S_{pq}(\varphi) \\ 0 & I_n \end{pmatrix}, \tag{2.4}$$

where $S_{pq}(\varphi) \in \mathbb{R}^{n \times n}$ is the matrix

$$S_{pq}(\varphi) = (s_{ij}) = \begin{cases} s_{pq} = \varphi, \\ s_{ij} = 0, \quad (i,j) \notin \{(p,q)\}. \end{cases}$$

In this case the parameter φ is computed by the formula

$$\varphi = \frac{2\, e_{pq}^{(k)}}{\max\left(2|e_{pq}^{(k)}|, M_{qp}^{(k)}\right)}, \tag{2.5}$$

where

$$\begin{aligned} M_{qp}^{(k)} &= 2\sum_j \left(\left(d_{qj}^{(k)}\right)^2 + \left(d_{pj}^{(k)}\right)\right)^2 + 2\sum_{j\neq q}\left(a_{jq}^{(k)}\right)^2 2\sum_{j\neq p}\left(a_{jp}^{(k)}\right)^2 \\ &+ 2\left(a_{pp}^{(k)} + a_{qq}^{(k)}\right)^2 + 4\left(a_{pq}^{(k)}\right)^2 + 4\left(a_{qp}^{(k)}\right)^2 \\ &+ 2\,\tau^2\left(\left(b_{pp}^{(k)}\right)^2 + \left(b_{qq}^{(k)}\right)^2 + \left(b_{pq}^{(k)}\right)^2 + \left(d_{pp}^{(k)}\right)^2 + \left(d_{qq}^{(k)}\right)^2 + \left(d_{pq}^{(k)}\right)^2\right). \end{aligned}$$

A.3. $|e_{pp}^{(k)}|^{1/2} = c^{(k)} \geq h^{(k)},\ 1 \leq p = p_k = q = q_k \leq n,\ \varphi_k = \varphi.$
Then the matrix $U = U_{pq}(\varphi)$ is of the form

$$U = \begin{pmatrix} I_n & S_p(\varphi) \\ 0 & I_n \end{pmatrix}, \tag{2.6}$$

where

$$S_p(\varphi) = \mathrm{diag}[I_{p-1}, \varphi, I_{n-p}]$$

and φ is computed by

$$\varphi := \frac{e_{pp}^{(k)}}{\max\left(|e_{pp}^{(k)}|, M_{pp}^{(k)}\right)} \tag{2.7}$$

and

$$M_{pp}^{(k)} = 2\sum_j \left(d_{pj}^{(k)}\right)^2 + 2\sum_{j\neq p}\left(a_{jp}^{(k)}\right)^2 + 4\left(a_{pp}^{(k)}\right)^2 + \tau^2\left(\left(b_{pp}^{(k)}\right)^2 + \left(d_{pp}^{(k)}\right)^2\right).$$

A.4. $|a_{pq}^{(k)} + a_{qp}^{(k)}| = h^{(k)} > c^{(k)},\ 1 \leq p = p_k < q = q_k \leq n,\ \varphi = \varphi_k.$
In this case we choose the matrix $U_k = U_{pq}(\varphi)$,

$$U_{pq}(\varphi) = \mathrm{diag}[T_{pq}(\varphi), T_{pq}(\varphi)], \tag{2.8}$$

where $T_{pq}(\varphi) \in \mathbb{R}^{n\times n}$ is a matrix of the form

$$T_{pq}(\varphi) = (t_{\beta\gamma}) = \begin{cases} t_{pp} = t_{qq} = \cos\ \varphi, \\ t_{pq} = -t_{qp} = -\sin\ \varphi, \\ t_{\beta\gamma} = \delta_{\beta\gamma}, \quad (\beta,\gamma) \notin \{(p,p),(p,q),(q,p),(q,q)\}. \end{cases}$$

The parameter φ is computed from the equation

$$\mathrm{tg}(2\varphi) = \frac{a_{pq}^{(k)} + a_{qp}^{(k)}}{a_{pp}^{(k)} - a_{qq}^{(k)}}.$$

A.5. $|b_{pq}^{(k)} + b_{qp}^{(k)}| = h^{(k)} > c^{(k)},\ 1 \le p = p_k < q = q_k \le n,\ \varphi = \varphi_k$.
We choose the matrix $U_k = U_{pq}(\varphi)$,

$$U_{pq}(\varphi) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \tag{2.9}$$

where $C,\ S \in \mathbb{R}^{n\times n}$ and

$$\begin{aligned} C &= \mathrm{diag}[I_{p-1}, \cos\varphi, I_{q-p}, \cos\varphi, I_{m-q}], \\ S &= (s_{\beta\gamma}) = \begin{cases} s_{pq} = s_{qp} = \sin\varphi, \\ s_{\beta\gamma} = 0, \quad (\beta,\gamma) \notin \{(p,q),(q,p)\}. \end{cases} \end{aligned}$$

In this case the parameter φ is computed from the equation

$$\mathrm{tg}(2\varphi) = \frac{b_{pq}^{(k)} + d_{qp}^{(k)}}{a_{pp}^{(k)} + a_{qq}^{(k)}}.$$

