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TIKHONOV'S THEOREM FOR FUNCTIONAL-DIFFERENTIAL INCLUSIONS*

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We investigate differential inclusions and equations of a retarded type with a small real parameter $\varepsilon > 0$ in part of the derivatives. Analogues of the well-known in the theory of singularly perturbed ordinary differential equations theorem of Thikhonov are obtained. We prove lower semicontinuity of the solution set for inclusions and continuity of the solution for equations in the most appropriate topology when $\varepsilon \to 0$.

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1. INTRODUCTION

Suppose that the functional-differential inclusion

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in F(t, x(t), y(t), x_t, y_t), \quad x_0 = \phi,\ y_0 = \psi, \quad t \in I = [0,1], \qquad (1)$$

is given, where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and $\varepsilon > 0$ is a real parameter.

In the sequel, $C(I, X)$ and $L^1(I, X)$ are the usual spaces of respectively continuous and integrable functions on I with values in X. For any $z \in C([-\tau, 1], \mathbf{R}^k)$ and $t \in I$ we let $z_t \in C([-\tau, 0], \mathbf{R}^k)$ be defined by $z_t(s) = z(t+s)$, $-\tau \leq s \leq 0$.

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Here $\tau \in (0,1)$ and F is a map from $I \times \mathbf{R}^{n+m} \times C([-\tau,0],\mathbf{R}^n) \times L^1([-\tau,0],\mathbf{R}^m)$ into $\mathbf{R}^n \times \mathbf{R}^m$, while $\phi \in C([-\tau,0],\mathbf{R}^n)$, $\psi \in C([-\tau,0],\mathbf{R}^m)$.

There is a fundamental theorem refered as Tikhonov's theorem [10] dealing with the continuity of the (unique) solution of (1) when F is single valued and does not contain (x_t, y_t). Namely, continuous dependence of the solution with respect to $C(I,\mathbf{R}^n) \times C([\delta,1],\mathbf{R}^m)$ topology $(0 < \delta < 1)$ when $\varepsilon \to 0$ is stated. Our considerations differ from the situation in [10] also in the fact that we assume only measurable on t right hand side. Then it is natural to define the solution set $Z(\varepsilon)$ of (1) when $\varepsilon > 0$ as the collection of all absolutely continuous functions (x,y) satisfying (1) for a.e. $t \in I$. When $\varepsilon = 0$, inclusion (1) becomes

$$\begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \in F(t, x(t), y(t), x_t, y_t), \quad x_0 = \phi,\ y_0 = \psi, \quad t \in I = [0,1]. \qquad (2)$$

Here solutions are all pairs (x,y) of absolutely continuous functions $x(\cdot)$ and L^1-functions $y(\cdot)$ such that (2) holds for a.e. $t \in I$. As in the ordinary differential case, $y(\cdot)$ can differ from the initial condition $\psi(\cdot)$ at $t = 0$.

It is too restrictive to assume the y-part of the solutions of the above "degenerate" inclusion to be continuous in view of the following simple example:

$$\varepsilon \dot{y}(t) = -2ay(t) + ay\left(t - \frac{1}{2}\right), \ y(s) = 1, \ s \in \left[-\frac{1}{2}, 0\right), \ a > 0.$$

For $\varepsilon = 0$ one has $0 = -2y(t) + y(t - 1/2)$, i.e. $y(t) = (1/2)y(t - 1/2)$. Thus $y^0(t) = 1/2$ for $t \in [0, 1/2)$ and $y^0(t) = 1/4$ for $t \in [1/2, 1)$. For this reason the C-topology used in [10] is not suitable and must be replaced with another one. In Examples 2.1 and 2.2 we show that when the delay is not fixed it happens the classical Tikhonov's theorem not to be valid. So it must be reformulated in the functional-differential case when it holds at all.

Here we examine first the lower semicontinuity properties of the solution map $Z(\varepsilon)$ as $\varepsilon \to 0^+$ and then derive on this base the continuity dependence of the solution for equations. For inclusions without the functional arguments (x_t, y_t) the lower semicontinuity is studied initially in [11]. The results then are extended under weaker type of assumptions in [3] for functional-differential inclusions with fixed time delay. The main assumption in the last paper is a version of the one-side Lipschitz condition used first for multivalued maps in [2]. Since singular perturbations are not presented in [2], this key condition is modified in [3] and here in a suitable way. We do not consider upper semicontinuous properties since, as shown in [3], the solution set is not upper semicontinuous in used here $C(I,\mathbf{R}^n) \times L^1(I,\mathbf{R}^m)$ topology, even for linear control system. Moreover, in the case considered in [3], redefining the solution set of (2) to obtain upper semicontinuity one will lose lower semicontinuity. Some upper semicontinuous results under restrictive assumptions are obtained in [3–5].

