ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 106

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 106

SMOOTHEST INTERPOLATION WITH BOUNDARY CONDITIONS IN $W_2^3[a,b]$

VELINA IVANOVA, RUMEN ULUCHEV

We study the problem on the smoothest interpolant with boundary conditions in the Sobolev space $W_2^3[a,b]$. Characterization and uniqueness of the best interpolant with free knots of interpolation, satisfying boundary conditions, are proved. Based on our proofs we present an algorithm for finding the unique oscillating spline interpolant. Numerical results are given.

Keywords: Smoothest interpolation, splines, Birkhoff interpolation.

2010 Math. Subject Classification: 65D05, 65D07.

1. INTRODUCTION

Let $[a,b]$ be a closed finite subinterval of the real line, r be a natural number, and $1 < p < \infty$. As usual, by $W_p^r[a,b]$ we denote the Sobolev space

$$W_p^r[a,b] = \left\{ f \,:\, f^{(r-1)} \text{ is abs. continuous in } [a,b],\ f^{(r)} \in L_p[a,b] \right\},$$

and by $\|\cdot\|_p$ the norm in $L_p[a,b]$,

$$\|g\|_p = \left(\int_a^b |g(t)|^p \, dt \right)^{1/p}, \qquad g \in L_p[a,b].$$

Suppose that we are given real numbers $\mathbf{y} = (y_0, y_1, \ldots, y_{N+1})$. We shall use the notation $\mathbf{x} = (x_0, x_1, \ldots, x_{N+1})$ for the elements of the set

$$X_N := \big\{(x_0, x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+2} \,:\, a = x_0 < x_1 < \cdots < x_{N+1} = b\big\}.$$

In 1988, Pinkus [12] considered the problem on existence, characterization, and uniqueness of knots $\mathbf{x}^* \in X_N$ and a function $f^* \in W_p^r[a,b]$ for which the following quantity is attained:

$$\inf_{\mathbf{x}\in X_N}\ \inf_{f\in W_p^r[a,b]} \big\{\|f^{(r)}\|_p : f(x_i) = y_i,\ i = 0, \ldots, N+1\big\}. \tag{1.1}$$

That is, we seek for the *smoothest* interpolant in $W_p^r[a,b]$ with free interpolation knots in $[a,b]$. The paper of Pinkus [12] may be regarded as a further development of de Boor's study [6] on the "best" interpolant with fixed interpolation knots.

Henceforth we assume that the data $\mathbf{y} = (y_0, y_1, \ldots, y_{N+1})$ satisfy the inequalities

$$(y_i - y_{i-1})(y_{i+1} - y_i) < 0, \qquad i = 1, \ldots, N. \tag{1.2}$$

Note that conditions (1.2) are not essential restrictions. Indeed, if $y_{i-1} < y_i < y_{i+1}$ or $y_{i-1} > y_i > y_{i+1}$ for some i and f takes values y_{i-1}, y_{i+1} at knots $x_{i-1} < x_{i+1}$, respectively, then by the continuity of the functions from $W_p^r[a,b]$, f takes the intermediate value y_i at some point $x_i \in (x_{i-1}, x_{i+1})$. It means that if there exists a solution to (1.1) in the case of oscillating data, we easily obtain a solution when the data $\mathbf{y}$ do not oscillate by taking the maximal subsequence of values in $\mathbf{y}$ satisfying (1.2). We also assume that

$$N + 2 > r, \tag{1.3}$$

for otherwise a trivial solution to (1.1) is given by the Lagrange interpolation polynomial of degree $r-1$ with arbitrary knots from the set X_N.

Taking into account the above remarks we henceforth assume that r, N, and the data $\mathbf{y}$ satisfy (1.2) and (1.3).

We give below a brief account on the results on problem (1.1).

The case $r = 1$ is elementary (see [12]).

In 1984 Marin [10] completely solved (1.1) for $r = p = 2$. He first characterized the solution $(\mathbf{x}^*, f^*)$ as follows:

$$f^* \text{ is strictly monotone in } [x_i^*, x_{i+1}^*], \quad i = 0, \ldots, N, \tag{1.4}$$

and explicitly found the optimal knots $\mathbf{x}^*$ and the interpolant f^*. The extremal function is actually the unique interpolating natural cubic spline with knots $\mathbf{x}^*$ satisfying (1.4).

For $p \in (1, \infty)$, Pinkus [12] proved the existence and characterization of the solution to (1.1) for all r, but the uniqueness for $p = 1$ and $r = 2$ only. The following result is a keystone in the survey on the smoothest interpolation, where as usual $f[x_i, \dots, x_{i+r}]$ is the divided difference of the function f at knots $x_i, \dots, x_{i+r}$.

Theorem A (Pinkus [12]) *Let $1 < p < \infty$, $\mathbf{y} = (y_0, y_1, \dots, y_{N+1})$ be real numbers satisfying* (1.2) *and* (1.3)*, and let f^* be a solution of* (1.1). *There exist $a = x_0^* < \cdots < x_{N+1}^* = b$, such that $f^*(x_i^*) = y_i$, $i = 0, \dots, N+1$. Furthermore,*

(a)

$$f^{*(r)}(t) = \left| \sum_{i=0}^{N+1-r} \eta_i B_i(t) \right|^{q-1} \operatorname{sign} \left(\sum_{i=0}^{N+1-r} \eta_i B_i(t) \right),$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $B_i(t)$ is the B-spline of degree $r-1$ with knots $x_i^*, \dots, x_{i+r}^*$, and the coefficients $\{\eta_i\}_{i=0}^{N+1-r}$ satisfy

$$\int_a^b B_i(t) f^{*(r)}(t)\, dt = f[x_i^*, \dots, x_{i+r}^*], \qquad i = 0, \dots, N+1-r;$$

(b) f^* is strictly monotone in $[x_i^*, x_{i+1}^*]$, $i = 0, \dots, N$.

The uniqueness of the smoothest interpolant in general was conjectured but it is still an open problem. There are a few particular cases where it was proved, e.g., for $p = 2$ and $r = 2$ by Marin [10], for $p = 2$ and $r = 3$ by Uluchev [20], for $p \in (1, \infty)$ and $r = 2$ by Rademacher and Scherer [14] and independently by Uluchev [20]. Based on key results of Bojanov [1] concerning interpolation by perfect splines, Pinkus [12] proved the uniqueness of the smoothest interpolant, which is actually a perfect spline, for the case $p = \infty$ and $r \in \mathbb{N}$. In 1995, Naidenov [11] proposed an algorithm for construction of the unique smoothest perfect spline. The case $p = 1$ was studied by Pinkus [12].

Various modifications of the problem have been studied by Bojanov [4], Draganova in [7] for the periodic case and on interpolation in mean values in [8]. Multidimensional aspects of the problem (1.1) have been considered by Marin [10], Rademacher and Scherer [14], Scherer and Smith [16], Scherer [15].

A short summary of the results on the topic was presented by Pinkus in [13].

Here we study a problem on the smoothest interpolant with free knots in the space $W_2^3[a, b]$ with additional boundary conditions imposed on the interpolant. The paper is organized as follows. We state our main results in Section 2. Section 3 consists of preliminaries on Birkhoff interpolation and B-splines with Bitkhoff type of knots. In Section 4 we study an extremal problem for interpolation at fixed knots with functions from $W_2^3[a, b]$ satisfying boundary conditions. Then we give characterization of the smoothest interpolant for our problem with free interpolation knots. Applying a constructive approach used in [20] by the second author we prove that there exists a unique fifth degree oscillating spline interpolant in Section 5. A direct consequence is the uniqueness of the smoothest interpolant with boundary conditions. In the final Section 6 we suggest a numerical algorithm for finding the oscillating spline interpolant. We conclude this section with results of numerical experiment for a given data.

2. MAIN RESULTS

Suppose that $[a,b] \subset \mathbb{R}$, $r \in \mathbb{N}$, and $\mathbf{y} = (y_0, \ldots, y_{N+1})$ are arbitrary real numbers. For a fixed $\mathbf{x} = (x_0, \ldots, x_{N+1}) \in X_N$, we denote by $F(\mathbf{x}, \mathbf{y})$ the set of all functions $f \in W_2^3[a,b]$, such that

$$f(x_i) = y_i, \qquad i = 0, \ldots, N+1, \tag{2.1}$$

$$f'(x_0) = 0, \quad f'''(x_0) = 0, \quad f'(x_{N+1}) = 0, \quad f'''(x_{N+1}) = 0. \tag{2.2}$$

In addition to the usual interpolation conditions we impose boundary conditions for the first and third derivative of the function at the endpoints $a = x_0$ and $b = x_{N+1}$. At first glance it seems that conditions (2.2) are very restrictive. Note that in the case of smoothest interpolation in $W_2^3[a,b]$ satisfying (2.1) only, the extremal function is a natural fifth degree spline whose third and fourth derivatives a priori vanish at the endpoints of the interval $[a,b]$, see [20]. Henceforth, $S_m(x_1, \ldots, x_N)$ will stand for the space of spline functions of degree m with knots $x_1, \ldots, x_N$.