A.6. $|b_{pp}^{(k)} + b_{pp}^{(k)}| = h^{(k)} > c^{(k)},\ 1 \le p = p_k = q = q_k \le n,\ \varphi = \varphi_k$.
The matrix $U_k = U_{pq}(\varphi)$ has the form

$$U_{pq}(\varphi) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \tag{2.10}$$

where $C,\ S \in \mathbb{R}^{n\times n}$ and

$$\begin{aligned} C &= \mathrm{diag}[I_{p-1}, \cos\varphi, I_{m-p}], \\ S &= (s_{\beta\gamma}) = \begin{cases} s_{pp} = \sin\varphi, \\ s_{\beta\gamma} = 0, \quad (\beta,\gamma) \notin \{(p,p)\}, \end{cases} \end{aligned}$$

with the parameter φ computed now from the equation

$$\mathrm{tg}(2\varphi) = \frac{b_{pp}^{(k)} + d_{pp}^{(k)}}{a_{pp}^{(k)} + a_{pp}^{(k)}}.$$

We shall prove now the following

Lemma 2.1. Let p, q, $1 \leq p \leq q \leq n$, and $H_{k+1} = U^{-1}H_kU$, where $U = U_{pq}(\varphi)$ is given by (2.2), (2.4) or (2.6); the parameter φ is computed by means of Eqs. (2.3), (2.5) or (2.7), respectively. Then

$$||H_k||^2 - ||H_{k+1}||^2 \geq \frac{2}{\tau^2}\frac{(\tau^2-\tau-1)(c^{(k)})^4}{\max(2(c^{(k)})^2, M_{qp}^{(k)})} \geq \frac{1}{\tau^2}\frac{(c^{(k)})^4}{||H_k||^2}, \qquad (2.11)$$

where $\tau \geq (1+\sqrt{5})/2$.

Proof. Let us choose p, q so that $1 \leq p < q \leq n$. We shall provide a detailed proof of the lemma in the case **A.1.** only, since the reasoning in the rest of the cases is fully similar.

We compute φ from (2.3), choose the matrix U from (2.2) and construct $H_{k+1} = U^{-1}H_kU$. Then for $\Delta(\varphi)$ we get

$$\begin{aligned} \Delta(\varphi) &= ||H_k||^2 - ||H_{k+1}||^2 \\ &= -G\varphi^4 - Q\varphi^3 - W\varphi^2 + 4f_{pq}^{(k)}\varphi, \end{aligned}$$

where

$$\begin{aligned} G &= 2\left(a_{pq}^{(k)}\right)^2 + \left(b_{pp}^{(k)}\right)^2 + \left(d_{qq}^{(k)}\right)^2, \\ Q &= 4a_{pq}^{(k)}\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right) - 4b_{pp}^{(k)}b_{pq}^{(k)} + 4d_{qq}^{(k)}d_{pq}^{(k)}, \\ W &= 2\sum_{i\neq q}\left(\left(a_{jq}^{(k)}\right)^2 + \left(b_{pj}^{(k)}\right)^2\right) + 2\sum_{j\neq p}\left(\left(a_{pj}^{(k)}\right)^2 + \left(d_{qj}^{(k)}\right)^2\right) \\ &+ \left(a_{pp}^{(k)} - a_{qq}^{(k)}\right)^2 + 4\left(b_{pq}^{(k)}\right)^2 + 4\left(d_{pq}^{(k)}\right)^2 \\ &+ 2b_{pp}^{(k)}b_{qq}^{(k)} + 2d_{pp}^{(k)}d_{qq}^{(k)} - 4a_{pq}^{(k)}a_{qp}^{(k)}. \end{aligned}$$

We find

$$\frac{|W|}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)} \leq \frac{|W|}{M_{qp}^{(k)}} \leq 1.$$

Consider the inequality

$$2xy \leq \frac{t^4x^2 + y^2}{t^2} \qquad (2.12)$$

which holds for all real numbers t, x, y; setting, in particular, $t = \sqrt{\tau}$, $x = \sqrt{2}|a_{pq}^{(k)}|$, $y = \sqrt{2}|a_{pp}^{(k)} - a_{qq}^{(k)}|$, we obtain

$$4a_{pq}^{(k)}\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right) \leq \frac{\tau^2 2\left(a_{pq}^{(k)}\right)^2 + 2\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right)^2}{\tau}.$$

Similarly, we get

$$4b_{pp}^{(k)}b_{pq}^{(k)} \le \frac{\tau^2\left(b_{pp}^{(k)}\right)^2 + 4\left(b_{pq}^{(k)}\right)^2}{\tau}.$$

Thus the expression $|Q|$ becomes

$$|Q| \le \frac{\tau^2 2\left(a_{pq}^{(k)}\right)^2 + 2\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right)^2}{\tau}$$

$$+ \frac{\tau^2\left(b_{pp}^{(k)}\right)^2 + 4\left(b_{pq}^{(k)}\right)^2}{\tau} + \frac{\tau^2\left(d_{qq}^{(k)}\right)^2 + 4\left(d_{pq}^{(k)}\right)^2}{\tau}.$$