At the end of the section we shall give some notations and definitions. Introduce the subspaces $\Omega_i = \{\alpha \in C([-\tau,0],\mathbf{R}^{k_i}) : |\alpha(0)| = \max_{-\tau \le s \le 0} |\alpha(s)|\}$, $k_1 = n$, $k_2 = m$, which are used in Razumikhin type conditions [7]. The norms in $C(I,X)$ and $L^1(I,X)$ are denoted with $\|\cdot\|_C$ and $\|\cdot\|_{L^1}$, respectively. For the sake of simplicity we will denote by $\|\alpha_t\|_C$ and $\|\alpha_t\|_{L^1}$, respectively, the norms $\max_{-\tau\leq s\leq 0}|\alpha(t+s)|$ and $\int\limits_{-\tau}^{0}|\alpha(t+s)|\,ds$. For a set $A\subset \mathbf{R}^k$ and a vector $l\in\mathbf{R}^k$ we let $\sigma(l,A)=\sup_{a\in A}\langle l,a\rangle$ be the support function, where $\langle\cdot,\cdot\rangle$ is the scalar product. If $A\subset\mathbf{R}^{n+m}$, we denote by $\hat{A}$ the projection of A on $\mathbf{R}^n$, by $\bar{A}$ the projection of A on $\mathbf{R}^m$, and by clA ($clcoA$) the closed (the closed convex) hull of A. The set-valued map $G: I\times Z\to Z$ is called: a) lower semicontinuous (LSC) when for every (t,z) and every $u\in G(t,z)$ there exists $u_i\in G(t_i,z_i)$ such that $u_i\to u$ when $t_i\to t$, $z_i\to z$; b) upper semicontinuous (USC) if for every (t,z) and every $\nu>0$ there exists $\delta>0$ such that $G(s,w)\subset G(t,z)+\nu U$ (here U is the unit ball in Z) when $|t-s|+|z-w|<\delta$; c) continuous when G is LSC and USC. G is called *almost* continuous (resp. LSC, USC) when for every $\delta>0$ there is a compact set $I_\delta\subset I$ with $meas(I\setminus I_\delta)<\delta$ such that G is continuous (resp. LSC, USC) on $I_\delta\times Z$. For more detailed considerations of definitions and concepts used bellow we refer to [1] and [7].

2. LOWER SEMICONTINUITY IN $C\times L^1$–TOPOLOGY

We take an example which tells us that for continuity with respect to $C[\delta,1]$ topology on $y(\cdot)$ there have to be restrictive assumptions.

Example 2.1. Consider the following equation:

$$\varepsilon\dot{y}(t)=-2y(t)+\max_{s\in I_t}y(t+s),\quad y(0)=1,$$

where $I_t=[\max\{-1/2,-t\},0]$ for $t\in[0,1]$. For $\varepsilon>0$ one can find

$$y^\varepsilon(t)\geq\frac{1}{2}\left(1+\exp\left(-\frac{1}{\varepsilon}\right)\right),\quad 0\leq t\leq\frac{1}{2},$$

$$y^\varepsilon(t)\geq\frac{1}{4}\left(1+\exp\left(-\frac{2}{\varepsilon}\left(t-\frac{1}{2}\right)\right)\right),\quad \frac{1}{2}\leq t\leq 1.$$

For $\varepsilon=0$ we get the "degenerate" equation

$$2y(t)=\max_{s\in I_t}y(t+s).$$

Obviously, $\bar{y}^0(t)=1/2$, $t\in(0,1/2]$; $\bar{y}^0(t)=1/4$, $t\in(1/2,1]$ with $\bar{y}^0(0)=1$ is a solution of the above equation. Also it is not difficult to see that $y^\varepsilon(t)\to\bar{y}^0(t)$, $\varepsilon\to 0$ for $t\in I$ and that this convergence is uniform on $[\delta,1/2)\cup[1/2+\delta,1]$. On the other hand, $y^0(t)\equiv 0$ on $t\in I$ is other solution of the "degenerate" equation. The last implies that there is no continuous in $C[\delta,1]$ but only USC dependence in $C([\delta,1/2)\cup[1/2+\delta,1])$ topology.

