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ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
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A MOTION OF A FAST SPINNING RIGID BODY ABOUT A FIXED POINT IN A SINGULAR CASE

ABDEL ISMAIL

In this paper the problem of motion of a rigid body about a fixed point under the action of a Newtonian force field is studied for a singular value of the natural frequency ($\omega = 1/3$). This singularity deals with different bodies being classified according to the moments of inertia. Using Poincaré's small parameter method, the periodic solutions — with non-zero basic amplitudes — of the quasi-linear autonomous system are obtained in the form of power series expansions, up to the third approximation, containing assumed small parameter. Also, the quasi-linear autonomous system is integrated numerically using any of the numerical integration methods, such as the fourth order Runge – Kutta method. At the end, a comparison between the analytical and the numerical solutions is given aiming to get a small deviation between them.

Keywords: rigid body motion, small parameter method, periodic solutions

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1. INTRODUCTION

In [1] the motion of a fast spinning rigid body about a fixed point in a central Newtonian force field is considered. The nonlinear differential equations of motion have been reduced to a quasi-linear autonomous system having one first integral. In the case of the rational value of the natural frequency ω (except $\omega = 1/2,\ 1,\ 2,\ 1/3,\ 3$) the periodic solutions of the initial system are obtained. Here, the analytical and the numerical solutions for the case when $\omega = 1/3$ are constructed. Let us consider a rigid body of mass (M) with one fixed point (O), whose ellipsoid of inertia is arbitrary, and acted upon by a central Newtonian force field arising from attracting centre being located on a vertical downwards axis (Z) passing through the fixed point. Let us assume ($OXYZ$) to be the fixed frame in space and ($Oxyz$) to be the moving frame (fixed of the body). It is taken into consideration that the principal axis (z) of the ellipsoid of inertia makes an angle $\theta_o \neq m\pi/2$ ($m = 0, 1, 2, \ldots$) with Z-axis and that the body spins about z-axis with a high angular velocity r_o. Without a loss of generality we select the positive branches of the z-axis and of the x-axis do not make an obtuse angle with the direction of the Z-axis. According to the restriction on θ_o and the selection of the co-ordinate system one gets

$$\gamma_o \geq 0, \quad 0 < \gamma_o'' < 1,$$

the limiting case $\gamma_o'' \approx 0$ has been studied in [2]. The following system of equations of motion and its first integral can be deduced:

$$\ddot{p}_2 + \frac{1}{9}p_2 = \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \qquad \ddot{\gamma}_2 + \gamma_2 = \mu^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \tag{1}$$

$$\gamma_o''^{-2} - 1 = \gamma_2^2 + \dot{\gamma}_2^2 + 2\mu(\nu p_2\gamma_2 + \nu_2\dot{p}_2\dot{\gamma}_2 + s_{21}) + \mu^2\Big[\nu_2^2\dot{p}_2^2 - 2\dot{\gamma}_2(e_2A_1^{-1}\dot{\gamma}_2$$
$$+A_1^{-1}\dot{p}_2 s_{21} + \frac{1}{2}\dot{\gamma}_2 s_{11} - y_o'a^{-1}A_1^{-1}) + \nu^2 p_2^2 + s_{21}^2 + 2\Big(s_{22} - \frac{1}{2}s_{11}\Big)\Big] + \mu^3(\ldots), \tag{2}$$

where

$$\begin{aligned}
F &= F_2 + \mu F_3 + \cdots, & \Phi &= \Phi_2 + \mu\Phi_3 + \cdots,\\
F_2 &= f_2 - \frac{8}{9}\nu e_1 p_2, & \Phi_2 &= \phi_2 + \frac{8}{9}\nu(e + e_1\gamma_2),\\
F_3 &= f_3 - e_1\phi_2 - \frac{8}{9}\nu e_1(e + e_1\gamma_2), & \Phi_3 &= \phi_3 - \nu f_2 + \frac{8}{9}\nu^2 e_1 p_2,
\end{aligned}$$

$$\begin{aligned}
f_2 = {} & A_1 b^{-1}x_o' s_{21} - \frac{1}{9}p_2 s_{11} + C_1 A_1^{-1} p_2\dot{p}_2^2 - y_o'a^{-1}p_2\dot{\gamma}_2\\
& + x_o'\dot{p}_2\dot{\gamma}_2 - z_o'a^{-1}p_2 - y_o'A_1^{-1}(A_1 + a^{-1})\gamma_2\dot{p}_2\\
& - k[(1 - C_1)\gamma_2\dot{p}_2\dot{\gamma}_2 + A_1(1 + B_1)\gamma_2 s_{21} - A_1 p_2(1 - \dot{\gamma}_2^2)],\\
\phi_2 = {} & -\gamma_2 s_{11} + (1 + B_1)p_2 s_{21} - (1 - C_1)A_1^{-1}p_2\dot{p}_2\dot{\gamma}_2 + x_o'\dot{\gamma}_2{}^2 - y_o'\gamma_2\dot{\gamma}_2\\
& - z_o'b^{-1}\gamma_2 + x_o'b^{-1} - A_1^{-2}\gamma_2\dot{p}_2^2 + k(C_1\dot{\gamma}_2^2 - B_1)\gamma_2,\\
f_3 = {} & C_1 A_1^{-1}\dot{p}_2[e\dot{p}_2 + e_1\gamma_2\dot{p}_2 - 2p_2(y_o'a^{-1} - e_2\dot{\gamma}_2)] - \frac{1}{9}(es_{11} + e_1\gamma_2 s_{11} + 2p_2 s_{12})\\
& + A_1 b^{-1}x_o' s_{22} + x_o'[\nu_2\dot{p}_2^2 - \dot{\gamma}_2(y_o'a^{-1} - e_2\dot{\gamma}_2)] - y_o'a^{-1}[\dot{\gamma}_2(e + e_1\gamma_2) + \nu_2 p_2\dot{p}_2]\\
& + y_o'(1 + A_1^{-1}a^{-1})[\gamma_2(y_o'a^{-1} - e_2\dot{\gamma}_2) - \nu p_2\dot{p}_2] + \frac{1}{2}z_o'(a^{-1} - A_1 b^{-1})\gamma_2 s_{11}\\
& - z_o'a^{-1}(e + e_1\gamma_2 + p_2 s_{21}) + k\Big[(1 - C_1)(y_o'a^{-1} - e_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2\\
& - \nu(1 - C_1)p_2\dot{p}_2\dot{\gamma}_2 - 2\nu_2 A_1 p_2\dot{p}_2\dot{\gamma}_2 - \nu_2(1 - C_1)\gamma_2\dot{p}_2^2 - \nu A_1(1 + B_1)p_2 s_{21}\\
& + 2A_1 p_2 s_{21} + \Big(\frac{1}{9} - A_1\Big)\gamma_2 s_{22} + A_1(e + e_1\gamma_2)(1 - \dot{\gamma}_2^2)\Big],
\end{aligned}$$