Consequently,

$$\frac{|Q|}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)} \le \frac{|Q|}{M_{qp}^{(k)}} \le \frac{1}{\tau}.$$

Similarly, we have

$$\frac{|G|}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)} \le \frac{|G|}{M_{qp}^{(k)}} \le \frac{2\left(a_{pq}^{(k)}\right)^2 + \left(b_{pp}^{(k)}\right)^2 + \left(d_{qq}^{(k)}\right)^2}{M_{qp}^{(k)}} \le \frac{1}{\tau^2}.$$

Using the above inequalities, we obtain

$$\begin{aligned}
\Delta(\varphi) &\ge 4f_{pq}^{(k)}\varphi - |W|\varphi^2 - |Q|\varphi^3 - |G|\varphi^4 \\
&\ge 2\varphi^2 \max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right) - |W|\varphi^2 - |Q|\varphi^2 - |G|\varphi^2 \\
&= \max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)\left(2\varphi^2 - \frac{|W| + |Q| + |G|}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)}\varphi^2\right) \\
&\ge \max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)\left(2\varphi^2 - (1 + \frac{1}{\tau} + \frac{1}{\tau^2})\varphi^2\right) \\
&= \max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)\left(\frac{\tau^2 - \tau - 1}{\tau^2}\right)\varphi^2
\end{aligned}$$

Then

$$\Delta(\varphi) \ge \frac{\tau^2 - \tau - 1}{\tau^2}\frac{4(f_{pq}^{(k)})^2}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)}.$$

Since

$$\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right) < 4\left(\tau^2 - \tau - 1\right)||H_k||^2, \quad \tau > (1 + \sqrt{5})/2,$$

and therefore

$$\Delta(\varphi) \geq \frac{1}{\tau^2} \frac{(f_{pq}^{(k)})^2}{\|H_k\|^2} = \frac{1}{\tau^2} \frac{(c^{(k)})^4}{\|H_k\|^2}.$$

This completes the proof of the case **A.1**. The proofs of cases **A.2** and **A.3** are fully similar, as already pointed out. □

Lemma 2.2. Let p, q be natural numbers, $1 \leq p \leq q \leq n$, and $H_{k+1} = U^{-1} H_k U$, where H_k is a matrix from the sequence (2.1). Let the matrix $U = U_{p_k q_k}(\varphi_k)$ be given by (2.2), (2.4) or (2.6), and φ be computed by (2.3), (2.5) or (2.7), respectively. Then

$$|h_{rs}^{(k+1)} - h_{rs}^{(k)}| \leq 4|c_{pq}^{(k)}|^2$$

for $r, s = 1, \ldots, 2n$.

Proof. Let p, q be integers, $1 \leq p < q \leq n$, and let U be a matrix of the type (2.2) with φ computed by means of (2.3). Then

$$|h_{rs}^{(k+1)} - h_{rs}^{(k)}| = \begin{cases} |a_{rs}^{(k+1)} - a_{rs}^{(k)}|, & 1 \leq r, s \leq n, \\ |b_{rs-n}^{(k+1)} - b_{rs-n}^{(k)}|, & 1 \leq r \leq n,\ n+1 \leq s \leq 2n, \\ |d_{r-ns}^{(k+1)} - d_{r-ns}^{(k)}|, & n+1 \leq r \leq 2n,\ 1 \leq s \leq n, \\ |-a_{r-ns-n}^{(k+1)} + a_{r-ns-n}^{(k)}|, & n+1 \leq r, s \leq 2n. \end{cases}$$

For the expression $|a_{rs}^{(k+1)} - a_{rs}^{(k)}|$ we obtain in turn

$$|a_{rs}^{(k+1)} - a_{rs}^{(k)}| = \begin{cases} |a_{rp}^{(k)} + \varphi a_{rq}^{(k)} - a_{rp}^{(k)}|, & r = 1, \ldots, n, r \neq q, \\ |a_{qs}^{(k)} - \varphi a_{ps}^{(k)} - a_{qs}^{(k)}|, & s = 1, \ldots, n, s \neq p, \\ |a_{qp}^{(k)} - \varphi\left(a_{pp}^{(k)} - a_{qq}^{(k)}\right) - \varphi^2 a_{pq}^{(k)} - a_{qp}^{(k)}|, & r = p,\ s = q, \\ 0, & \text{otherwise}. \end{cases}$$

Hence

$$\begin{aligned} |a_{rp}^{(k+1)} - a_{rp}^{(k)}| &\leq |\varphi a_{rq}^{(k)}| \\ &\leq \frac{2|f_{pq}^{(k)}|}{\max\left(2|f_{pq}^{(k)}|, M_{qp}^{(k)}\right)} |a_{rq}^{(k)}| \\ &\leq 2|f_{pq}^{(k)}| = 2|c_{pq}^{(k)}|^2 \end{aligned}$$

for $r = 1, \ldots, n$ and $r \neq q$.