Here we study the problem

$$\inf_{\mathbf{x} \in X_N} \inf_{f \in F(\mathbf{x}, \mathbf{y})} \|f'''\|_2. \tag{2.3}$$

The following result answers some questions concerning (2.3), including a characterization of the smoothest interpolant.

Theorem 1. *Let $\mathbf{y} = (y_0, \ldots, y_{N+1})$, $N > 1$, be real numbers satisfying conditions* (1.2) *and let f^* be a solution to problem* (2.3). *Then, there exist knots $\mathbf{x}^* = (x_0^*, \ldots, x_{N+1}^*) \in X_N$ such that $f^* \in F(\mathbf{x}^*, \mathbf{y})$. Furthermore,*

(a) $f^* \in S_5(x_1^*, \ldots, x_N^*)$;

(b) f^* *is strictly monotone in* $[x_i^*, x_{i+1}^*]$, *for all* $i = 0, \ldots, N$.

Therefore, the smoothest interpolant with free knots is strictly monotone in each interval between two consecutive knots, thus its first derivative must vanish at the interior knots.

In Section 5 we show that there exists a unique fifth degree spline interpolant with knots in X_N satisfying the above characterization conditions for the smoothest interpolant to the problem (2.3). More precisely, we prove:

Theorem 2. *Let $N > 1$ and the real numbers $\mathbf{y} = (y_0, y_1, \ldots, y_{N+1})$ oscillate in the sense that $(y_i - y_{i-1})(y_{i+1} - y_i) < 0$, $i = 1, \ldots, N$. Then, there exists unique spline $s^* \in S_5(x_1^*, \ldots, x_N^*)$ and knots $\mathbf{x}^* = (x_0^*, \ldots, x_{N+1}^*) \in X_N$, such that*

$$\begin{aligned} &s^*(x_i^*) = y_i, && i = 0, \ldots, N+1, \\ &{s^*}'(x_i^*) = 0, && i = 0, \ldots, N+1, \\ &{s^*}'''(x_0^*) = 0, && {s^*}'''(x_{N+1}^*) = 0. \end{aligned} \tag{2.4}$$

A direct consequence of Theorem 1 and Theorem 2 is the next statement.

Theorem 3. *Let* $N > 1$ *and* $\mathbf{y} = (y_0, \ldots, y_{N+1})$ *be real numbers satisfying inequalities* (1.2). *If* $(f^*, \mathbf{x}^*)$, $\mathbf{x}^* = (x_0^*, \ldots, x_{N+1}^*) \in X_N$ *is a solution to problem* (2.3), *then* f^* *is the unique spline interpolant from the set* $S_5(x_1^*, \ldots, x_N^*) \cap F(\mathbf{x}^*, \mathbf{y})$ *strictly oscillating at the knots* $\mathbf{x}^*$.

3. PRELIMINARIES ON BIRKHOFF INTERPOLATION

We need some basic definitions and results concerning Birkhoff interpolation and B-splines with Birkhoff type of knots, see for details [3, 5, 9]. Let $\mathbf{t} = (t_1, \ldots, t_m)$, $t_1 < \cdots < t_m$,

$$E = \begin{pmatrix} e_{10} & \ldots & e_{1,r-1} \\ \ldots & \ldots & \ldots \\ e_{m0} & \ldots & e_{m,r-1} \end{pmatrix}$$

be an *incidence matrix* (E consists of 0's and 1's only), and $|E|$ be the total number of 1-entries in E. By π_r we denote the set of algebraic polynomials with real coefficients of degree at most r.

The incidence matrix E satisfies *Strong Pólya condition*, if $\sum_{j \le k} \sum_i e_{ij} > k+1$ for all $k = 0, \ldots, r-2$.

A sequence of 1-entries $e_{ij}, \ldots, e_{i,j+\ell-1}$ in i-th row of the matrix E is said to be *supported odd block* if ℓ is an odd number and there exist i_1, i_2, j_1, j_2, such that

$$e_{i_1 j_1} = e_{i_2 j_2} = 1, \qquad i_1 < i < i_2, \qquad j_1 < j, \qquad j_2 < j.$$

The matrix E is *conservative* if it does not contain supported odd blocks of 1's. The pair $(\mathbf{t}, E)$ is *s-regular*, if E is conservative and satisfies Strong Pólya condition.

Based on Birkhoff interpolation by polynomials, B-splines with Birkhoff type of knots were introduced preserving most important properties of the usual B-splines with simple (or multiple) knots (see [3]). Namely, for points $\mathbf{t} = (t_1, \ldots, t_m)$, $t_1 < \cdots < t_m$, and an s-regular incidence matrix E with $|E| = r+1$, the B-spline of degree $r-1$ with Birkhoff knots $(\mathbf{t}, E)$ is defined by

$$B\big((\mathbf{t}, E); t\big) = \frac{1}{(r-1)!} D\big[(\mathbf{t}, E); (\cdot - t)_+^{r-1}\big],$$

where $D[(\mathbf{t}, E); f]$ is the divided difference of the function f at $(\mathbf{t}, E)$, i.e. the coefficient of t^r in the polynomial $p(t) \in \pi_r$ which interpolates f at $(\mathbf{t}, E)$ in the sense

$$p^{(j)}(t_i) = f^{(j)}(t_i), \qquad e_{ij} = 1.$$

Given $r, N \in \mathbb{N}$ and a pair $(\mathbf{t}, E)$, $t_1 < \cdots < t_m$, $E = \{e_{ij}\}_{i=1,j=0}^{m\ r-1}$, $|E| = r+N$, we define a $(r+1)$-*partition of* $(\mathbf{t}, E)$ as a sequence of pairs $\{(\mathbf{t}_i, E_i)\}_{i=1}^{N}$, obtained in the following way. Let us order the elements of E in the manner

$$e_{10}, \ldots, e_{1,r-1}, e_{20}, \ldots, e_{2,r-1}, \ldots, e_{m0}, \ldots, e_{m,r-1}$$

and enumerate the 1's in the latter sequence from 1 to $r+N$. Let $\mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_q$ be the rows of E containing $r+1$ consecutive 1's starting with the i-th one. Suppose that the number of 1's of this $(r+1)$-sample in the row $\mathbf{e}_p$ is μ and the number of 1's in the sample in the row $\mathbf{e}_q$ is ν. We denote by $\mathbf{t}_i$ the set of knots $t_p < \cdots < t_q$ and by E_i the matrix composed from $\mathbf{e}_p, \ldots, \mathbf{e}_q$ in which all 1's in the first (resp., last) row of E_i except the first μ (resp., ν) ones are replaced by 0's.

It is said that the $(r+1)$-partition $\{(\mathbf{t}_i, E_i)\}_{i=1}^{N}$ of $(\mathbf{t}, E)$ is s-*regular* if all pairs $(\mathbf{t}_i, E_i)$, $i = 1, \ldots, N$ are s-regular.

In our study we need conditions for solvability of the Birkhoff interpolation problem by splines. The following general necessary and sufficient condition is due to Borislav Bojanov.

Theorem B (Bojanov [3], [5, Theorem 4.20]) *Let* $\mathbf{x} = (x_0, \ldots, x_{m+1})$, $a = x_0 < \cdots < x_{m+1} = b$, $E = \big\{e_{ij}\big\}_{i=0,\,j=0}^{m+1\ r-1}$ *and integers* $\{\nu_i\}_{i=1}^{n}$ *be given such that* $N = \nu_1 + \cdots + \nu_n$, $1 \le \nu_i \le r$, $i = 1, \ldots, n$, *and* $|E| = N + r$. *Assume that* $(\mathbf{x}, E)$ *has an s-regular* $(r+1)$*-partition* $\big\{(\mathbf{x}_i, E_i)\big\}_{i=1}^{N}$. *Then the interpolation problem*

$$s^{(j)}(x_i) = f_{ij}, \qquad e_{ij} = 1$$

by splines s *of degree* $r-1$ *with knots* $\xi_1, \ldots, \xi_n$ *of multiplicities* $\nu_1, \ldots, \nu_n$, *respectively, has a unique solution for each given data* $\{f_{ij}\}$ *if and only if*

$$B\big((\mathbf{x}_i, E_i); \theta_i\big) \neq 0, \qquad i = 1, \ldots, N,$$

where $(\theta_1, \ldots, \theta_N) = \big((\xi_1, \nu_1), \ldots, (\xi_n, \nu_n)\big)$.