$$
\begin{aligned}
\phi_3 &= 2x'_o\nu_2\dot{p}_2\dot{\gamma}_2 - 2\gamma_2 s_{12} - \nu p_2 s_{11} + (1+B_1)[p_2 s_{22} + (e + e_1\gamma_2)s_{21}] \\
&\quad +(1-C_1)A_1^{-1}[p_2\dot{\gamma}_2(y'_o a^{-1} - e_2\dot{\gamma}_2) - \nu_2 p_2\dot{p}_2^2 - (e+e_1\gamma_2)\dot{p}_2\dot{\gamma}_2] \\
&\quad -z'_o b^{-1}(\nu p_2 + \gamma_2 s_{21}) + 2x'_o b^{-1} s_{21} + A_1^{-2}[2\gamma_2\dot{p}_2(y'_o a^{-1} - e_2\dot{\gamma}_2) - \nu p_2\dot{p}_2^2] \\
&\quad -y'_o(\nu p_2\dot{\gamma}_2 + \nu_2\gamma_2\dot{p}_2) + k[\nu p_2(C_1\dot{\gamma}_2^2 - B_1) + 2\gamma_2(\nu_2 C_1\dot{p}_2\dot{\gamma}_2 - B_1 s_{21})]; \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
p_2 &= p_1 - \mu e - \mu e_1\gamma_2, \qquad \gamma_2 = \gamma_1 - \mu\nu p_2, \\
q_1 &= -A_1^{-1}\dot{p}_2 + \mu A_1^{-1}(y'_o a^{-1} - e_2\dot{\gamma}_2) + \mu^2\Big[(aA_1)^{-1}y'_o s_{21} + \frac{1}{2}A_1^{-1}\dot{p}_2 s_{11} + k\dot{\gamma}_2 s_{21} \\
&\quad -\nu_2\dot{p}_2(a^{-1}A_1^{-1}z'_o - k)\Big] + \mu^3\Big[(aA_1)^{-1}y'_o s_{22} + \frac{1}{2}A_1^{-1}e_1\dot{\gamma}_2 s_{11} + A_1^{-1}\dot{p}_2 s_{12} \\
&\quad +a^{-1}A_1^{-2}e_1 z'_o\dot{\gamma}_2 + a^{-1}A_1^{-1}s_{11}(z'_o\dot{\gamma}_2 - y'_o) + (k - a^{-1}A_1^{-1}z'_o)(a^{-1}A_1^{-1}y'_o \\
&\quad -a^{-1}A_1^{-1}z'_o\dot{\gamma}_2 + k\dot{\gamma}_2) + a^{-1}A_1^{-2}z'_o\dot{p}_2 s_{21} + k(\nu\dot{p}_2 s_{21} + \dot{\gamma}_2 s_{22} \\
&\quad -A_1^{-1}e_1\dot{\gamma}_2 - \frac{3}{2}\dot{\gamma}_2 s_{11} - 2A_1^{-1}\dot{p}_2 s_{21})\Big] + \cdots, \\
r_1 &= 1 + \frac{1}{2}\mu^2 s_{11} + \mu^3 s_{12} + \cdots, \\
\gamma'_1 &= \dot{\gamma}_2 + \mu\nu_2\dot{p}_2 + \mu^2\left[(aA_1)^{-1}y'_o - A_1^{-1}(e_2\dot{\gamma}_2 + \dot{p}_2 s_{21}) - \frac{1}{2}\dot{\gamma}_2 s_{11}\right] \\
&\quad +\mu^3\Big[-A_1^{-1}(e_1\dot{\gamma}_2 s_{21} + \dot{p}_2 s_{22}) + \frac{1}{2}(3A_1^{-1} - \nu)\dot{p}_2 s_{11} - \dot{\gamma}_2 s_{12} \\
&\quad +\nu_2(k - a^{-1}A_1^{-1}z'_o)\dot{p}_2 + 2a^{-1}A_1^{-1}y'_o s_{21} + (2k - a^{-1}A_1^{-1}z'_o)\dot{\gamma}_2 s_{21}\Big] + \cdots, \\
\gamma''_1 &= 1 + \mu s_{21} + \mu^2\Big(s_{22} - \frac{1}{2}s_{11}\Big) - \mu^3\Big(s_{12} + \frac{1}{2}s_{11}s_{21}\Big) + \cdots; \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
p_1 &= p/c\sqrt{\gamma''_o}, \qquad q_1 = q/c\sqrt{\gamma''_o}, \qquad r_1 = r/r_o, \qquad \gamma_1 = \gamma/\gamma''_o, \\
\gamma'_1 &= \gamma'/\gamma''_o, \qquad \gamma''_1 = \gamma''/\gamma''_o, \qquad \tau = r_o t, \qquad (\dot{\ } \equiv d/d\tau); \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
s_{11} &= a(p_{2o}^2 - p_2^2) + b(\dot{p}_{2o}^2 - \dot{p}_2^2)/A_1^2 - 2[x'_o(\gamma_{2o} - \gamma_2) + y'_o(\dot{\gamma}_{2o} - \dot{\gamma}_2)] \\
&\quad +k[a(\gamma_{2o}^2 - \gamma_2^2) + b(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)], \\
s_{12} &= a[e(p_{2o} - p_2) + e_1(p_{2o}\gamma_{2o} - p_2\gamma_2)] - bA_1^{-2}[y'_o a^{-1}(\dot{p}_{2o} - \dot{p}_2) \\
&\quad -e_2(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)] - \nu x'_o(P_{2o} - p_2) - \nu_2 y'_o(\dot{p}_{2o} - \dot{p}_2) + (z'_o - k)s_{21} \\
&\quad +k[\nu a(p_{2o}\gamma_{2o} - p_2\gamma_2) + \nu_2 b(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)], \\
s_{21} &= a(p_{2o}\gamma_{2o} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2), \\
s_{22} &= a[\nu(p_{2o}^2 - p_2^2) + e(\gamma_{2o} - \gamma_2) + e_1(\gamma_{2o}^2 - \gamma_2^2)] + bA_1^{-1}[-\nu_2(\dot{p}_{2o}^2 - \dot{p}_2^2) \\
&\quad +a^{-1}y'_o(\dot{\gamma}_{2o} - \dot{\gamma}_2) - e_2(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)]; \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \frac{C-B}{A}, \quad B_1 = \frac{A-C}{B}, \quad C_1 = \frac{B-A}{C}, \quad a = \frac{A}{C}, \quad b = \frac{B}{C}, \\
c^2 &= \frac{Mg\ell}{C}, \quad \mu = \frac{c\sqrt{\gamma''_o}}{r_o}, \quad x_o = \ell x'_o, \quad y_o = \ell y'_o, \quad z_o = \ell z'_o, \\
\ell^2 &= x_o^2 + y_o^2 + z_o^2, \quad A_1 B_1 = -\frac{1}{9}, \quad e = 9x'_o A_1 b^{-1}, \quad \nu = \frac{9}{8}(1 + B_1), \\
e_1 &= \frac{9}{8}\left[k\left(A_1 - \frac{1}{9}\right) + z'_o(A_1 b^{-1} - a^{-1})\right], \quad e_2 = e_1 + a^{-1} z'_o - kA_1, \\
\nu_2 &= \nu - A_1^{-1}, \quad k = N\gamma''_o/c^2, \quad N = 3g/R, \quad g = \lambda/R^2; \qquad (7)
\end{aligned}
$$

here λ is the constant of gravity of the attracting centre, R is the distance from the fixed point to such centre, (p_o, q_o, r_o) and $(\gamma_o, \gamma'_o, \gamma''_o)$ are the initial values of projections of the angular velocity vector (p, q, r) of the body on the principal axes of inertia and the direction cosines $(\gamma, \gamma', \gamma'')$ of Z-axis, respectively, A, B and C are the principal moments of inertia and x_o, y_o and z_o are the co-ordinates of the centre of mass in the moving co-ordinate system.

2. PROPOSED METHOD

In this section Poincaré's small parameter method is applied to investigate the non-zero basic amplitude periodic solutions of system (1). The generating system ($\mu = 0$) of (1) is

$$\ddot{p}_2^{(0)} + \frac{1}{9}p_2^{(0)} = 0, \qquad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0, \tag{8}$$

which admits periodic solutions in the forms

$$p_2^{(0)} = M_1 \cos\frac{1}{3}\tau + M_2 \sin\frac{1}{3}\tau, \qquad \gamma_2^{(0)} = M_3 \cos\tau \tag{9}$$

with period $T_o = 6\pi$, and M_1, M_2 and M_3 are constants which have to be determined. Since the system (1) is autonomous, the condition

$$\dot{\gamma}_2(0, \mu) = 0 \tag{10}$$

does not restrict the generality of the required solutions [3].