In the same manner it can be shown that

$$\begin{aligned}
|a_{qs}^{(k+1)} - a_{qs}^{(k)}| &\leq 2|f_{pq}^{(k)}| = 2|c_{pq}^{(k)}|^2, \qquad s = 1,\ldots,n, \; s \neq p, \\
|a_{qp}^{(k+1)} - a_{qp}^{(k)}| &\leq |\varphi||a_{pp}^{(k)} - a_{qq}^{(k)}| + |\varphi|^2|a_{pq}^{(k)}| \\
&\leq |\varphi| \left(|a_{pp}^{(k)} - a_{qq}^{(k)}| + |a_{pq}^{(k)}|\right) \\
&\leq 4|f_{pq}^{(k)}| = 4|c_{pq}^{(k)}|^2
\end{aligned}$$

and

$$\begin{aligned}
|b_{rs-n}^{(k+1)} - b_{rs-n}^{(k)}| &\leq 4|c_{pq}^{(k)}|^2, \\
|d_{r-ns}^{(k+1)} - d_{r-ns}^{(k)}| &\leq 4|c_{pq}^{(k)}|^2.
\end{aligned}$$

This completes the proof of the case **A.1**. The proofs of cases **A.2.** and **A.3.** are again similar and omitted. □

Theorem 2.3. The iterative method (2.1) has the following properties:

I. $C(H_k) \to 0, \quad k \to \infty$.

II. The symmetric matrix $\frac{1}{2}(H_k + H_k^T)$ tends to the diagonal matrix $\frac{1}{2}(H_0 + H_0^T)$, where $H_0 = H(A_0, B_0, D_0) = (h_{rs}^{(0)})$, and

$$\frac{1}{2}(H_0 + H_0^T) = \text{diag}[h_{11}^{(0)}, \ldots, h_{2n2n}^{(0)}],$$

where $h_{ii}^{(0)}$ are the real parts of the eigenvalues of the matrix H.

III. Let p, q be natural numbers, $1 \leq p \neq q \leq 2n$, and $h_{pp}^{(0)} \neq h_{qq}^{(0)}$. Then

$$h_{pq}^{(k)} \to 0, \quad k \to \infty.$$

IV. Let p, q be natural numbers, $1 \leq p \neq q \leq 2n$, $h_{pp}^{(0)} = h_{qq}^{(0)}$, and for each t, $1 \leq t \leq 2n$, $t \neq p, q$, we have $h_{tt}^{(0)} \neq h_{pp}^{(0)}$. Then

$$h_{pq}^{(k)} \to h_{pq}^{(0)}, \quad k \to \infty,$$

where $h_{pq}^{(0)}$ is the imaginary part of the eigenvalues of H with real part $h_{pp}^{(0)}$.

Proof. I. We consider the sequence $||H_k||^2$, $k = 1, 2, 3, \ldots$ The similarity transformations with matrices of the form (2.8), (2.9) and (2.10) preserve the Euclidean norm and the similarity transformations with matrices of the form (2.2), (2.4) and (2.6) decrease or preserve the Euclidean norm implying that the sequence $||H_1||^2, ||H_2||^2, \ldots$ is monotonically decreasing. Let for each matrix H_k a number α_k be introduced such that

$$\alpha_k = \begin{cases} 0, & \text{if } U_k \text{ is of the form (2.8), (2.9), (2.10)}, \\ 1, & \text{if } U_k \text{ is of the form (2.2), (2.4), (2.6)}. \end{cases}$$

The sequence of matrices $\{H_k\}$ is bounded. From this sequence we choose a convergent subsequence $\{H_s\}$, where $s \in S \subset N$ and N is the set of natural numbers. Suppose that $\{\alpha_s\}$ contains an infinite number of ones and $\{H_m\}$ is such a subsequence of $\{H_s\}$ that for each $m \in M \subset S \subset N$ we have $\alpha_m = 1$. Then from Lemma 1 it follows

$$||H_m||^2 - ||H_{m+1}||^2 \geq \frac{1}{\tau^2} \frac{(c^{(m)})^4}{||H||^2}.$$

Hence $c^{(m)} \to 0$, $m \to \infty$. From the inequality $h^{(m)} \leq c^{(m)}$ it follows that $h^{(m)} \to 0$, $m \to \infty$.

Let $H_0 = H(A_0, B_0, D_0) = (h_{rs}^{(0)})$ be a limit of the sequence $\{H_m\}$. For $H_0 + H_0^T$ and $C(H_0)$ we obtain

$$\begin{aligned} H_0 + H_0^T &= H(A_0 + A_0^T, B_0 + D_0^T, D_0 + B_0^T) = (h_{rs}^{(0)} + h_{sr}^{(0)}), \\ C(H_0) &= H(F_0, E_0, E_0) = (c_{rs}^{(0)}), \end{aligned}$$

where

$$\begin{aligned} F_0 &= F_0^T = A_0 A_0^T + B_0 B_0 - A_0^T A_0 - D_0 D_0 = (f_{\beta\gamma}^{(0)}), \\ E_0 &= E_0^T = A_0 D_0 - B_0 A_0 - A_0^T B_0 + D_0 A_0^T = (e_{\beta\gamma}^{(0)}). \end{aligned}$$