4. PROOF OF THE CHARACTERIZATION THEOREM

Using the notations from Section 2, let $\mathbf{y} = (y_0, \ldots, y_{N+1})$ be arbitrary real numbers, $\mathbf{x} = (x_0, \ldots, x_{N+1}) \in X_N$ and $F(\mathbf{x}, \mathbf{y})$ be the set of all functions $f \in W_2^3[a,b]$ satisfying (2.1) and (2.2).

Lemma 1. *There exists a unique spline function* $s \in S_5(x_1, \ldots, x_N)$ *satisfying the interpolation conditions* (2.1) *and* (2.2).

Proof. The assertion follows immediately from Theorem B setting $r = 6$, $m = N$, $\nu_1 = \cdots = \nu_n = 1$, $n = N$, $\theta_i = \xi_i = x_i$, $i = 1, \ldots, N$. Indeed, the

$(N+2) \times r$ incidence matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad |E| = N+6,$$

of the Birkhoff interpolation problem (2.1)–(2.2) has an s-regular $(r+1)$-partition $\big\{(\mathbf{t}_i, E_i)\big\}_{i=1}^{N}$. Obviously, $\theta_i = x_i \in \operatorname{supp} B\big((\mathbf{t}_i, E_i); t\big)$, $i = 1, \ldots, N$. Then Theorem B yields that there exists a unique spline s of degree $r-1 = 5$ with knots $(\xi_1 \ldots, \xi_N) = (x_1 \ldots, x_N)$ satisfying (2.1) and (2.2), i.e. $s \in S_5(x_1, \ldots, x_N)$. $\square$

Remark 1. Note that the spline s in Lemma 1 is a function from the class $F(\mathbf{x}, \mathbf{y})$.

The following is a modified version of the classical Holladay's theorem.

Lemma 2. *Let s be the unique spline in the space $S_5(x_1, \ldots, x_N)$ satisfying the interpolation conditions* (2.1) *and* (2.2)*. Then, for each function $f \in F(\mathbf{x}, \mathbf{y})$,*

$$\|s'''\|_2 \le \|f'''\|_2.$$

The equality holds if and only if $f = s$ in $[a, b]$.

Proof. We follow the standard line taking into account that both f and s satisfy the interpolation conditions (2.1) and (2.2), $x_0 = a$, $x_{N+1} = b$, and $s^{V}(t)\big|_{(x_i, x_{i+1})} = c_i = const.$, $i = 0, \ldots, N$:

$$\begin{aligned} \int_a^b s'''(t)\big(f'''(t) - s'''(t)\big)\,dt &= \int_a^b s'''(t)\,d\big(f''(t) - s''(t)\big) \\ &= s'''(t)\big(f''(t) - s''(t)\big)\Big|_a^b - \int_a^b s^{IV}(t)\big(f''(t) - s''(t)\big)\,dt \\ &= -\int_a^b s^{IV}(t)\,d\big(f'(t) - s'(t)\big) \\ &= -s^{IV}(t)\big(f'(t) - s'(t)\big)\Big|_a^b + \int_a^b s^{V}(t)\big(f'(t) - s'(t)\big)\,dt \\ &= \int_a^b s^{V}(t)\,d\big(f(t) - s(t)\big) = \sum_{i=0}^{N} \int\limits_{x_i}^{x_{i+1}} s^{V}(t)\,d\big(f(t) - s(t)\big) \\ &= \sum_{i=0}^{N} \int\limits_{x_i}^{x_{i+1}} c_i\,d\big(f(t) - s(t)\big) = \sum_{i=0}^{N} c_i\big(f(t) - s(t)\big)\Big|_{x_i}^{x_{i+1}} \\ &= 0. \end{aligned}$$

Then

$$\begin{aligned}\int_a^b \big(s'''(t)\big)^2\,dt &\le \int_a^b \Big[\big(f'''(t)-s'''(t)\big)^2+\big(s'''(t)\big)^2\Big]\,dt\\ &= \int_a^b \Big[\big(f'''(t)-s'''(t)\big)+s'''(t)\Big]^2\,dt\\ &= \int_a^b \big(f'''(t)\big)^2\,dt,\end{aligned}$$

i.e.

$$\|s'''\|_2 \le \|f'''\|_2,$$

where the equality holds if and only if $f'''(t)-s'''(t)=0$ in $[a,b]$. The last identity yields $f-s\in\pi_2$. Since $f,s\in F(\mathbf{x},\mathbf{y})$, the quadratic polynomial $f-s$ vanishes at the endpoins of $[a,b]$ and at least in one interior knot, hence $f-s=0$ in $[a,b]$. The proof of the lemma is complete. □

Remark 2. Lemma 2 claims that the only function for which

$$\inf_{f\in F(\mathbf{x},\mathbf{y})} \|f'''\|_2$$

is attained is the unique spline interpolant s from Lemma 1.

Remark 3. A general result on the existence, characterization and uniqueness of a function $f\in W_p^r[a,b]$ satisfying Birkhoff type interpolation conditions with minimal L_p norm of $f^{(r)}$ for fixed knots was proved by Bojanov [2]. However our case does not fall in the scope of Theorem 1 in [2].

Lemma 3. *Let $s\in S_5(x_1,\dots,x_N)$ be the unique spline satisfying* (2.1) *and* (2.2)*. If the data $\mathbf{y}=(y_0,\dots,y_{N+1})$ satisfies condition* (1.2) *and $N>1$, then*

(a) s''' has exactly $N+2$ simple zeros in $[a,b]$;

(b) s' has exactly N simple zeros in (a,b).

Proof. (a) Since the data oscillates, s has at least N local extrema in (a,b). Then, the derivative s' has at least N zeros in (a,b). The interpolation conditions (2.2) give two additional zeros at the endpoints of the interval $[a,b]$ which means that s' has totally at least $N+2$ non-coinciding zeros in $[a,b]$. Applying Rolle's theorem for s', it follows that the second derivative s'' has at least $N+1$ non-coinciding zeros in (a,b), which give N zeros of s''' in (a,b). Because of (2.2), s''' has two more zeros at the endpoints of $[a,b]$. Therefore, s''' has at least $N+2$ zeros in $[a,b]$.

Observe that s''' is a spline function from the space $S_2(x_1,\dots,x_N)$. A well-known result (see [17, Theorem 4.53]) says that any spline from $S_2(x_1,\dots,x_N)$ has no more than $N+2$ zeros counting multiplicities, i.e. s''' has at most $N+2$ zeros in $[a,b]$.

So, we conclude that s''' has exactly $N+2$ simple zeros in $[a,b]$.

(b) From the proof of (a) it follows that s' has exactly N simple zeros in (a,b). Otherwise Rolle's theorem would give more than $N+2$ zeros for s''' in $[a,b]$, a contradiction. □

Proof of Theorem 1. (a) Let f^* solve the extremal problem (2.3). Therefore there exist $\mathbf{x}^* \in X_N$, such that $f^* \in F(\mathbf{x}^*, \mathbf{y})$. Since f^* solves (2.3), then f^* must solve the extremal problem for fixed knots at $\mathbf{x}^*$, namely

$$\inf_{f\in F(\mathbf{x}^*,\mathbf{y})} \|f'''\|_2.$$

By Lemma 1 and Lemma 2 it follows that f^* is the unique spline in $S_5(x_1^*,\dots,x_N^*)$, satisfying the interpolation conditions (2.1) and (2.2).

(b) From Lemma 3, we obtain that $f^{*\prime}$ has exactly N simple zeros in (a,b) which are the extremal points of f^* as well. Denote by $a < \eta_1 < \cdots < \eta_N < b$ all the extremal points of f^* in (a,b) and set $\eta_0 = a$, $\eta_{N+1} = b$. It is clear that the function f^* is strictly monotone in each interval $[\eta_i, \eta_{i+1}]$, $i = 0,\dots,N$. We remark that due to the oscillation of the data $\mathbf{y}$ we have $\eta_i \in (x_{i-1}^*, x_{i+1}^*)$, $i = 1,\dots,N$.

We will show that $\eta_i = x_i^*$ for all $i = 0,\dots,N+1$. Let us assume to the contrary that $\eta_j \neq x_j^*$ for some j. We set $z_i = f^*(\eta_i)$, $i = 0,\dots,N+1$ and consider the extremal problem

$$\inf_{f\in F(\boldsymbol{\eta},\mathbf{z})} \|f'''\|_2$$

for fixed interpolation knots $\boldsymbol{\eta} = (\eta_0,\dots,\eta_{N+1}) \in X_N$ and $\mathbf{z} = (z_0,\dots,z_{N+1})$.