Applying Poincaré's method, the periodic solutions for system (1) are considered in the forms [4]

$$
\begin{aligned}
p_2(\tau, \mu) &= \tilde{M}_1 \cos\frac{1}{3}\tau + \tilde{M}_2 \sin\frac{1}{3}\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau), \\
\gamma_2(\tau, \mu) &= \tilde{M}_3 \cos\tau + \sum_{k=2}^{\infty} \mu^k H_k(\tau) \qquad (11)
\end{aligned}
$$

with period $T = 6\pi + \alpha(\mu)$ and initial conditions

$$p_2(0, \mu) = \tilde{M}_1, \quad \dot{p}_2(0, \mu) = \frac{1}{3}\tilde{M}_2, \quad \gamma_2(0, \mu) = \tilde{M}_3, \quad \dot{\gamma}_2(0, \mu) = 0, \tag{12}$$

where $\alpha(\mu) = 0$ at $\mu = 0$ and "$\sim$" denotes the result of substitution

$$M_i \rightarrow \tilde{M}_i = M_i + \beta_i \quad (i = 1, 2, 3), \tag{13}$$

here β_1, $\frac{1}{3}\beta_2$ and β_3 denote the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of system (1) from their initial values of the generating system (8), these deviations are functions of μ and equal zero when $\mu = 0$. Let us define the functions $G_k(\tau)$ and $H_k(\tau)$ by the operator [5]

$$U = u + \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 + \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \cdots \quad \left\{ \begin{array}{l} U = G_k, H_k \\ u = g_k, h_k \end{array} \right\}. \tag{14}$$

The functions $g_k(\tau)$ and $h_k(\tau)$ take the forms

$$\begin{aligned} g_k(\tau) &= 3\int_0^\tau F_k^{(0)}(t_1)\sin\frac{1}{3}(\tau - t_1)\,dt_1, \\ h_k(\tau) &= \int_0^\tau \Phi_k^{(0)}(t_1)\sin(\tau - t_1)\,dt_1, \quad k = 2, 3. \end{aligned} \tag{15}$$

The solutions (9) are written as follows:

$$p_2^{(0)} = E\cos\left(\frac{1}{3}\tau - \epsilon\right), \qquad \gamma_2^{(0)} = M_3\cos\tau, \tag{16}$$

where $E = \sqrt{M_1^2 + M_2^2}$, $M_1 = E\cos\epsilon$ and $M_2 = E\sin\epsilon$.

Substituting (16) into (6), one obtains

$$s_{ij}^{(0)} = s_{ij}^{(0)}\left(p_2^{(0)}, \dot{p}_2^{(0)}, \gamma_2^{(0)}, \dot{\gamma}_2^{(o)}\right), \quad i, j = 1, 2. \tag{17}$$

Making use of (16), (17) and (3), the functions $F_k^{(0)}$ and $\Phi_k^{(0)}$ are obtained, then using (15), one gets $g_k(6\pi)$, $h_k(6\pi)$, $\dot{g}_k(6\pi)$ and $\dot{h}_k(6\pi)$. Substituting the initial conditions (12) into the integral (2), evaluated at $\tau = 0$, the quantity $\tilde{M}_3$ is determined as follows:

$$\tilde{M}_3 = (\gamma_o'')^{-1}(1-\gamma_o''^2)^{1/2} - \mu a\tilde{M}_1 - 9\mu^2\nu_2^2\tilde{M}_2^2/2M_3 - 3\mu^3 y_o'\nu_2\tilde{M}_2/aA_1M_3 + \cdots, \tag{18}$$

where $\nu_2 = 9b/(9 - 8b)$ and $0 < b < 1$ or $b > 9/8$. The independent conditions [6] for periodicity of the solutions $p_2(\tau,\mu)$, $\dot{p}_2(\tau,\mu)$, $\gamma_2(\tau,\mu)$ and $\dot{\gamma}_2(\tau,\mu)$ are reduced to the forms

$$\begin{aligned} \left(\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21}\right)\tilde{M}_2 &= -\mu\tilde{M}_2\Big\{\left(\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31}\right) - \tilde{M}_1\Big[2\tilde{L}_{34} \\ &\quad + \frac{1}{9}N_{33}\tilde{M}_3^{-1}(\tilde{M}_1^2 - 3\tilde{M}_2^2)\Big]\Big\} + \cdots, \\ \left(\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21}\right)\tilde{M}_1 &= -\mu\Big\{\tilde{M}_1\left(\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31}\right) + \Big[\tilde{L}_{34}(\tilde{M}_1^2 - \tilde{M}_2^2) \\ &\quad - \frac{1}{9}N_{33}\tilde{M}_1^2(\tilde{M}_1^2 - 3\tilde{M}_2^2)\tilde{M}_3^{-1}\Big]\Big\} + \cdots; \end{aligned} \tag{19}$$

$$\alpha(\mu) = \mu^2 \tilde{M}_3^{-1}[\dot{H}_2(6\pi) + \mu \dot{H}_3(6\pi) + \cdots], \tag{20}$$

where

$$\begin{aligned}
\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21} &= a_1(\tilde{M}_1^2 + \tilde{M}_2^2) - [a_2 + ka_3(2M_3\beta_3 + \beta_3^2)], \\
\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31} &= a_4 \tilde{M}_1 \tilde{M}_3, \\
a_1 &= (a-1)(a+b-2)/2b, \\
a_2 &= z_o'(ab)^{-1}[3(a+b) - 2(2ab+1)] + \frac{2}{9}k\left[1 - (a+b) + \frac{1}{2}bM_3^2\right], \\
a_3 &= \frac{1}{9}b, \qquad a_4 = \frac{1}{9}[20a - 9b^{-1} + ak(b-1)(32a-41)], \\
\tilde{L}_{34} &= -\frac{14}{3}\tilde{M}_3 z_o' b^{-1}(b-1)^{-1}(8b-9)^{-1}\left(b - \frac{3}{2}\right)\left(b - \frac{3}{4}\right)\left(b - \frac{33}{28}\right) \\
&\quad -\frac{1}{2}k\tilde{M}_3(b-1)(8b-9)^{-1}\Big\{(2b-3)[(9-4b) + (8b-15)(8b-9)^{-1}] \\
&\quad +\frac{1}{3}(4b-3)(4b-5)(9-7b)(b-1)^{-1}(8b-9)^{-1}\Big\}, \\
N_{33} &= \frac{1}{4}\Big\{\frac{1}{9}A_1^{-1}[\nu(A_1^{-1} + C_1) + \nu_2(1 - C_1)] - \frac{8}{9}\nu\left(\frac{1}{9}bA_1^{-2} - a\right) \\
&\quad -(1 + B_1)\left(\nu a + \frac{1}{9}bA_1^{-1}\nu_2\right)\Big\}.
\end{aligned} \tag{21}$$

Equating to zero the terms of zero power of μ for equations (19), one gets two equations for determining M_1 and M_2. Solving the resulting equations, when $M_1 M_2 = 0$, we obtain

$$\begin{aligned}
&\text{(i)}\ \ M_1 = M_2 = 0, \\
&\text{(ii)}\ \ M_1 = 0, \qquad M_2 = \pm\sqrt{\frac{a_2}{a_1}}, \\
&\text{(iii)}\ \ M_1 = \pm\sqrt{\frac{a_2}{a_1}}, \qquad M_2 = 0.
\end{aligned} \tag{22}$$

If $M_1 M_2 \neq 0$, subtracting from the first equation of (19), multiplied by $\tilde{M}_1$, the second equation, multiplied by $\tilde{M}_2$, and dividing by μ, we get a new form for the periodicity conditions

$$3\tilde{M}_1^2 - \tilde{M}_2^2 + \mu[\cdots] = 0, \qquad \tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21} + \mu[\cdots] = 0. \tag{23}$$