Since H_0 is a limit, then if $r \neq s$, we have $h_{rs}^{(0)} + h_{sr}^{(0)} = 0$ and $c_{rs}^{(0)} = 0$. From $h_{rs}^{(0)} + h_{sr}^{(0)} = 0$, $r \neq s$, it follows that

$$a_{\beta\gamma}^{(0)} + a_{\gamma\beta}^{(0)} = 0, \quad \beta \neq \gamma, \tag{2.13}$$

$$b_{\beta\gamma}^{(0)} - d_{\gamma\beta}^{(0)} = 0, \quad \beta, \gamma = 1, \ldots, n. \tag{2.14}$$

For the elements $f_{\beta\gamma}^{(0)}$ of $C(H_0)$ we obtain

$$f_{\beta\gamma}^{(0)} = \sum_j \left(a_{j\beta}^{(0)} a_{j\gamma}^{(0)} - a_{\beta j}^{(0)} a_{\gamma j}^{(0)} + d_{\beta j}^{(0)} d_{j\gamma}^{(0)} - b_{\beta j}^{(0)} b_{j\gamma}^{(0)} \right).$$

From (2.13), (2.14) and $c_{rs}^{(0)} = 0$ we compute

$$f_{\beta\gamma}^{(0)} = 2a_{\beta\gamma}^{(0)} \left(a_{\beta\beta}^{(0)} - a_{\gamma\gamma}^{(0)} \right) = 0. \tag{2.15}$$

Consequently,

$$f_{\beta\beta}^{(0)} = 0, \quad \beta = 1, \ldots, n.$$

Similarly, we find

$$e_{\beta\gamma}^{(0)} = 2b_{\beta\gamma}^{(0)} \left(a_{\beta\beta}^{(0)} + a_{\gamma\gamma}^{(0)} \right) = 0, \quad \beta, \gamma = 1, \ldots, n. \tag{2.16}$$

Hence $c_{rs}^{(0)} = 0$ for each r and s, i.e. H_0 is a normal matrix and $H_0 + H_0^T$ is a diagonal matrix. Then the diagonal elements $h_{11}^{(0)}, \ldots, h_{2n2n}^{(0)}$ are the real parts of eigenvalues of H.

For the subsequence $\{H_m\}$ we have $C(H_m) \to 0$, $m \to \infty$, where $\|H_m\|^2 \to 2\sum_{j=1}^{n} |\nu_j|^2$, $\nu_j = \lambda_j + i\mu_j$. Since the whole sequence $\|H_k\|$ is non-increasing, it follows that $\|H_k\| \to 2\sum_{j=1}^{n} |\nu_j|^2$, i.e. $C(H_k) \to 0$, $k \to \infty$. Hence the sequence H_k tends to a normal matrix.

Let the sequence $\{\alpha_k\}$ contains only a finite number of ones. Then there is a natural number s_0 and for each s, so that $s \geq s_0$, $\alpha_s = 0$, i.e. the matrices U_s are of the form (2.8) or (2.9) or (2.10). Then

$$h^{(s)} \to 0, \ s \to \infty.$$

Since $c^{(s)} \leq h^{(s)}$, then

$$c^{(s)} \to 0, \ s \to \infty.$$

Hence the convergent subsequence H_s has a limit $H_0 = H(A_0, B_0, D_0)$ with the properties (2.13) – (2.16). This proves I.

II. We will prove that $h^{(k)} \to 0$, $k = 1, 2, \ldots$, for the sequence $\{H_k\}$. In proving I we have found a subsequence $\{H_s\}$ of H_k. We consider the case when the sequence $\{\alpha_k\}$ contains an infinite number of both zeros and ones. Let $\{\alpha_p\}$ be a subsequence of $\{\alpha_k\}$. If $\alpha_p = 1$, then $h^{(p)} \to 0$, $p \to \infty$. We consider the sequence of all indices $k_1, \ldots, k_s, \ldots$, so that $\alpha_{k_s} = 0$ and $\alpha_{k_s - 1} = 1$ for $s = 1, 2, \ldots$ In the case $m = k_s$, according to Lemma 2 we obtain

$$\begin{aligned} |h^{(m)}| &\leq |h^{(m-1)}| + |h^{(m)} - h^{(m-1)}| \\ &\leq |h^{(m-1)}| + 8(c^{(m-1)})^4. \end{aligned}$$

Since $c^{(m-1)} \to 0$ and $h^{(m-1)} \to 0$ for $m = k_s$ and $s = 1, 2, \ldots$ $(m - 1 = k_s - 1)$, it follows that $h^{(m)} \to 0$. Let σ_k^2 denote a sum of the squares of off-diagonal elements in blocks of the symmetric matrix $H_k + H_k^T$. Then we have

$$h^{(k)} \leq \sigma_k^2 \leq 2n(2n-1)h^{(k)}.$$

For the subsequence $\{H_m\}$ from $h^{(m)} \to 0$ it follows that $\sigma_m^2 \to 0$.