From Lemma 2 it follows that there exists a unique function $\hat{f} \in F(\boldsymbol{\eta},\mathbf{z})$ for which the infimum is attained. Since by Lemma 2, $\hat{f} \in S_5(\eta_1,\dots,\eta_N)$ and $f^* \in S_5(x_1^*,\dots,x_N^*)$, and by assumption $\boldsymbol{\eta} \neq \mathbf{x}^*$, then it follows that $\hat{f} \neq f^*$. Note that $f^* \in F(\boldsymbol{\eta},\mathbf{z})$ but the extremal interpolant in $F(\boldsymbol{\eta},\mathbf{z})$ is the function $\hat{f}$. Therefore

$$\|\hat{f}'''\|_2 < \|f^{*\prime\prime\prime}\|_2.$$

Now, observe that $|z_i| = |f^*(\eta_i)| \geq |y_i|$, $i = 1,\dots,N$. Then for the continious function $\hat{f}$ there exist points $a = \zeta_0 < \zeta_1 < \cdots < \zeta_N < \zeta_{N+1} = b$ such that $\hat{f}(\zeta_i) = y_i$, $i = 0,\dots,N+1$. This means that $\hat{f} \in F(\boldsymbol{\zeta},\mathbf{y})$, $\boldsymbol{\zeta} = (\zeta_0,\dots,\zeta_{N+1}) \in X_N$, and $\|\hat{f}'''\|_2 < \|f^{*\prime\prime\prime}\|_2$ which contradicts the minimality property of the function f^* for the extremal problem (2.3).

Thus, we proved that the extremal points of f^* coincide with interpolation knots, i.e. $\eta_i = x_i^*$ for all $i = 0,\dots,N+1$. Therefore, f^* is strictly monotone in $[x_i^*, x_{i+1}^*]$, $i = 1,\dots,N$. □

5. PROOF OF THEOREM 2

Let $x_i < x_{i+1}$ and arbitrary real numbers y_i, y_{i+1}, s''_i, s''_{i+1} be given. We set $\Delta_i = x_{i+1} - x_i$, $\Delta y_i = y_{i+1} - y_i$ and denote by $P_i(t) \in \pi_5$ the polynomial satisfying

$$\begin{aligned} &P_i(x_i) = y_i, && P_i'(x_i) = 0, && P_i''(x_i) = s_i'', \\ &P_i(x_{i+1}) = y_{i+1}, && P_i'(x_{i+1}) = 0, && P_i''(x_{i+1}) = s_{i+1}''. \end{aligned} \tag{5.1}$$

We can find explicitly the polynomial P_i solving Hermite iterpolation problem (5.1). Standard calculations show that the following relations hold true:

$$\begin{aligned} P_{i-1}'''(x_i) &= \frac{6}{\Delta_{i-1}^3}\big(10\Delta y_{i-1} - \frac{1}{2}\Delta_{i-1}^2 s_{i-1}'' + \frac{3}{2}\Delta_{i-1}^2 s_i''\big), \\ P_i'''(x_i) &= \frac{6}{\Delta_i^3}\big(10\Delta y_i - \frac{3}{2}\Delta_i^2 s_i'' + \frac{1}{2}\Delta_i^2 s_{i+1}''\big), \\ P_{i-1}^{IV}(x_i) &= \frac{24}{\Delta_{i-1}^4}\big(15\Delta y_{i-1} - \Delta_{i-1}^2 s_{i-1}'' + \frac{3}{2}\Delta_{i-1}^2 s_i''\big), \\ P_i^{IV}(x_i) &= \frac{24}{\Delta_i^4}\big(-15\Delta y_i + \frac{3}{2}\Delta_i^2 s_i'' - \Delta_i^2 s_{i+1}''\big). \end{aligned} \tag{5.2}$$

We seek for a spline $s \in S_5(x_1, \dots, x_N)$ with

$$s \in C^4[a,b], \qquad P_i = s\big|_{(x_i,x_{i+1})} \in \pi_5, \quad i = 0, \dots, N, \tag{5.3}$$

satisfying the interpolation conditions (2.4).

Let us set

$$\begin{aligned} & && s_i'' = s''(x_i), && i = 0, \dots, N+1, \\ &\Delta_i = x_{i+1} - x_i, && \Delta y_i = y_{i+1} - y_i, && i = 0, \dots, N, \\ &\alpha_i = \frac{\Delta_i}{\Delta_{i-1}}, && \delta_{i-1} = \frac{\Delta y_i}{\Delta y_{i-1}}, && i = 1, \dots, N, \\ &\beta_i = \frac{\Delta_i^2 s_i''}{2\Delta y_i}, && \gamma_i = \frac{\Delta_i^2 s_{i+1}''}{2\Delta y_i}, && i = 0, \dots, N. \end{aligned} \tag{5.4}$$

The boundary conditions for $s'''(t)$ at the endpoints and the continuity conditions for $s'''(t)$ and $s^{IV}(t)$ at the knots $\{x_i\}_{i=1}^N$ can be written for the polynomial pieces P_i as follows:

$$\begin{aligned} P_0'''(x_0) = 0, \quad & P_{i-1}'''(x_i) = P_i'''(x_i), \quad i = 1, \dots, N, \quad P_N'''(x_{N+1}) = 0, \\ & P_{i-1}^{IV}(x_i) = P_i^{IV}(x_i), \quad i = 1, \dots, N, \end{aligned} \tag{5.5}$$

From (5.2)–(5.4) we obtain for $P_0'''(x_0) = 0$ in (5.5):

$$10 - 3\beta_0 + \gamma_0 = 0,$$

i.e.

$$\gamma_0 = 3\beta_0 - 10. \tag{5.6}$$

Using (5.2)–(5.4) we have for the continuity conditions (5.5) at x_1:

$$\begin{aligned} (20 - 2\beta_0 + 6\gamma_0)\alpha_1^3 &= (20 - 6\beta_1 + 2\gamma_1)\delta_0, \\ (60 - 8\beta_0 + 12\gamma_0)\alpha_1^4 &= (-60 + 12\beta_1 - 8\gamma_1)\delta_0. \end{aligned} \tag{5.7}$$

Now, taking into account (5.6) we rewrite (5.7) in the form

$$\begin{aligned} (8\beta_0 - 20)\alpha_1^3 &= (10 - 3\beta_1 + \gamma_1)\delta_0, \\ (7\beta_0 - 15)\alpha_1^4 &= (-15 + 3\beta_1 - 2\gamma_1)\delta_0, \end{aligned} \tag{5.8}$$

Note that $\beta_1\delta_0 = \gamma_0\,\alpha_1^2$ from (5.4). Then by elimination of γ_1 in (5.8) we get

$$(7\beta_0 - 15)\alpha_1^4 + (16\beta_0 - 40)\alpha_1^3 + (9\beta_0 - 30)\alpha_1^2 - 5\delta_0 = 0. \tag{5.9}$$

On the other hand, from equalities (5.8) it follows that

$$\begin{aligned} 15 - 2\beta_1 + 3\gamma_1 &= \frac{-1}{12\delta_0}\big[28(7\beta_0 - 15)\alpha_1^4 + 10(16\beta_0 - 40)\alpha_1^3 + 40\delta_0\big], \\ 20 - 2\beta_1 + 6\gamma_1 &= \frac{-1}{3\delta_0}\big[16(7\beta_0 - 15)\alpha_1^4 + 7(16\beta_0 - 40)\alpha_1^3 + 40\delta_0\big], \\ 3\gamma_1 &= \frac{-3}{8\delta_0}\big[8(7\beta_0 - 15)\alpha_1^4 + 4(16\beta_0 - 40)\alpha_1^3 + 40\delta_0\big]. \end{aligned} \tag{5.10}$$

Similarly, for $i = 2, \dots, N$ we obtain from (5.2)–(5.5):

$$\begin{aligned} (10 - \beta_{i-1} + 3\gamma_{i-1})\alpha_i^3 &= (10 - 3\beta_i + \gamma_i)\delta_{i-1}, \\ (15 - 2\beta_{i-1} + 3\gamma_{i-1})\alpha_i^4 &= (-15 + 3\beta_i - 2\gamma_i)\delta_{i-1}. \end{aligned} \tag{5.11}$$

Since $\delta_{i-1}\beta_i = \gamma_{i-1}\alpha_i^2$ by (5.4), equalities (5.11) give

$$(15 - 2\beta_{i-1} + 3\gamma_{i-1})\alpha_i^4 + (20 - 2\beta_{i-1} + 6\gamma_{i-1})\alpha_i^3 + 3\gamma_{i-1}\alpha_i^2 - 5\delta_{i-1} = 0, \tag{5.12}$$

and

$$\begin{aligned} &15 - 2\beta_i + 3\gamma_i \\ &\qquad = \frac{-1}{12\delta_{i-1}}\big[28(15 - 2\beta_{i-1} + 3\gamma_{i-1})\alpha_i^4 + 10(20 - 2\beta_{i-1} + 6\gamma_{i-1})\alpha_i^3 + 40\delta_{i-1}\big], \\ &20 - 2\beta_i + 6\gamma_i \\ &\qquad = \frac{-1}{3\delta_{i-1}}\big[16(15 - 2\beta_{i-1} + 3\gamma_{i-1})\alpha_i^4 + 7(20 - 2\beta_{i-1} + 6\gamma_{i-1})\alpha_i^3 + 40\delta_{i-1}\big], \\ &3\gamma_i = \frac{-3}{8\delta_{i-1}}\big[8(15 - 2\beta_{i-1} + 3\gamma_{i-1})\alpha_i^4 + 4(20 - 2\beta_{i-1} + 6\gamma_{i-1})\alpha_i^3 + 40\delta_{i-1}\big]. \end{aligned} \tag{5.13}$$