The equations of the basic amplitudes of (23) are

$$3M_1^2 - M_2^2 = 0, \qquad a_1(M_1^2 + M_2^2) - a_2 = 0, \tag{24}$$

the following solutions for M_1 and M_2 are obtained:

$$M_1 = \pm\frac{1}{2}\sqrt{\frac{a_2}{a_1}}, \qquad M_2 = \pm\frac{\sqrt{3}}{2}\sqrt{\frac{a_2}{a_1}}, \tag{25}$$

where M_1 and M_2 are real under the condition

$$a_2 > 0; \tag{26}$$

this condition can be satisfied by choice of M_3, while $a_1 > 0$ is satisfied at all, since the initial fast spin r_o is assumed to be given about the major or the minor axis of the ellipsoid of inertia ($a > 1$, $b > 1$ or $a < 1$, $b < 1$). For this case β_1 and β_2 are assumed in the forms

$$\beta_1 = \sum_{k=1}^{3} \mu^k \ell_k + O(\mu^4), \qquad \beta_2 = \sum_{k=1}^{3} \mu^k m_k + O(\mu^4). \tag{27}$$

Considering (27), (23) and the substitution (13), one gets

$$\begin{aligned}
\ell_1 &= -aa_3kM_3/4a_1, \qquad m_1 = 3\ell_1 M_1/M_2, \\
\ell_2 &= \frac{1}{8}M_1^{-1}\left[-4\ell_1^2 + ka_3a_1^{-1}(a^2M_1^2 - 9M_2^2\nu_2^2)\right], \\
m_2 &= \frac{1}{2}M_2^{-1}\left(6M_1\ell_2 + 3\ell_1^2 - m_1^2\right), \\
\ell_3 &= \frac{1}{4}M_1^{-1}\Big[-4\ell_1\ell_2 + aa_1^{-1}a_3M_1k\Big(a\ell_1 + \frac{9}{2}M_3^{-1}\nu_2^2M_2^2\Big) \\
&\quad -a_1^{-1}a_3k(aM_3\ell_2 + 9M_2\nu_2^2m_1 + 3M_2y_o'\nu_2a^{-1}A_1^{-1})\Big], \\
m_3 &= M_2^{-1}\Big[aa_1^{-1}a_3M_1k\Big(a\ell_1 + \frac{9}{2}\nu_2^2M_2^2M_3^{-1}\Big) - a_1^{-1}a_3k(a\ell_2M_3 + 9\nu_2^2m_1M_2 \\
&\quad +3y_o'\nu_2M_2a^{-1}A_1^{-1}) - m_1m_2 - \ell_1\ell_2 - M_1\ell_3\Big].
\end{aligned} \tag{28}$$

The equations (14) and (15) give the functions $G_k(\tau)$ and $H_k(\tau)$, then the periodic solutions (11) are obtained up to the third approximation of μ. Making use of (4) and (5), we get the required periodic solutions as follows:

$$\begin{aligned}
p &= c\sqrt{\gamma_o''}\Big\{M_1\cos\frac{1}{3}\tau + M_2\sin\frac{1}{3}\tau + \mu\Big(e + \ell_1\cos\frac{1}{3}\tau + m_1\sin\frac{1}{3}\tau + e_1M_3\cos\tau\Big) \\
&\quad +\mu^2\sum_{i=0}^{7}\Big(X_{1i}\cos\frac{i}{3}\tau + X_{1i}'\sin\frac{i}{3}\tau\Big) \\
&\quad +\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{1j}\cos\frac{j}{3}\tau + Y_{1j}'\sin\frac{j}{3}\tau\Big) + Y_{19}\cos 3\tau\Big] + \cdots\Big\}, \quad i \neq 6, \\
q &= c\sqrt{\gamma_o''}\Big\{\frac{1}{3}A_1^{-1}\Big(M_1\sin\frac{1}{3}\tau - M_2\cos\frac{1}{3}\tau\Big) + \mu A_1^{-1}\Big[y_o'a^{-1} + e_2M_3\sin\tau \\
&\quad -\frac{1}{3}\Big(\ell_1\sin\frac{1}{3}\tau + m_1\cos\frac{1}{3}\tau\Big)\Big] + \mu^2\sum_{i=0}^{7}\Big(X_{2i}\cos\frac{i}{3}\tau + X_{2i}'\sin\frac{i}{3}\tau\Big)
\end{aligned}$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{2j}\cos\frac{j}{3}\tau+Y'_{2j}\sin\frac{j}{3}\tau\Big)+Y'_{29}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 6,$$

$$r = r_o\Big\{1+\frac{1}{2}E^2\mu^2\Big[\sum_{i=0}^{3}\Big(X_{3i}\cos\frac{i}{3}\tau+X'_{3i}\sin\frac{i}{3}\tau\Big)+X_{36}\cos 2\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{4}\Big(Y_{3j}\cos\frac{j}{3}\tau+Y'_{3j}\sin\frac{j}{3}\tau\Big)+Y_{36}\cos 2\tau\Big]+\cdots\Big\},\quad i\neq 1,$$

$$\gamma = \gamma''_o\Big\{M_3\cos\tau+\mu a\Big[\Big(M_1\cos\frac{1}{3}\tau+M_2\sin\frac{1}{3}\tau\Big)-M_1\cos\tau\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{6}\Big(X_{4i}\cos\frac{i}{3}\tau+X'_{4i}\sin\frac{i}{3}\tau\Big)+X_{49}\cos 3\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{4j}\cos\frac{j}{3}\tau+Y'_{4j}\sin\frac{j}{3}\tau\Big)+Y'_{49}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 2,4,$$

$$\gamma' = \gamma''_o\Big\{-M_3\sin\tau+\mu\Big[-\frac{1}{2}\nu_2\Big(M_1\sin\frac{1}{3}\tau-M_2\cos\frac{1}{3}\tau\Big)+aM_1\sin\tau\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{6}\Big(X_{5i}\cos\frac{i}{3}\tau+X'_{5i}\sin\frac{i}{3}\tau\Big)+X'_{59}\sin 3\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{5j}\cos\frac{j}{3}\tau+Y'_{5j}\sin\frac{j}{3}\tau\Big)+Y'_{59}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 2,4,$$

$$\gamma'' = \gamma''_o\Big\{1+\mu M_3E\Big[a\cos\epsilon+\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}-a\Big)\Big(\cos\epsilon\cos\frac{2}{3}\tau-\sin\epsilon\sin\frac{2}{3}\tau\Big)$$

$$-\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}+a\Big)\Big(\cos\epsilon\cos\frac{4}{3}\tau+\sin\epsilon\sin\frac{4}{3}\tau\Big)\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{4}\Big(X_{6i}\cos\frac{i}{3}\tau+X'_{6i}\sin\frac{i}{3}\tau\Big)+X_{66}\cos 2\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{10}\Big(Y_{6j}\cos\frac{j}{3}\tau+Y'_{6j}\sin\frac{j}{3}\tau\Big)\Big]+\cdots\Big\},\quad i\neq 1,\ j\neq 9, \qquad (29)$$

the correction of the period $\alpha(\mu)$ becomes

$$\alpha(\mu) = 3\pi\mu^2\Big\{-\Big(aM_1^2+\frac{bM_2^2}{9A_1^2}\Big)-\frac{1}{2}(M_1^2+M_2^2)\Big[\frac{1}{9}A_1^{-2}(1-b)+aB_1\Big]$$

$$+2M_3x'_o+\frac{8}{9}ae_1+k(M_3^2C_1-B_1)-\Big(aM_1\ell_1+\frac{1}{9}bA_1^{-2}M_2m_1\Big)$$

$$-(M_1\ell_1+m_1M_2)\Big[\frac{1}{9}A_1^{-2}(1-b)+aB_1\Big]-2z'_ob^{-1}+\frac{8}{9}ae_1$$

$$
\begin{aligned}
&-2aM_1e + \frac{2}{3}M_2y_o'(\nu_2 + ba^{-1}A_1^{-2}) + aM_1M_3[e_1(1+B_1) - z_o'b^{-1}] \\
&-2aM_1M_3(e_1 + z_o' + ak - k) + akM_1M_3[aA_1(1+B_1) - 2B_1 - 2C_1] \\
&-kB_1 + \frac{1}{36}M_1M_3^{-1}(M_1^2 - 3M_2^2)\Big[aA_1^{-1}(A_1^{-1} + C_1) - 8a\Big(\frac{1}{9}bA_1^{-2} - a\Big) \\
&+A_1^{-1}\nu_2(1 - C_1) - (1+B_1)(9a^2 + bA_1^{-1}\nu_2)\Big]\Bigg\} + \cdots, \qquad (30)
\end{aligned}
$$

where the constants X's, X''s, Y's and Y''s are determined in terms of the rigid body motion parameters and are written in about twenty pages. The symbol $(\cdots)$ means terms of order higher than $O(\mu^3)$.