Consider the indices $m + t$ of $\{\alpha_k\}$. For $m = k_s$ it is true for $\alpha_{m-1} = 1$, $\alpha_{m-1+t} = 0$, for $t = 1, 2, \ldots, p$ and $\alpha_{m+p} = 1$ for $s = 1, 2, \ldots$ For these indices the number sequence σ_{m+t}^2 is monotonically decreasing, because for the matrix $H_{m+t} + H_{m+t}^T$ a step is used from a modification of Jacobi's method for a symmetric Hamiltonian matrix [9].

It thus follows that $\sigma_k^2 \to 0$ for $k = 1, 2, \ldots$, and $h^{(k)} \to 0$ for the same k. Hence from I we obtain that each convergent subsequence of H_k has a limit with the properties (2.13) – (2.16) and its diagonal elements are the real parts of the eigenvalues of H.

III. Now we will prove that if $h_{pp}^{(0)} \neq h_{qq}^{(0)}$, $p \neq q$, then $h_{pq}^{(k)} \to 0$, $k \to \infty$. There are three possible cases.

Let p, q be natural numbers, $1 \leq p \neq q \leq n$. Then $h_{pp}^{(0)} = a_{pp}^{(0)}$ and $h_{qq}^{(0)} = a_{qq}^{(0)}$. Since $h_{pq}^{(k)} \to h_{pq}^{(0)}$, $a_{pp}^{(0)} \neq a_{qq}^{(0)}$, from (2.15) we have $a_{pq}^{(0)} = 0$, i.e. $h_{pq}^{(0)} = 0$.

Let p, q be natural numbers, $1 \leq p \leq n$, $n+1 \leq q \leq 2n$. Then $h_{pp}^{(0)} = a_{pp}^{(0)}$, $h_{qq}^{(0)} = -a_{q-nq-n}^{(0)}$, $h_{pq}^{(0)} = b_{pq-n}^{(0)}$. Since $h_{pq}^{(k)} \to h_{pq}^{(0)}$, from (2.16) when $p \neq q-n$ we have

$$e_{pq-n}^{(0)} = 2b_{pq-n}^{(0)} \left(h_{pp}^{(0)} - h_{qq}^{(0)} \right) = 0.$$

Hence $b_{pq-n}^{(0)} = 0$. When $p = q-n$, from (2.16) we obtain

$$e_{pp}^{(0)} = 4b_{pp}^{(0)}\, a_{pp}^{(0)} = 0.$$

If $a_{pp}^{(0)} = 0$, it follows that $h_{pp}^{(0)} = h_{qq}^{(0)}$, because $h_{pp}^{(0)} = a_{pp}^{(0)} = 0$, $h_{qq}^{(0)} = -a_{q-nq-n}^{(0)} = -a_{pp}^{(0)}$. Hence $b_{pp}^{(0)} = 0$.

In the case $n+1 \leq p \leq 2n$, $1 \leq q \leq n$, the proof is similar to that of the case $1 \leq p \leq n$, $n+1 \leq q \leq 2n$.

IV. Let $\{H_s\}$ be a convergent subsequence of $\{H_k\}$ with the limit

$$H_0 = H(A_0, B_0, D_0) = \left(h_{rs}^{(0)} \right).$$

The limit H_0 possesses the properties (2.13) – (2.16). Let $h_{pp}^{(0)} = h_{qq}^{(0)}$, $p \neq q$, and for each $t \neq p, q$, $h_{tt}^{(0)} \neq h_{pp}^{(0)}$. We choose the number t so that $1 \leq t \leq 2n$, $t \neq p, q$. Then, according to III, in the rows and columns of H_0 with numbers p, q there will exist only two nonzero off-diagonal elements $h_{pq}^{(0)}$, $h_{qp}^{(0)}$.

If $1 \leq p, q \leq n$, then the nonzero off-diagonal elements are $a_{pq}^{(0)}$, $a_{qp}^{(0)}$. Consequently, $a_{pp}^{(0)} \pm \mathrm{i}\ a_{pq}^{(0)}$ and $-a_{pp}^{(0)} \pm \mathrm{i}\ a_{pq}^{(0)}$ are eigenvalues of the Hamiltonian matrix H.

If $1 \leq p \leq n$, $n+1 \leq q \leq 2n$, $p \neq q-n$, then the nonzero off-diagonal elements are $b_{pq-n}^{(0)}$, $d_{q-np}^{(0)} = -b_{pq-n}^{(0)}$. This implies that $a_{pp}^{(0)} \pm \mathrm{i}\ b_{pq-n}^{(0)}$ are eigenvalues of the Hamiltonian matrix H. From the type of H_0 follows that $h_{q-nq-n}^{(0)} = h_{p+np+n}^{(0)}$. Hence $a_{q-nq-n}^{(0)} \pm \mathrm{i}\ b_{q-np}^{(0)}$ are eigenvalues of the Hamiltonian matrix H.