Finally, from (5.2), (5.4), and (5.13) we obtain

$$
\begin{aligned}
s'''(x_{N+1}) = P_N'''(x_{N+1}) &= \frac{3\Delta y_N}{\Delta_N^3}\big(20 - 2\beta_N + 6\gamma_N\big) \\
&= \frac{-\Delta y_N}{\Delta_N^3\,\delta_{N-1}}\big[16(15 - 2\beta_{N-1} + 3\gamma_{N-1})\alpha_N^4 \\
&\qquad + 7(20 - 2\beta_{N-1} + 6\gamma_{N-1})\alpha_N^3 + 40\delta_{N-1}\big].
\end{aligned}
\tag{5.14}
$$

Remark 4. We will make use of the equalities (5.10) and (5.13) as recurrence relations for the coefficients in the algebraic equations (5.9) and (5.12), and for the quantity (5.14).

Now we consider useful monotonicity properties of polynomial zeros under recurrence relations of the polynomial coefficients. The following two lemmas can be found in [20]; proofs in full details are given in the PhD Thesis of the second author [19].

By the classical *Descartes' rule* a polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ has no more positive zeros counting multiplicities than the number of strict sign changes in the sequence $a_0, \ldots, a_n$. In particular, if there is exactly one strict sign change in the sequence of coefficients then the polynomial has exactly one simple positive zero.

Lemma 4 (Uluchev [20, Lemma 3.3.1])**.** *Suppose that the coefficients $a_0(\tau)$, $a_1(\tau)$, $a_2(\tau)$, a_4 of the function*

$$Q(\tau, z) = a_0(\tau)z^4 + a_1(\tau)z^3 + a_2(\tau)z^2 + a_4$$

satisfy the conditions:

(i) $a_4 = const.,\ \ a_4 > 0$*;*

(ii) $a_i(\tau) \in C^1_{[t,T]},\ \ i = 0, 1, 2,\ \ t < T$*;*

(iii) $a_0(t) \le 0,\ \ a_1(t) < 0,\ \ a_2(t) < 0$*;*

(iv) $a_i'(\tau) > 0,\ \ i = 0, 1, 2,\ \ \tau \in (t, T)$*;*

(v) there exist $\{\tau_i\}_{i=0}^2,\ \ t \le \tau_0 < \tau_1 < \tau_2 < T$ *with* $a_i(\tau_i) = 0,\ \ i = 0, 1, 2.$

Then, there exist unique points t_1 and t_2, such that $t < t_1 < t_2 < T$, and the equation with respect to z,

$$Q(\tau, z) = 0,$$

(a) has exactly one positive simple root $z(\tau)$ if $\tau \in [t, t_1]$ is fixed;

(b) has exactly two positive simple roots $z(\tau) < \hat{z}(\tau)$ if $\tau \in (t_1, t_2)$ is fixed;

(c) has exactly one positive root $z(\tau) = \hat{z}(\tau)$ of multiplicity two if $\tau = t_2$;

(d) *has no positive root if* $\tau \in (t_2, T]$ *is fixed;*

(e) $z(\tau) \in C^1_{(t,t_2)}$ *and* $z'(\tau) > 0$ *for* $\tau \in (t, t_2)$.

Remark 5. More precisely, in Lemma 4, $t_1 = \tau_0$ and the larger positive zero $\hat{z}(\tau)$ of $Q(\tau, z)$ comes from $+\infty$ as τ runs to the right of t_1. For $\tau \in (t_1, t_2)$, $z(\tau)$ increases, $\hat{z}(\tau)$ decreases, and both positive zeros of $Q(\tau, z)$ coincide for $\tau = t_2$.

Let us set

$$\begin{aligned}
&Q(\tau, z) = a_0(\tau)z^4 + a_1(\tau)z^3 + a_2(\tau)z^2 + a_4,\\
&b_0(\tau, z) = A_0\big(28a_0(\tau)z^4 + 10a_1(\tau)z^3 - 8a_4\big), \quad A_0 = const., \quad A_0 > 0,\\
&b_1(\tau, z) = A_1\big(16a_0(\tau)z^4 + 7a_1(\tau)z^3 - 8a_4\big), \quad A_1 = const., \quad A_1 > 0,\\
&b_2(\tau, z) = A_2\big(8a_0(\tau)z^4 + 4a_1(\tau)z^3 - 8a_4\big), \quad A_2 = const., \quad A_2 > 0,\\
&\qquad b_4 = const., \quad b_4 > 0.
\end{aligned} \tag{5.15}$$

Lemma 5 (Uluchev [20, Lemma 3.3.2])**.** *Suppose that the coefficients* $a_0(\tau)$, $a_1(\tau)$, $a_2(\tau)$, a_4 *of function*

$$Q(\tau, z) = a_0(\tau)z^4 + a_1(\tau)z^3 + a_2(\tau)z^2 + a_4$$

satisfy the conditions:

(i) $a_4 = const.$, $a_4 > 0$;

(ii) $a_i(\tau) \in C^1_{[\tau_0,\tau_2]}$;

(iii) $a_i(\tau_i) = 0$, $i = 0, 1, 2$, $\tau_0 < \tau_1 < \tau_2$;

(iv) $a_i'(\tau) > 0$, $i = 0, 1, 2$, $\tau \in (\tau_0, \tau_2)$.

Now, Lemma 4 applies and let t_1, t_2, $z(\tau)$, $\hat{z}(\tau)$, $\xi(\tau)$ *be as in Lemma 4. Then, for* b_0, b_1, b_2, b_4 *defined in* (5.15),

(a) *the algebraic equation with respect to* z,

$$b_0(\tau, \hat{z}(\tau))z^4 + b_1(\tau, \hat{z}(\tau))z^3 + b_2(\tau, \hat{z}(\tau))z^2 + b_4 = 0,$$

has no positive root if $\tau \in (t_1, t_2)$ *is fixed;*

(b) *there exist unique points* $\{\theta_i\}_{i=0}^2$ *such that* $t_1 < \theta_0 < \theta_1 < \theta_2 = t_2$ *and*

$$\begin{aligned}
b_i(\tau, z(\tau)) < 0, \quad &\tau \in (t_1, \theta_i), \quad i = 0, 1, 2,\\
b_i(\theta_i, z(\theta_i)) = 0, \quad &i = 0, 1, 2;
\end{aligned}$$

(c) *the functions* $b_j(\tau) = b_j(\tau, z(\tau))$, $j = 0, 1, 2$, *and* b_4 *satisfy conditions (i)–(iv) of Lemma 4 for the interval* $[\theta_0, \theta_2]$.

Proof of Theorem 2. Let us set

$$
\begin{aligned}
&\tau = 7\beta_0 - 15,\\
&a_{0,1}(\tau) = \tau, \qquad a_{1,1}(\tau) = \frac{16}{7}\Big(\tau - \frac{5}{2}\Big), \qquad a_{2,1}(\tau) = \frac{9}{7}\Big(\tau - \frac{25}{3}\Big), \qquad a_{4,1} = -5\delta_0,\\
&a_{0,i}(\tau) = \frac{-1}{12\delta_{i-2}}\big(28a_{0,i-1}(\tau)\alpha_{i-1}^4 + 10a_{1,i-1}(\tau)\alpha_{i-1}^3 - 8a_{4,i-1}(\tau)\big), \quad i = 2,\dots,N,\\
&a_{1,i}(\tau) = \frac{-1}{3\delta_{i-2}}\big(16a_{0,i-1}(\tau)\alpha_{i-1}^4 + 7a_{1,i-1}(\tau)\alpha_{i-1}^3 - 8a_{4,i-1}(\tau)\big), \qquad i = 2,\dots,N,\\
&a_{2,i}(\tau) = \frac{-3}{8\delta_{i-2}}\big(8a_{0,i-1}(\tau)\alpha_{i-1}^4 + 4a_{1,i-1}(\tau)\alpha_{i-1}^3 - 8a_{4,i-1}(\tau)\big), \qquad i = 2,\dots,N,\\
&a_{4,i} = -5\delta_{i-1}, \qquad i = 2,\dots,N.
\end{aligned}
\tag{5.16}
$$