3. GEOMETRIC INTERPRETATION OF MOTION

In this section the motion of the rigid body is investigated by introducing Euler's angles θ, ψ and ϕ, which can be determined through the obtained periodic solutions. Since the initial system is autonomous, then the periodic solutions still remain such if (t) is replaced by $(t + t_o)$, where (t_o) is an arbitrary interval of time. Euler's angles, in terms of time (t), take the forms [7]

$$
\begin{aligned}
&\theta = \cos^{-1}\gamma'', \quad \dot\psi = \frac{p\gamma + q\gamma'}{1 - \gamma''^2}, \quad \dot\phi = r - \dot\psi\cos\theta, \\
&\phi_o = \tan^{-1}\frac{\gamma_o}{\gamma_o'}, \quad (\dot u \equiv \frac{du}{dt}). \qquad (31)
\end{aligned}
$$

Assuming the initial instant of time corresponds to the instant $t = t_o$, substituting the solutions (29) into the equations (31), one gets

$$
\begin{aligned}
\phi_o &= \frac{\pi}{2} + r_o t_o + \cdots, \qquad \theta_o = \tan^{-1} M_3, \\
\theta &= \theta_o - \mu E[\theta_1(t + t_o) - \theta_1(t_o)] - \mu^2\cot\theta_o[\theta_2(t+t_o) - \theta_2(t_o)] \\
&\quad -\mu^3[\theta_3(t+t_o) - \theta_3(t_o)] + \cdots, \\
\psi &= \psi_o + Mg\ell C^{-1}r_o^{-1}\mathrm{cosec}\,\theta_o\Big(\frac{1}{2}e_1 + v_o\cot^2\theta_o\Big)t \\
&\quad + \frac{1}{2}\mu r_o\mathrm{cosec}\,\theta_o[\psi_1(t+t_o) - \psi_1(t_o)] \\
&\quad + \mu^2 r_o\cot\theta_o\mathrm{cosec}\,^2\theta_o[\psi_2(t+t_o) - \psi_2(t_o)] + \cdots, \\
\phi &= \phi_o + \Big\{r_o - \frac{1}{2}e_1MglC^{-1}r_o^{-1}(\cot\theta_o + aEr_o^{-1}c\cos\epsilon\sqrt{\cos\theta_o}) \\
&\quad + \frac{1}{2}Mg\ell C^{-1}r_o^{-1}E^2X_{3o}\cos\theta_o - v_o^*\sqrt{\cos\theta_o}[r_o + acE\cos\epsilon\tan\theta_o\sqrt{\cos\theta_o} \\
&\quad + Mg\ell C^{-1}r_o^{-1}X_{6o}\cos\theta_o] - \frac{1}{6}EMg\ell C^{-1}\sqrt{\cos\theta_o}\Big[\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1} - a\Big)(X_{72}'\cos\epsilon \\
&\quad + X_{72}\sin\epsilon) - \Big(\frac{1}{3}bA_1^{-1} + a\Big)(X_{74}'\cos\epsilon - X_{74}\sin\epsilon)\Big]\Big\}t
\end{aligned}
$$

$$+\frac{1}{2}\mu\sqrt{\cos\theta_o}[\phi_1(t+t_o)-\phi_1(t_o)]+\mu^2[\phi_2(t+t_o)-\phi_2(t_o)]+\cdots, \tag{32}$$

where

$$\begin{aligned}
\theta_1(t) &= a\cos\epsilon+\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}-a\Big)\Big(\cos\epsilon\cos\frac{2}{3}r_ot-\sin\epsilon\sin\frac{2}{3}r_ot\Big)\\
&\quad -\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}+a\Big)\Big(\cos\epsilon\cos\frac{4}{3}r_ot+\sin\epsilon\sin\frac{4}{3}r_ot\Big),\\
\theta_2(t) &= \sum_{i=0}^{4}\Big(X_{6i}\cos\frac{i}{3}r_ot+X'_{6i}\sin\frac{i}{3}r_ot\Big)+X_{66}\cos 2r_ot, \quad i\neq 1,\\
\theta_3(t) &= \sum_{i=0}^{10}\Big(Y_{6i}\cos\frac{i}{3}r_ot+Y'_{6i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 9,\\
\psi_1(t) &= \sum_{i=2}^{4}\Big(X_{7i}\cos\frac{i}{3}r_ot+X'_{7i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 3,\\
\psi_2(t) &= \sum_{i=2}^{8}\Big(Y_{7i}\cos\frac{i}{3}r_ot+Y'_{7i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 5,7,\\
\phi_1(t) &= \sum_{i=2}^{4}\Big(X_{8i}\cos\frac{i}{3}r_ot+X'_{8i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 3,\\
\phi_2(t) &= \sum_{i=2}^{8}\Big(Y_{8i}\cos\frac{i}{3}r_ot+Y'_{8i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 5,7,\\
v_o &= -\frac{1}{2}A_1^{-1}\Big[e\tan^2\theta_o+\frac{1}{6}\nu_2(M_1^2+M_2^2)\\
&\quad +\frac{1}{3}E(a+b)(M_1\cos\epsilon+M_2\sin\epsilon)\Big],\\
v_o^* &= Mg\ell C^{-1}r_o^{-2}v_o\operatorname{cosec}\theta_o\cot^2\theta_o,
\end{aligned} \tag{33}$$

the constants Y's, Y''s, X's and X''s are determined in terms of the motion parameters and are written in about three pages. The formula (32) shows that the expressions of Eulerian angles depend on four arbitrary constants θ_o, ψ_o, ϕ_o and r_o (sufficiently large).

4. ANALYTICAL AND NUMERICAL SOLUTIONS

This section is devoted to ascertain the accuracy of the obtained analytical solutions of the previous sections. That is the quasi-linear system (1) is integrated numerically using fourth order Runge – Kutta method [8] and the obtained results are compared with the analytical ones.

4.1. THE ANALYTICAL SOLUTIONS

In this case the analytical solutions p_2, γ_2 and their derivatives with respect to (t) are written in the following forms:

$$
\begin{aligned}
p_2 &= M_1\cos\frac{h}{3}+M_2\sin\frac{h}{3}+\mu\Big[\ell_1\cos\frac{h}{3}+m_1\sin\frac{h}{3}\Big]+\frac{9}{2}\mu^2\Big[2L_{2o}+i_{1o}\cos\frac{h}{3}\\
&\quad+i'_{1o}\sin\frac{h}{3}+i_2\cos\frac{2h}{3}+i'_2\sin\frac{2h}{3}+i_3\cos h+i'_3\sin h+i_4\cos\frac{4h}{3}\\
&\quad+i'_4\sin\frac{4h}{3}+i_5\cos\frac{5h}{3}+i'_5\sin\frac{5h}{3}+i_6\cos\frac{7h}{3}+i'_6\sin\frac{7h}{3}\Big],\\
x &= \frac{dp_2}{dt}=M_{1o}\sin\frac{h}{3}+M_{2o}\cos\frac{h}{3}+\frac{1}{3}\mu\Big[m_1\cos\frac{h}{3}-\ell_1\sin\frac{h}{3}\Big]+\frac{3}{2}\mu^2\Big[i'_{1o}\cos\frac{h}{3}\\
&\quad-i_{1o}\sin\frac{h}{3}-i_{2o}\sin\frac{2h}{3}+i'_{2o}\cos\frac{2h}{3}-i_{3o}\sin h+i'_{3o}\cos h-i_{4o}\sin\frac{4h}{3}\\
&\quad+i'_{4o}\cos\frac{4h}{3}-i_{5o}\sin\frac{5h}{3}+i'_{5o}\cos\frac{5h}{3}-i_{6o}\sin\frac{7h}{3}+i'_{6o}\cos\frac{7h}{3}\Big],\\
\gamma_2 &= M_3\cos h-\mu a M_1\cos h+\mu^2\Big[N_{2o}+v_{11}\cos\frac{h}{3}+v'_{11}\sin\frac{h}{3}+\Big(v_{12}-a\ell_1\\
&\quad-\frac{9}{2}\nu_2^2M_2^2M_3^{-1}\Big)\cos h+v'_{12}\sin h+v_{13}\cos 2h+v'_{13}\sin 2h+v_{14}\cos\frac{5h}{3}\\
&\quad+v'_{14}\sin\frac{5h}{3}+v_{15}\cos 3h\Big],\\
y &= \frac{d\gamma_2}{dt}=-M_3\sin h+\mu a M_1\sin h+\mu^2\Big[\frac{1}{3}\Big(v'_{11}\cos\frac{h}{3}-v_{11}\sin\frac{h}{3}\Big)\\
&\quad-\Big(v_{12}-a\ell_1-\frac{9}{2}\nu_2^2M_2^2M_3^{-1}\Big)\sin h+v'_{12}\cos h-2v_{13}\sin 2h+2v'_{13}\cos 2h\\
&\quad-\frac{5}{3}v_{14}\sin\frac{5h}{3}+\frac{5}{3}v'_{14}\cos\frac{5h}{3}-3v_{15}\sin 3h\Big],
\end{aligned}
\tag{34}
$$

where

$h = iT/300$ for $i = 0$ to 300 step 5 and T =max value of t-variable.

Let us assume

$$
\begin{aligned}
&A=B=1.2,\quad C=1.6,\quad x_o=5,\quad y_o=6,\\
&z_o=7,\quad R=1500,\quad \lambda=.6,\quad M=300,\\
&\gamma''_o=.5,\quad r_o=1100,\quad T=18.78775142.
\end{aligned}
\tag{35}
$$

In this case the following parameters are determined:

$$
\begin{aligned}
&A_1=-B_1=\omega=.3333333,\quad C_1=0,\quad \ell=10.48809,\\
&x'_o=.4767313,\quad y'_o=.5720776,\quad z'_o=.6674238,\\
&g=2.66667E-07,\quad c=2.28999E-02,\quad \mu=1.47206E-05,\\
&a=b=.75,\quad k=5.08513E-07,\quad e=1.906925,\\
&e_1=-.667424,\quad e_2=.222474,\quad \nu=.75,\quad \nu_2=-2.25.
\end{aligned}
\tag{36}
$$

Consider that $p_{2a}, \gamma_{2a}, \ldots$ denote the analytical solutions $p_2, \gamma_2, \ldots$, the graphical representations and the corresponding phase plane diagrams for these solutions are given in figures 1, 4, 7, 10, 13 and 16.

For this case the system of differential equations can be rewritten as follows:

$$\frac{dp_2}{dt} = x, \qquad \frac{d\gamma_2}{dt} = y, \qquad \frac{dx}{dt} = -\frac{1}{9}p_2 + \mu^2 f_1, \qquad \frac{dy}{dt} = -\gamma_2 + \mu^2 g_1, \tag{37}$$

where

$$\begin{aligned} f_1 &= cna_1 s_{21} + p_2\Big(cna_2 - \frac{1}{9}s_{11} - y'_o y a^{-1} + C_1 x^2 A_1^{-1}\Big) \\ &\quad + \gamma_2(cna_3 x + cna_4 xy + cna_5 s_{21}) + x'_o xy, \\ g_1 &= cnb_1 + \gamma_2(cnb_2 - s_{11} + cnb_3 y^2 + cnb_4 x^2 - y'_o y) \\ &\quad + x'_o y^2 + p_2(cnb_5 s_{21} + cnb_6 xy). \end{aligned} \tag{38}$$

The constants $cna_1 \to cna_5$ and $cnb_1 \to cnb_6$ are determined by the correspondence between the above system and the system (1). Assuming the same data (35) and (36) with the initial values of the analytical solutions, the numerical solutions are obtained by using the fourth order Runge – Kutta method. Supposing $p_{2n}, \gamma_{2n}, \ldots,$ denote the numerical solutions $p_2, \gamma_2, \ldots,$ and using the computer, the numerical solutions and their phase trajectories are obtained in figures 2, 5, 8, 11, 14 and 17.

The comparison between the analytical and the numerical solutions is given in figures 3, 6, 9, 12, 15 and 18. This comparison shows that the deviation between the analytical and the numerical solutions is very small and can be neglected, that is the numerical solutions are in full agreement with the analytical ones.

5. CONCLUSIONS

Poincaré's small parameter method is applied to investigate the periodic solutions, with non-zero basic amplitudes, for the singular case of the natural frequency ($\omega = 1/3$). This problem deals with the following bodies being classified according to the moments of inertia:

1. $C > A > B,\ B < \frac{3}{4}C,\ A > \frac{3}{4}C$;
2. $C > B > A,\ A < \frac{3}{4}C,\ B > \frac{3}{4}C$;
3. $A = B = \frac{3}{4}C$, which represents rapidly spinning Lagrange's gyroscope about the axis of symmetry ($x'_o = y'_o = 0$);
4. $A = B = \frac{3}{2}C$, another Lagrange's gyroscope;
5. $\frac{9}{8}C < B < A,\ A > \frac{3}{2}C,\ B < \frac{3}{2}C$;
6. $\frac{9}{8}C < A < B,\ B > \frac{3}{2}C,\ A < \frac{3}{2}C$.

This problem is a generalization of the corresponding one in the uniform gravity field, that is the solution of the latter problem is deduced from the solution of the considered one by putting $k = 0$. The geometric interpretation of motion is considered to describe the orientation of the body at any instant (t) of time. A computer program is carried out to obtain the graphical representations for the analytical solutions. Starting the initial values of the analytical solutions, the autonomous system is solved, using the fourth order Runge – Kutta method, to obtain the numerical solutions through another program. The obtained analytical and numerical solutions are represented graphically, using the computer, to show the difference between them. The deviations between both solutions are very small, which give powerful agreement of the obtained solutions.