If $1 \leq p \leq n$, $n+1 \leq q \leq 2n$, $p = q-n$, from the type H_0 it follows that $h_{pp}^{(0)} = h_{qq}^{(0)} = 0$. Then the nonzero elements are $b_{pp}^{(0)}$, $d_{pp}^{(0)} = -b_{pp}^{(0)}$, and $\pm\mathrm{i}\ b_{pp}^{(0)}$ are eigenvalues of the Hamiltonian matrix H. □

3. APPLICATIONS AND NUMERICAL EXPERIMENTS

Numerical experiments for solving the spectral problem for Hamiltonian matrices and for numerical computing of the solution of the algebraic Riccati equation are performed and will be reported in this section. All numerical experiments were made on a PENTIUM computer using the algorithmic language *Turbo Pascal* and the real arithmetic having an 11 sedecimal digit mantissa. The code of our algorithm uses a cyclic choice on the pivot indices (p, q).

The presented method for computing the spectral problem of a Hamiltonian matrix is the Jacobi type method for solving the eigenproblem of real nonsymmetric matrices. The reason is that in the Jacobi method for finding the eigenvalues only two rows and columns are involved in each iteration step of our method. The parallel implementation of our algorithm can be followed of those for Jacobi algorithm for symmetric eigenvalue on the hypercube or a linear array of processors [4] and on distributed memory multiprocessors [5].

3.1. THE SPECTRAL PROBLEM FOR HAMILTONIAN MATRICES

The code of our algorithm computes the eigenvalues of an $(n \times n)$-Hamiltonian matrix $H = H(A, B, D)$. Let us denote $\varepsilon = \max_i |\lambda_i - \tilde{\lambda}_i|$, where λ_i are the exact eigenvalues and $\tilde{\lambda}_i$ are the computed eigenvalues obtained by our algorithm.

Example 1 [1]. Consider the matrix

$$H = U \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} U^T, \tag{3.1}$$

where $A \in \mathbb{R}^{n \times n}$,

$$A = (a_{ij}) = \begin{cases} a_{11} = -10^3, \\ a_{ii} = n + 1 - i, \quad i = 2, \ldots, n-2, \\ a_{n-1n-1} = a_{nn} = 2, \\ a_{n-1n} = -a_{nn-1} = 1, \\ a_{ij} = 0, \quad \text{otherwise}, \end{cases}$$

and the matrix U is the product of randomly generated symplectic matrices of the form (2.2), (2.4), (2.8). The following results are obtained in this case:

TABLE 1

n	5	10	15	20	25
ε	$1.3245E-7$	$4.2331E-7$	$2.1289E-7$	$1.5673E-7$	$5.3289E-6$

Example 2. Consider the matrix (3.1), where A is a diagonal matrix with randomly chosen elements and the matrix U is the product of randomly generated symplectic matrices of the form (2.2), (2.4), (2.8). The results are shown in the following table:

TABLE 2

n	5	10	15	20	25
ε	$2.5463E-10$	$1.3568E-10$	$6.8452E-8$	$3.4562E-8$	$1.2344E-6$

Example 3. We have executed numerical experiments of random strict diagonal dominant Hamiltonian matrices using Byers' algorithm [2] and the algorithm proposed here. Table 3 displays the average number of sweeps necessary for convergence. Each trial includes 10 matrices of the different dimensions.

TABLE 3

n	Byers' algorithm	Our algorithm
10	14	12
15	15	15
20	18	16
30	19	16

We compare Byers' method with our method. Byers' algorithm computes $2n^2$ similarity transformations per sweep, the method proposed here computes $n(n+1)/2$ similarity transformations. Byers' algorithm makes $32n^3 + O(n^2)$ flops for computing the $2n^2$ transformations. Our algorithm makes $20n^3 + O(n^2)$ flops for computing the $n(n+1)/2$ transformations. Hence we obtain that one sweep of the Byers' algorithm is more expensive than a sweep of the algorithm proposed here. Our algorithm is faster than Byers' algorithm for the above set of examples (Example 3). Moreover, Byers' method uses complex arithmetic, while in our method real arithmetic is solely utilized.

In the case of a symmetric Hamiltonian matrix our method uses similarity transformations with orthogonal symplectic matrices of the form (2.2), (2.4), (2.6). The amount of work for performing the transformations per sweep is $12n^3 + O(n^2)$ flops. We have made numerical experiments for randomly generating symmetric Hamiltonian matrices for the same dimensions as in Example 3. We have obtained that the average number of sweeps of the method proposed here is equal to the average number of sweeps of Byers' method.

3.2. NUMERICAL SOLUTION OF THE ALGEBRAIC RICCATI EQUATION

The algorithm for computing the eigenvalues and eigenvectors of a real Hamiltonian matrix presented here can be successfully used to calculate the solution of the Riccati equation

$$L(X) = XBX - XA - A^T X - D = 0, \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B = B^T \in \mathbb{R}^{n\times n}$, $D = D^T \in \mathbb{R}^{n\times n}$ and B is a positive definite matrix, D is a positive semidefinite matrix.