Using the recurrence relations (5.10) and (5.13) in view of the notations (5.16), we rewrite equations (5.9), (5.12) in the form

$$a_{0,i}(\tau)\alpha_i^4 + a_{1,i}(\tau)\alpha_i^3 + a_{2,i}(\tau)\alpha_i^2 + a_{4,i} = 0, \qquad i = 1,\dots,N, \tag{5.17}$$

and we seek for a solution $\tau, \alpha_1, \dots, \alpha_N$ of the nonlinear system (5.17) such that

$$\alpha_i > 0, \qquad i = 1,\dots,N. \tag{5.18}$$

In addition, by the interpolation conditions (2.4) the spline $s \in S_5(x_1,\dots,x_N)$ defined in (5.3) has to satisfy (5.14), which in view of notations (5.16) takes the form

$$s'''(x_{N+1}) = \frac{-\Delta y_N}{\Delta_N^3\,\delta_{N-1}}\big(16a_{0,N}(\tau)\alpha_N^4 + 7a_{1,N}(\tau)\alpha_N^3 + 40\delta_{N-1}\big) = 0. \tag{5.19}$$

Observe that $\delta_{i-1} < 0$, $i = 1,\dots,N$ and then

$$\frac{-1}{12\delta_{i-1}} > 0, \qquad \frac{-1}{3\delta_{i-1}} > 0, \qquad \frac{-3}{8\delta_{i-1}} > 0, \qquad i = 1,\dots,N.$$

We briefly sketch the idea of our proof. Let us denote the i-th equation of the system (5.17) by (5.17.i). We will bound τ for which the system (5.17) has a solution, satisfying (5.18), to a finite interval J. Moreover, for each fixed $\tau \in J$ we can uniquely determine $\alpha_i > 0$ satisfying (5.17.i), $i = 1,\dots,N-1$, and (5.17.N) would have positive roots $\alpha_N(\tau) < \hat\alpha_N(\tau)$. We will show that the function

$$\hat\sigma(\tau) = 16a_{0,N}(\tau)\hat\alpha_N^4(\tau) + 7a_{1,N}(\tau)\hat\alpha_N^3(\tau) + 40\delta_{N-1}$$

does not vanish in J, i.e. (5.19) cannot be satisfied if we choose the larger positive zero of (5.17.N). Using the smaller positive zero $\alpha_N(\tau)$ of (5.17.N) we will prove that

$$\sigma(\tau) = 16a_{0,N}(\tau)\alpha_N^4(\tau) + 7a_{1,N}(\tau)\alpha_N^3(\tau) + 40\delta_{N-1} \tag{5.20}$$

is monotone and has a unique zero in J. Hence, we will obtain a procedure and numerical algorithm for solving the system (5.17)–(5.19).

First, for any solution of (5.17)–(5.19), the relation $\tau \in [0, \frac{25}{3}]$ must hold. Otherwise we have two cases.

If $\tau < 0$, then $a_{j,1}(\tau) < 0$, $j = 0, 1, 2$ and $a_{4,1} > 0$. Hence equation (5.17.1) has only one positive root $\alpha_1(\tau)$. Recurrence formulae (5.16) yield that $a_{j,k}(\tau) < 0$, $j = 0, 1, 2$ and $a_{4,k} > 0$, hence (5.17.k) has a unique positive root $\alpha_k(\tau)$ for all $k = 2, \dots, N$. Then $\sigma(\tau) < 0$ which means that $s'''(x_{N+1}) \neq 0$, i.e. (5.19) is not satisfied.

In case of $\tau > \frac{25}{3}$, we have $a_{j,1}(\tau) > 0$, $j = 0, 1, 2$ and $a_{4,1} > 0$. Then (5.17.1) has no positive root, hence the system (5.17) has no solution satisfying (5.18).

Let us set $\tau_0^{(1)} = 0$, $\tau_1^{(1)} = \frac{5}{2}$ and $\tau_2^{(1)} = \frac{25}{3}$. Since $\tau_0^{(1)} < \tau_1^{(1)} < \tau_2^{(1)}$, the coefficients $a_{j,1}(\tau)$, $j = 0, 1, 2$ and $a_{4,1}$ satisfy conditions (i)–(iv) of Lemma 5.

Suppose that for a fixed $k \in \{1, \dots, N-1\}$ we have proved that any solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19) is such that $\tau \in \left[\tau_0^{(k)}, \tau_2^{(k)}\right]$ and the coefficients $a_{j,k}(\tau)$, $j = 0, 1, 2$ and $a_{4,k}$ satisfy conditions (i)–(iv) of Lemma 5 for all $\tau \in \left[\tau_0^{(k)}, \tau_2^{(k)}\right]$. By Lemma 4 there exist points $t_1^{(k)}$ and $t_2^{(k)}$ such that $\tau_0^{(k)} = t_1^{(k)} < t_2^{(k)} < \tau_2^{(k)}$ and the equation (5.17.k) has:

- exactly one simple positive root $\alpha_k(\tau)$ if $\tau \le t_1^{(k)}$;
- exactly two positive roots $\alpha_k(\tau) < \hat{\alpha}_k(\tau)$ if $\tau \in \left(t_1^{(k)}, t_2^{(k)}\right)$;
- exactly one positive root of multiplicity two $\alpha_k(\tau) = \hat{\alpha}_k(\tau)$ if $\tau = t_2^{(k)}$;
- no positive root if $\tau \in \left(t_2^{(k)}, \tau_2^{(k)}\right)$.

Assume that there exist a solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19), such that $\alpha_k = \hat{\alpha}_k(\tau)$ for some $\tau \in \left(t_1^{(k)}, t_2^{(k)}\right)$. That is, α_k is the larger positive zero $\hat{\alpha}_k(\tau)$ of (5.17.k). From Lemma 5 (a) it follows that (5.17.$k+1$) has no positive root with respect to α_{k+1}, hence (5.17) has no solution satisfying (5.18).

Therefore for any solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19) with $\tau \in \left(t_1^{(k)}, t_2^{(k)}\right)$, there holds $\alpha_k = \alpha_k(\tau)$ which is the smaller positive zero of (5.17.k). Application of Lemma 5 (b) gives that there exist unique points $\tau_j^{(k+1)}$, $j = 0, 1, 2$, with $t_1^{(k)} < \tau_0^{(k+1)} < \tau_1^{(k+1)} < \tau_2^{(k+1)} < t_2^{(k)}$ and $a_{j,k+1}(\tau_j^{(k+1)}) = 0$, $j = 0, 1, 2$. Now Lemma 5 (c) yields that the coefficients $a_{j,k+1}(\tau)$, $j = 0, 1, 2$ and $a_{4,k+1}$ satisfy conditions (i)–(iv) of Lemma 4 for $\tau \in \left[\tau_0^{(k+1)}, \tau_2^{(k+1)}\right]$.

Similar arguments as for $k = 1$ above show that for any solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19) there holds $\tau \in \left[\tau_0^{(k+1)}, \tau_2^{(k+1)}\right]$. The arguments are the same as for $k = 1$ above.

For $\tau \in \left[t_1^{(k)}, \tau_0^{(k+1)}\right]$ we have $a_{j,k+1}(\tau) < 0$, $j = 0, 1, 2$ and $a_{4,k+1} > 0$. Then equation (5.17.$k+1$) has only one positive root $\alpha_{k+1} = \alpha_{k+1}(\tau)$. Recurrence formulae (5.16) yield that $a_{j,\ell}(\tau) < 0$, $j = 0, 1, 2$ and $a_{4,\ell} > 0$, hence (5.17.ℓ) has a unique positive root $\alpha_\ell(\tau)$ for all $\ell = k+1, \dots, N$. But then $\sigma(\tau) < 0$ which means that $s'''(x_{N+1}) \neq 0$, i.e. (5.19) is not satisfied.

In the case $\tau \in \big(\tau_2^{(k+1)}, t_2^{(k)}\big]$ we have $a_{j,k+1}(\tau) > 0$, $j = 0, 1, 2$ and $a_{4,k+1} > 0$. Then $(5.17.k+1)$ has no positive root and the system (5.17) has no solution satisfying (5.18).

By Lemma 4 there exist unique points $t_1^{(N)}$ and $t_2^{(N)}$ such that $\tau_0^{(N)} = t_1^{(N)} < t_2^{(N)} < \tau_2^{(N)}$ and the equation $(5.17.N)$ has:

- exactly one simple positive root $\alpha_N(\tau)$ if $\tau \le t_1^{(N)}$;
- exactly two positive roots $\alpha_N(\tau) < \hat{\alpha}_N(\tau)$ if $\tau \in \big(t_1^{(N)}, t_2^{(N)}\big)$;
- exactly one positive root of multiplicity two $\alpha_N(\tau) = \hat{\alpha}_N(\tau)$ if $\tau = t_2^{(N)}$;
- no positive root if $\tau \in \big(t_2^{(N)}, \tau_2^{(N)}\big)$.