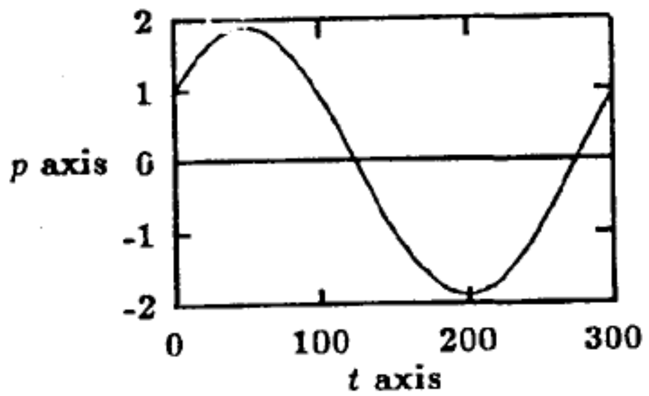

Fig. 1. $p_{2a} - t$ diagram

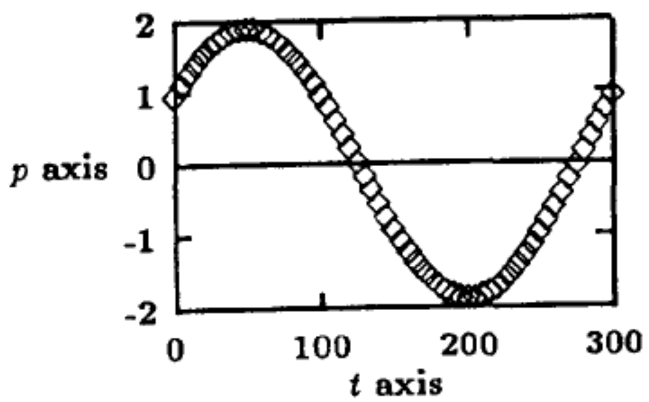

Fig. 2. $p_{2n} - t$ diagram

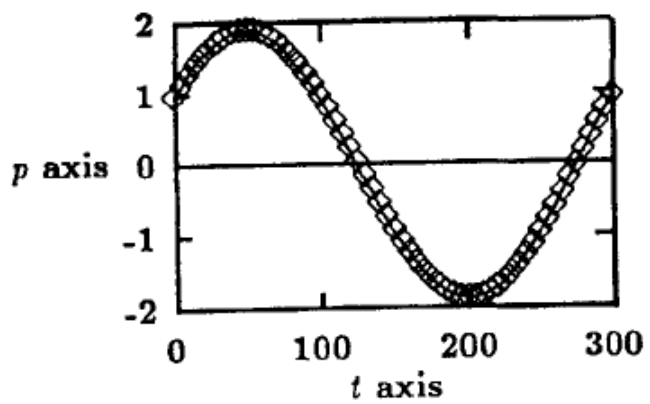

Fig. 3. $p_{2a,n} - t$ diagram

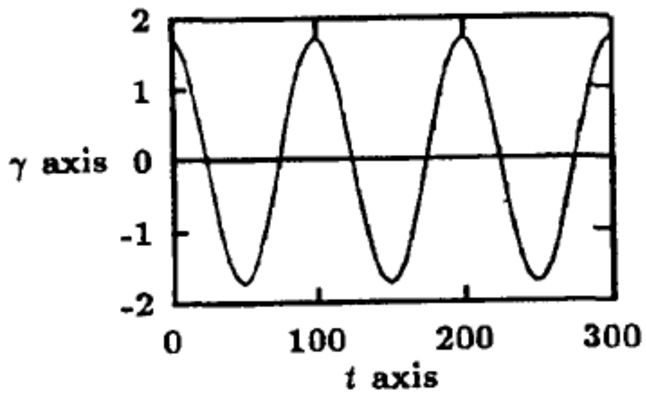

Fig. 4. $\gamma_{2a} - t$ diagram

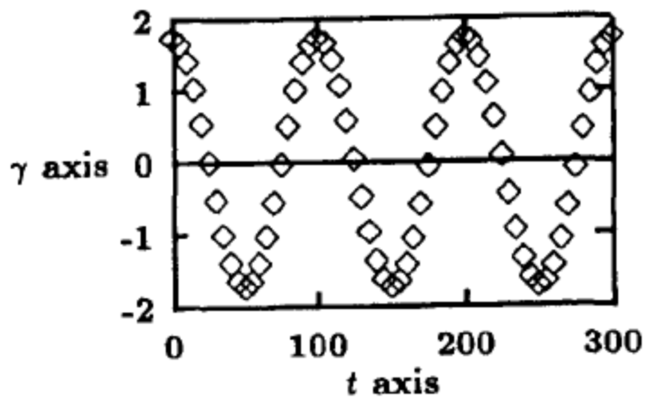

Fig. 5. $\gamma_{2n} - t$ diagram

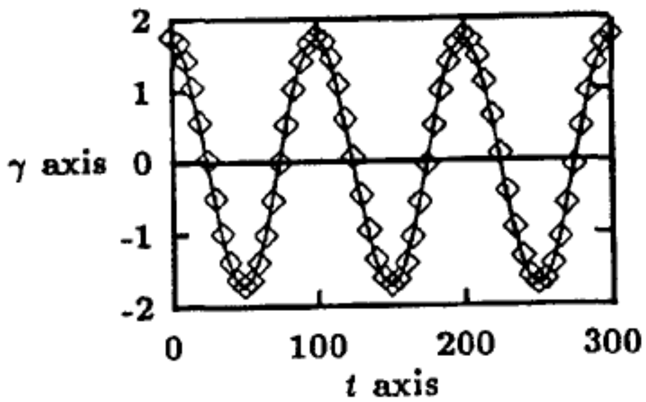

Fig. 6. $\gamma_{2a,n} - t$ diagram

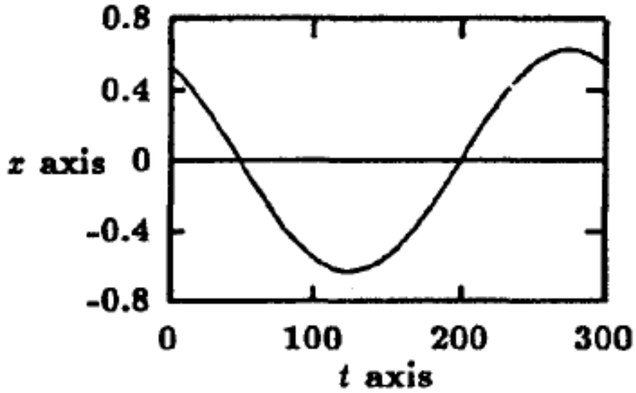

Fig. 7. $x_a - t$ diagram

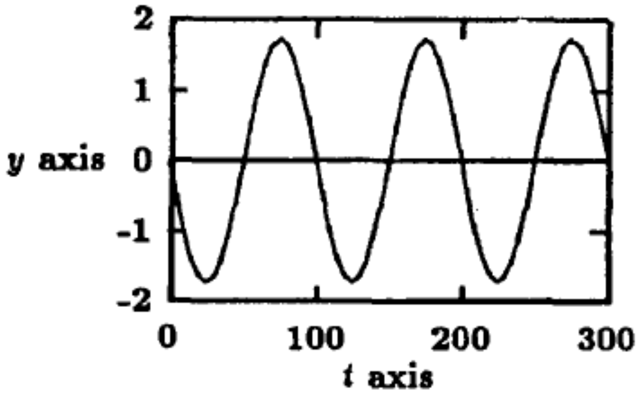

Fig. 10. $y_a - t$ diagram

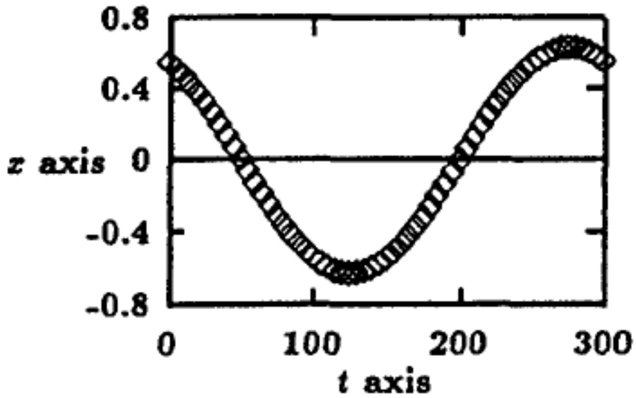

Fig. 8. $x_n - t$ diagram

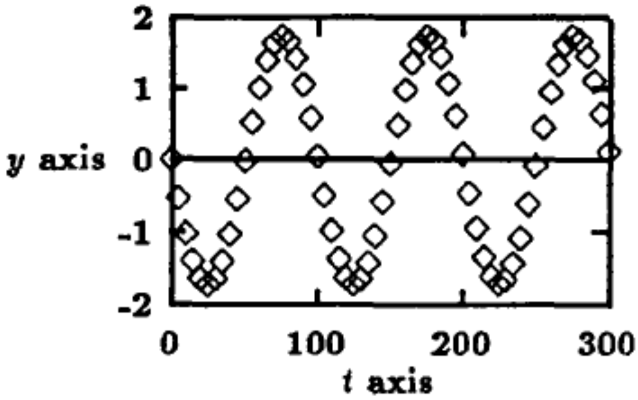

Fig. 11. $y_n - t$ diagram

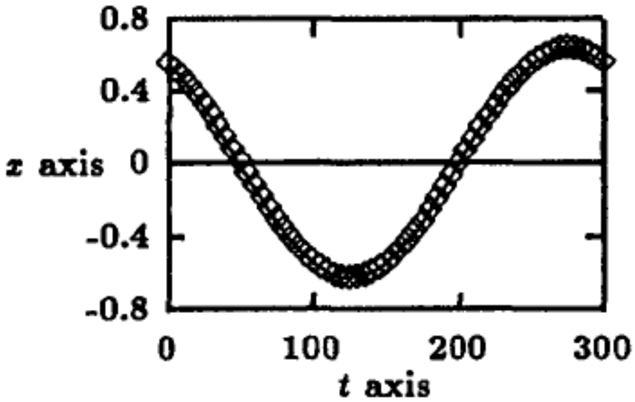