The computation of the solution P of (3.2) leads to the solving of the spectral problem for the Hamiltonian matrix $H = H(A, -B, -D)$. An algorithm for computing the solution P is described in [8, 10]. The matrix $H = H(A, -B, -D)$ is reduced in Schur's form $\tilde{H}$ with the QR-algorithm

$$U^T H U = \tilde{H}, \quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

For the solution P of the equation (3.2) we have $P = U_{21}U_{11}^{-1}$ [8, 10].

We propose the following algorithm for solving the algebraic Riccati equation (3.2). We compute the eigenvalues and eigenvectors of the Hamiltonian matrix $H = H(A, -B, -D)$ with the algorithm described in Section 2. The matrix U of the eigenvectors of H is partitioned into four $(n \times n)$-blocks

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

and then we compute the solution $P = U_{21}U_{11}^{-1}$ of the matrix equation (3.2).

We have made numerical experimens for computing the solution of the equation (3.2). For these experiments three algorithms have been used. The first algorithm $W1$ uses the QR-method [8, 10]. The second algorithm $W2$ uses the iterative method described by Petkov and Ivanov [13]. This method computes the eigenvalues and eigenvectors of $H = H(A, -B, -D)$ and then the solution $P = U_{21}U_{11}^{-1}$ is found. The third algorithm $W3$ uses the iterative method wich solves the spectral problem of a Hamiltonian matrix. In the programs of algorithms $W2$ and $W3$ we use a cyclic choice on the pivot indices (p, q).

The matrices A, B, D, for which the solution of the equation (3.2) is computed, are the matrices from Example 1 and Example 2 from Section 3.1 and the examples described below. On each trial we compute the accuracy of the computed solution — $\|L(P)\|_\infty$.

The results from Example 1 and Example 2 are given in Table 4.

TABLE 4

Example	$W1$	$W2$	$W3$
Example 1 $n = 5$ $\|L(P)\|_\infty$	$1.0862E-7$	$1.8703E-7$	$6.2748E-8$
Example 1 $n = 10$ $\|L(P)\|_\infty$	$3.3222E-7$	$4.1609E-7$	$3.3191E-7$
Example 1 $n = 20$ $\|L(P)\|_\infty$	$7.9954E-4$	$3.1569E-4$	$1.4978E-4$
Example 2 $n = 5$ $\|L(P)\|_\infty$	$2.8741E-10$	$5.2502E-9$	$1.2554E-9$
Example 2 $n = 10$ $\|L(P)\|_\infty$	$2.3272E-5$	$5.0117E-8$	$2.6402E-7$
Example 2 $n = 20$ $\|L(P)\|_\infty$	$7.5153E-7$	$1.3549E-7$	$2.0637E-8$

Example 4. The blocks A, B and D in the Riccati equation are of the type

$$\begin{aligned} A &= (a_{ij}) = \begin{cases} ij, & \text{if } i = j, \\ i + j, & \text{if } i \neq j, \end{cases} \\ B &= \operatorname{diag}[1, 2^2, \ldots, n^2], \\ D &= \operatorname{diag}[1, 2, \ldots, n]. \end{aligned}$$

The results from this example are shown in the following table:

TABLE 5

n	$W1$	$W2$	$W3$
5 $\|\|L(P)\|\|_\infty$	$6.7767E-5$	$6.9849E-9$	$6.9028E-8$
10 $\|\|L(P)\|\|_\infty$	$1.3256E-3$	$1.6763E-8$	$2.5378E-8$
20 $\|\|L(P)\|\|_\infty$	$3.4268E-3$	$1.6938E-7$	$1.2096E-7$

Example 5 (Example 5 in [10]). We compute the solution of the Riccati equation with $n = 5, 10, 20$. The results are shown in the following table:

TABLE 6

n	$W1$	$W2$	$W3$
5 $\|\|L(P)\|\|_\infty$	$2.7048E-8$	$9.8542E-7$	$9.8556E-8$
10 $\|\|L(P)\|\|_\infty$	$2.1153E-8$	$1.1726E-9$	$1.3171E-9$
20 $\|\|L(P)\|\|_\infty$	$1.2149E-4$	$5.0361E-9$	$5.1435E-9$

Example 6 (Example 6 in [10]). We compute the solution of the Riccati equation with $n = 21$ and $q = r = 1$. For the correct results $x_{1n} = 1$ we receive the value $x_{1n} = 1.0792769258E+00$.

There are examples (Example 4 and Example 5) for which the iterative methods $W2$ and $W3$ for computing the solution of the Riccati equation are more accurate than the QR-method ($W1$).

4. CONCLUSION

We have presented and investigated a new method for solving the spectral problem of Hamiltonian matrices. The method is a generalization of the Jacobi-like method for arbitrary real matrices, as proposed by Veselić [15]. It allows us to construct a new algorithm for solving the algebraic Riccati equation. Our method preserves the special structure of a Hamiltonian matrix and uses less memory than the algorithm $W1$ (QR-method). The method offers simpler computational schemes and gives better options for parallel modifications.

We note finally that the algorithm proposed here can be modified as well for solving the spectral problem for a symplectic matrix. But we were not able to prove a convergence theorem in this case.

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