So, we obtain a sequence of nested intervals

$$\big[t_1^{(N)}, t_2^{(N)}\big] \subset \big[t_1^{(N-1)}, t_2^{(N-1)}\big] \subset \cdots \subset \big[t_1^{(1)}, t_2^{(1)}\big] \subset \big[0, \tfrac{25}{3}\big],$$

and for any solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19) there holds $\tau \in \big[t_1^{(N)}, t_2^{(N)}\big]$.

Now we study functions $\hat{\sigma}(\tau)$ and $\sigma(\tau)$, $\tau \in \big[t_1^{(N)}, t_2^{(N)}\big]$. Observe that in view of notations (5.15),

$$\hat{\sigma}(\tau) = b_1(\tau, \hat{\alpha}_N(\tau)) \qquad \text{with} \quad A_1 = 1.$$

Also, the proof of Lemma 5 (a) relies on the inequalities $b_j(\tau, \hat{z}(\tau)) > 0$, $\tau \in (t_1, t_2)$ for each $j = 0, 1, 2$ (see [20, Eq. (3.3.6)]). If for some $\tau \in (t_1^{(N)}, t_2^{(N)})$ there is a solution $\alpha_1, \dots, \alpha_N$ of (5.17)–(5.18) with $\alpha_N = \hat{\alpha}_N(\tau)$, being the larger positive zero of the equation $(5.17.N)$, then $\hat{\sigma}(\tau) > 0$. Hence, $s'''(x_{N+1}) \neq 0$ and condition (5.19) is not satisfied.

It follows that for any solution $\tau, \alpha_1, \dots, \alpha_N$ of (5.17)–(5.19) there holds $\alpha_k = \alpha_k(\tau)$, being the smaller positive zero of the equation $(5.17.k)$ for all $k = 1, \dots, N$, and $\tau \in (t_1^{(N)}, t_2^{(N)})$. By the notations in (5.15) we have

$$\sigma(\tau) = b_1(\tau, \alpha_N(\tau)) \qquad \text{with} \quad A_1 = 1.$$

According to Lemma 5 (c) the function $\sigma(\tau)$ satisfies condition (iv) of Lemma 4, i.e. $\sigma'(\tau) > 0$, $\tau \in (t_1^{(N)}, t_2^{(N)})$. Then $\sigma(\tau)$ is strictly monotone for $\tau \in (t_1^{(N)}, t_2^{(N)})$. By Lemma 5 (b) there exists a unique point $\tau^* \in (t_1^{(N)}, t_2^{(N)})$ such that $\sigma(\tau^*) = 0$, which implies $s'''(x_{N+1}) = 0$, i.e. (5.19).

So, we have proved that there exists a unique spline function s^* and knots $\mathbf{x}^* = (x_1^*, \dots, x_N^*) \in X_N$, such that $s^* \in S_5(x_1^*, \dots, x_N^*) \cap F(\mathbf{x}^*, \mathbf{y})$ and s^* satisfies the characterization of the smoothest interpolant to the problem (2.3) given in Theorem 1. This completes the proof of the theorem. $\square$

6. NUMERICAL ALGORITHM AND RESULTS

Here we discuss computational aspects of finding the unique oscillating spline interpolant from Theorem 2. We follow the procedure described in the proof of Theorem 2.

Let us fix τ as a point from an equidistant mesh for $\left[0, \frac{25}{3}\right]$. If the first equation (5.17.1) of the system (5.17) has not two simple positive roots we skip this value of τ and go to the next point of the mesh. If we do not succeed for that mesh, we decrease the mesh step and repeat. Thus, we find interval J_1 such that for each $\tau \in J_1$, (5.17.1) has two simple positive roots and we set α_1 to be the smaller of them. We represent the coefficients of the next algebraic equation (5.17.2) by α_1. If for a fixed τ from an equidistant mesh of J_1 the equation (5.17.2) of the system (5.17) has not two simple positive roots we skip this value of τ and go to the next point of the mesh in J_1. If we do not succeed for that mesh we decrease the mesh step and repeat. In this way we find interval $J_2 \subset J_1$ such that for each $\tau \in J_2$, (5.17.2) has two simple positive roots and we set α_2 to be the smaller of them. Repeating this process for each $i = 1, \ldots, N$ we find an interval J_i such that for $\tau \in J_i$ all the equations (5.17.1)–(5.17.i) have two positive roots. Moreover, $J_N \subset J_{N-1} \subset \cdots \subset J_1 \subset \left[0, \frac{25}{3}\right]$. Here intervals J_i are related to the intervals $\left[t_1^{(i)}, t_2^{(i)}\right]$, $i = 1, \ldots, N$ in the proof of Theorem 2.

Observe that (5.19)–(5.20) yield that $s'''(x_{N+1}) = s'''(\tau, x_{N+1}) = 0$ if $\sigma(\tau) = 0$. But the function $\sigma(\tau)$ defined in (5.20) is a monotone function of $\tau \in J_N$ and it changes sign in J_N. Then we find approximately τ^* by an equidistant mesh of the interval J_N, for which $\sigma(\tau)$ is minimal by absolute value.

Then we solve (5.17) and find $\alpha_i^* = \alpha_i(\tau^*)$, $i = 1, \ldots, N$.

In the next step we find Δ_i, $i = 0, \ldots, N$ using the formulae

$$\Delta_0 = \frac{b-a}{1 + \sum_{i=1}^{N} \prod_{j=1}^{i} \alpha_j^*}, \qquad \Delta_{i+1} = \alpha_i^* \Delta_i, \qquad i = 0, \ldots, N. \tag{6.1}$$

Hence, the optimal knots for the extremal problem (2.3) are

$$x_0^* = a, \qquad x_{i+1}^* = x_i^* + \Delta_i, \quad i = 0, \ldots, N-1, \qquad x_{N+1}^* = b. \tag{6.2}$$

From (5.6), (5.10), (5.13), and (5.16) we find recurrently β_i, $i = 0, \ldots, N$ and γ_i, $i = 0, \ldots, N$. Now, applying (5.4) we find

$$s''(x_i) = s_i'' = \frac{2\beta_i \Delta y_i}{\Delta_i^2}, \quad i = 0, \ldots, N, \quad s''(x_{N+1}) = s_{N+1}'' = \frac{2\gamma_N \Delta y_N}{\Delta_N^2}. \tag{6.3}$$

Next, we find the polynomial pieces $P_i \in \pi_5$ for $[x_i^*, x_{i+1}^*]$ by solving the Hermite interpolation problem (5.1), $i = 0, \ldots, N$. So, based on (5.3) we get the oscillating spline interpolant $s^*(t)$ satisfying (2.4).

Algorithm 1	Finding the Oscillating Spline Interpolant
Input:	Data $\mathbf{y} = (y_0, y_1, \dots, y_{N+1})$ with (1.2) and (1.3)
Step 1.	Find τ^* such that the system (5.17) has a solution satisfying (5.18) and $s'''(x_{N+1}) \approx 0$
Step 2.	For τ^* obtained in Step 1 find $\{\alpha_i\}_{i=1}^{N}$ solving (5.17) with (5.18)
Step 3.	For τ^* as in Step 1 find the knots $\mathbf{x}^* = (x_0^*, x_1^*, \dots, x_{N+1}^*)$ from (6.1)–(6.2)
Step 4.	For τ^* obtained in Step 1 find the quantities $\{s_i''\}_{i=0}^{N+1}$ from (6.3)
Step 5.	Construct the polynomial pieces $P_i \in \pi_5$ in $[x_i^*, x_{i+1}^*]$ by solving (5.1), $i = 0, \dots, N$
Step 6.	Construct the oscillating spline interpolant $s^*(t)$ based on (5.3)
Output:	The knots $\mathbf{x}^*$ of $s^*(t)$, polynomial pieces $\{P_i\}_{i=0}^{N}$ of the spline interpolant $s^*(t)$, graphs of $s^*(t)$ and its derivatives

We summarize in an algorithm the basic steps we pass to find the fifth degree oscillating spline interpolant with boundary conditions.

With the assistance of *Mathematica* (by Wolfram Research Inc.) computer algebra system, we implement the above algorithm to a numerical example.

Example. We show results of numerical experiments for the data

$$N = 9, \qquad \mathbf{y} = (1,\ -2,\ 3,\ -1,\ 5,\ 2,\ 4,\ 0,\ 1,\ -3,\ 2),$$

satisfying conditions (1.2) and (1.3).

According to the algorithm described in the previous section, $J_i = [\ell_i, r_i]$ is an interval such that for $\tau \in J_i$ all the equations (5.17.1)–(5.17.i) have two positive roots, $i = 1, \dots, 9$. Moreover, $J_9 \subset J_8 \subset \cdots \subset J_1 \subset \left[0, \frac{25}{3}\right]$. These nested intervals are given in Table 1.