Fig. 9. $x_{a,n} - t$ diagram

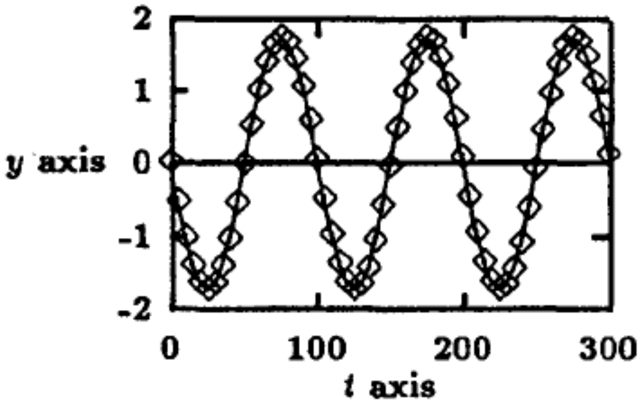

Fig. 12. $y_{a,n} - t$ diagram

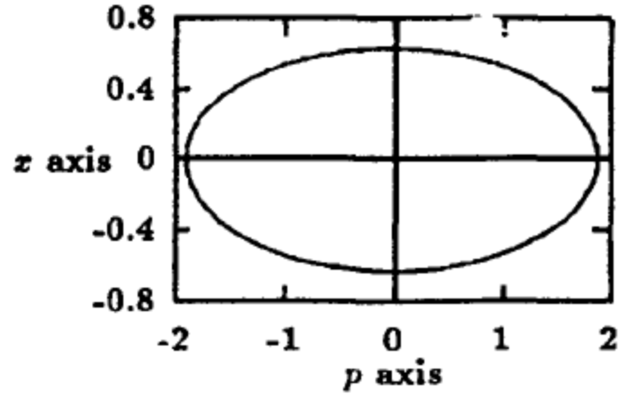

Fig. 13. $x_a - p_{2a}$ diagram

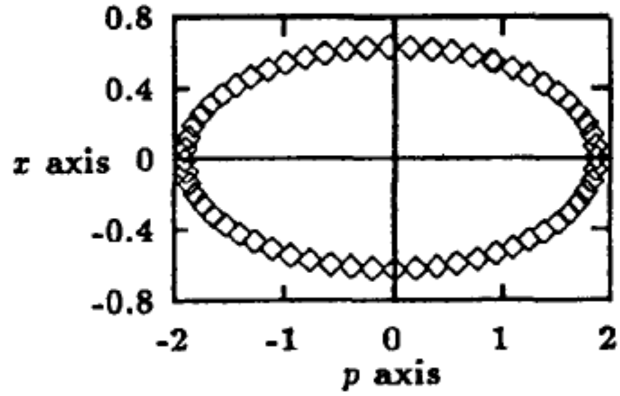

Fig. 14. $x_n - p_{2n}$ diagram

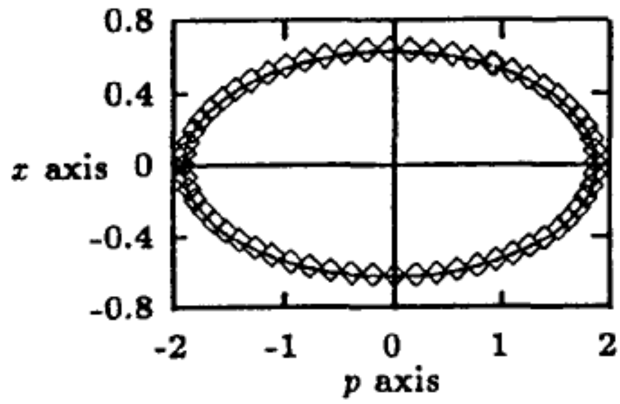

Fig. 15. $x_{a,n} - p_{2a,n}$ diagram

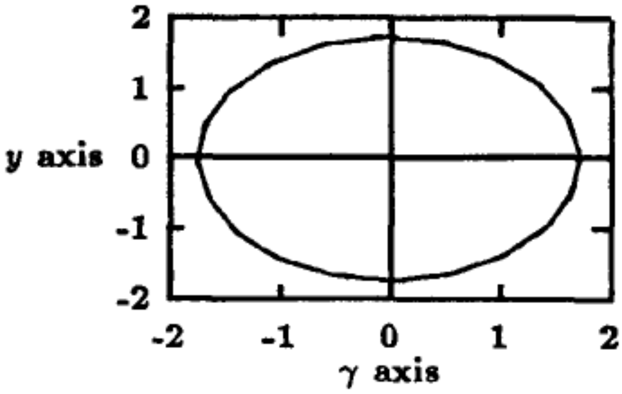

Fig. 16. $y_a - \gamma_{2a}$ diagram

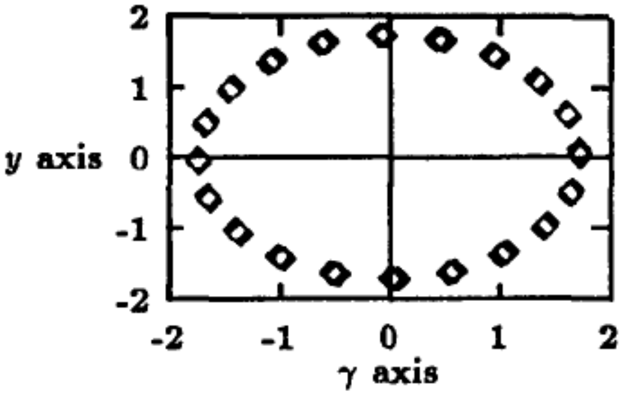

Fig. 17. $y_n - \gamma_{2n}$ diagram

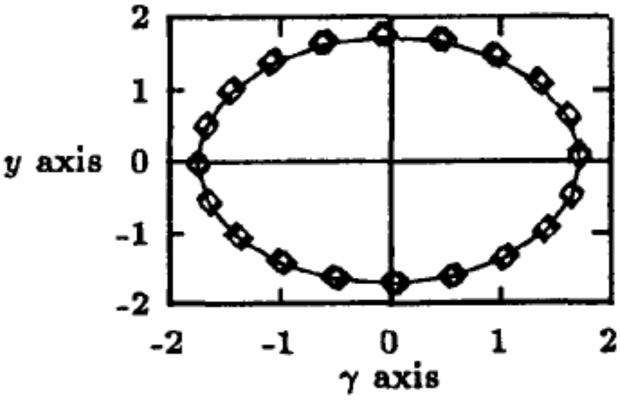

Fig. 18. $y_{a,n} - \gamma_{2a,n}$ diagram

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Abdel Aziz Ibrahim Ismail
Mathematics Department, Faculty of Science
Tanta University, Tanta, Egypt