J_i	ℓ_i	r_i
J_1	0.1	2.2
J_2	2.012	2.177
J_3	2.17	2.17495
J_4	2.1746	2.174867
J_5	2.174856	2.174864
J_6	2.1748639	2.174864
J_7	2.174864057	2.174864071
J_8	2.1748640706	2.17486407086
J_9	2.174864070844	2.1748640708585

Table 1: Nested intervals $J_9 \subset J_8 \subset \cdots \subset J_1 \subset \left[0, \frac{25}{3}\right]$

Our numerical results confirm monotonicity of the function $\sigma(\tau)$ for $\tau \in J_N$. Table 2 shows values of the function $\sigma(\tau)$ from (5.20), evaluated at equidistant points τ in a small interval $J \subset J_9$, where $\sigma(\tau) \approx 0$ and $\sigma(\tau)$ changes sign.

τ	$\sigma(\tau)$
2.17486407085837258	-0.00327161
2.17486407085837259	-0.00256242
2.1748640708583726	-0.00202345
2.17486407085837261	-0.00132029
2.17486407085837262	-0.00065045
2.17486407085837263	0.00001273
2.17486407085837264	0.00062011
2.17486407085837265	0.00132710
2.17486407085837266	0.00203236
2.17486407085837267	0.00274238

Table 2: $\sigma(\tau)$ for $\tau \in J = [2.17486407085837258, 2.17486407085837267]$

Now, we choose $\tau^* = 2.17486407085837263$ for which $\sigma(\tau) = 0.00001273$ is minimal in absolute value in Table 2, whence $s'''(x_{N+1}) \approx 0$. Solving the system (5.17) with (5.18) for $\tau = \tau^*$ we obtain the ratios $\alpha_i = \Delta_i/\Delta_{i-1}$, $i = 1, \dots, 9$. Hence, using (6.1) and (6.2) we find the interpolation knots $\{x_i^*\}_{i=0}^{10}$, being also knots of the oscillating spline interpolant, for the interval $[a, b] = [0, 1]$. The knots are listed in Table 3.

x_0^*	0
x_1^*	0.093572609937859
x_2^*	0.207960413155138
x_3^*	0.310783910315965
x_4^*	0.43351596313152
x_5^*	0.52765838534873
x_6^*	0.60829886372617
x_7^*	0.71795857706244
x_8^*	0.77954122488487
x_9^*	0.88685751448236
x_{10}^*	1

Table 3: Interpolation and spline knots $(x_0^*, x_1^*, \dots, x_{10}^*)$ for $[0, 1]$

Plot of the oscillating spline interpolant $s^*(t)$ satisfying (2.4), its first derivative, and its third derivative are shown in Figure 1, Figure 2, and Figure 3, respectively.

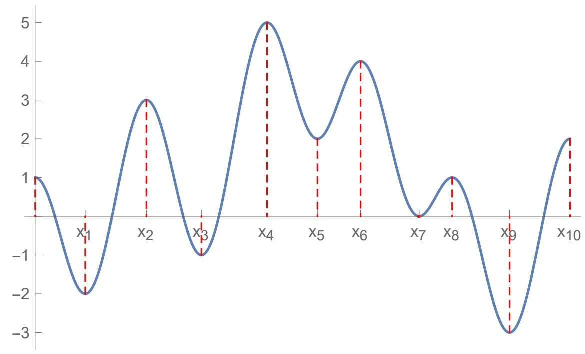

Figure 1: The smoothest interpolant $s^*(t)$

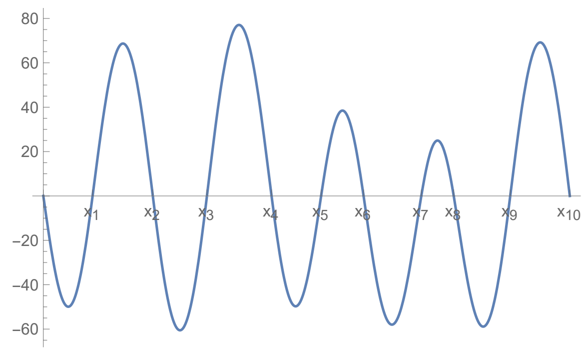

Figure 2: First derivative of the smoothest interpolant, $s^{*\prime}(t)$

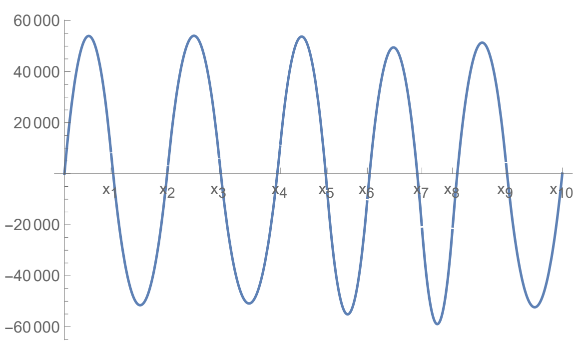

Figure 3: Third derivative of the smoothest interpolant, $s^{*\prime\prime\prime}(t)$

Acknowledgement. The authors are grateful to the referees for careful reading of the paper and their valuable suggestions and comments. This work was supported by the bilateral project KP-06-Austria/8/2019 (WTZ BG 03/2019), funded by Bulgarian National Science Fund and OeAD (Austria).

7. REFERENCES

[1] Bojanov, B.: Perfect splines of least deviation. *Anal. Math.* **16**, 1980, 185–197.

[2] Bojanov, B. σ-perfect splines and their applications to optimal recovery problems. *J. Complexity* **3**, 1987, 429–450.

[3] Bojanov, B.: B-splines with Birkhoff knots. *Constr. Approx.* **4**, 1988, 147–156.

[4] Bojanov, B.: Characterization of the smoothest interpolant. *SIAM J. Math. Anal.* **25**, 1994, No 6, 1642–1655.

[5] Bojanov, B., Hakopian, H., Sahakian, A.: *Spline Functions and Multivariate Interpolations*. Mathematics and Its Applications Vol. 248, Kluwer Academic Publishers, Dordrecht, 1993,

[6] de Boor, C.: On "Best" interpolation. *J. Approx. Theory* **16**, 1976, 28–42.

[7] Draganova, C.: Characterization of the smoothest periodic interpolant. *Math. Balkanica (N.S.)* **8**, 1994, 297–310.

[8] Draganova, C.: Smoothest interpolation in the mean. *J. Approx. Theory* **98** (1999), 223–247.

[9] Lorentz, G., Jetter, K., Riemenschneider, S.: *Birkhoff Interpolation*. Encyclopedia of Mathmatics and Its Applications, vol. 19, Addison & Wesley, 1983.

[10] Marin, S.: An approach to data parametrization in parametric cubic spline interpolation problems. *J. Approx. Theory* **41**, 1984, 64–86.

[11] Naidenov, N.: Algorithm for the construction of the smoothest interpolant. *East J. Approx.* **1**, 1995, 83–97.

[12] Pinkus, A.: On smoothest interpolants. *SIAM J. Math. Anal.* **19**, 1988, No 6, 1431–1441.

[13] Pinkus, A.: Uniqueness of the smoothest interpolants. *East J. Approx.* **3**, 1997, 377–380.

[14] Rademacher, C., Scherer, K.: Best parameter interpolation in L_p-norms. In: *Contributions to the Computation of Curves and Surfaces*, Monograf. Acad. Ciencia de Zaragoza, Puerto de la Cruz, 1989, pp. 67–80,

[15] Scherer, K.: Uniqueness of best parametric interpolation by cubic spline curves. *Constr. Approx.* **13**, 1997, 393–419.

[16] Scherer, K., Smith, P.: Existence of best parametric interpolation by curves. *SIAM J. Math. Anal.* **20**, 1989, 160–168.

[17] Schumaker, L. L.: *Spline Functios: Basic Theory*. Wiley Interscience, New York, 1981.

[18] Uluchev, R.: B-splines with Birkhoff knots. Applications in the approximations and shape-preserving interpolation. *Math. Balkanica (N.S.)* **3**, 1989, 225–239.

[19] Uluchev, R.: *Problems on Optimal Interpolation*, PhD Thesis, Sofia University, Sofia, 1990. (In Bulgarian)

[20] Uluchev, R.: Smoothest interpolation with free nodes in W_p^r, In: *Progress in Approximation Theory* (P. Nevai and A. Pinkus, Eds.), Academic Press, San Diego, 1991, pp. 787–806 (Special volume of *J. Approx. Theory*).

Received on April 18, 2020

Velina Ivanova, Rumen Uluchev

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5 James Bourchier Blvd.
BG 1164 Sofia
BULGARIA

E-mails: `vetotiv@abv.bg` , `rumenu@fmi.uni-sofia.bg`