Example 2.2. Let us combine the above equation with the control system from Example 2.5 of [3], i.e. consider

$$\dot{x}=|y_1-2y_2|,\quad x(0)=0,$$

$$\varepsilon \dot{y}_1 = -y_1 + u(t), \quad y_1(0) = 0,$$
$$\varepsilon \dot{y}_2 = -2y_2 + u(t), \quad y_2(0) = 0,$$
$$\varepsilon \dot{y}_3 = -2y_3(t) + \max_{s \in I_t} y_3(t+s), \quad y_3(0) = 1,$$

where $u(t) \in [-1, 1]$ is measurable. It is shown in [3] that the solution set of the subsystem consisting of the first three equations is not USC in $C([0,1], \mathbf{R}) \times L^1([0,1], \mathbf{R}^2)$ topology at $\varepsilon = 0$. Thus the solution set of the above inclusion is neither LSC nor USC.

These examples tell us that when the delay depends on time t it is hard to expect that Tikhonov's theorem is true. But still there are situations in which we could formulate a very close result. Consider first (1) under the following assumptions:

A1. The map F is almost continuous and bounded on the bounded sets. Moreover, there exist constants $a, b, \mu > 0$ such that for every $(x, y) \in \mathbf{R}^{n+m}$

$$\sigma(x, \hat{F}(t, x, y, \alpha, \beta)) \le a(1 + |x|^2 + |y|^2 + \|\beta\|_C^2), \quad \alpha \in \Omega_1, \ \beta \in C([-\tau, 0], \mathbf{R}^m),$$
$$\sigma(y, \bar{F}(t, x, y, \alpha, \beta)) \le b(1 + |x|^2 + \|\alpha\|_C^2) - \mu |y|^2, \quad \alpha \in C([-\tau, 0], \mathbf{R}^n), \ \beta \in \Omega_2,$$

for a.e. $t \in I$. Here $\alpha(0) = x$, $\beta(0) = y$.

A2. There exist positive constants A, B and μ such that if we choose arbitrary $(x_i, y_i, \alpha_i, \beta_i) \in \mathbf{R}^{n+m} \times C([-\tau, 0], \mathbf{R}^n) \times L^1([-\tau, 0], \mathbf{R}^m)$, $i = 1, 2$, then for every $(f_1, g_1) \in F(t, x_1, y_1, \alpha_1, \beta_1)$ there is $(f_2, g_2) \in F(t, x_2, y_2, \alpha_2, \beta_2)$ such that

$$\langle x_1 - x_2, f_1 - f_2 \rangle \le A(|x_1 - x_2|^2 + |y_1 - y_2|^2 + \|\beta_1 - \beta_2\|_{L^1}^2); \text{ for } \alpha_1 - \alpha_2 \in \Omega_1,$$
$$\langle y_1 - y_2, g_1 - g_2 \rangle \le B(|x_1 - x_2|^2 + \|\alpha_1 - \alpha_2\|_C^2 + \|\beta_1 - \beta_2\|_{L^1}^2) - \mu |y_1 - y_2|^2$$

for a.e. $t \in I$. Here $\alpha_i(0) = x_i$ and for β_i continuous $\beta_i(0) = y_i$, $i = 1, 2$.

The next result is proved in [3].

Lemma 2.3. *Under* A1 *there exists a constant $M > 0$ such that $|x^\varepsilon(t)| + |y^\varepsilon(t)| \le M$ for every $t \in I$, $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$ and $\varepsilon > 0$, and a.e. on I if $\varepsilon = 0$.*

By A1 it follows that there exists $L > 0$ such that $|F(t, x, y, \alpha, \beta)| \le L$ for every $t \in I$, $|x| + |y| \le M + 1$ and $\|\alpha\|_C + \|\beta\|_{L^\infty} \le M + 1$.

Theorem 2.4. *Under assumptions* A1 *and* A2 *the solution set $Z(\varepsilon)$ is LSC at $\varepsilon = 0^+$ with respect to $C([0,1], \mathbf{R}^n) \times L^1([0,1], \mathbf{R}^m)$ topology.*

Proof. Let (x^0, y^0) be a solution of (2) and $\delta > 0$ be given. Then there is a Lipschitz on I function z with a Lipschitz constant K_δ such that $z(s) = \psi(s)$, $s \in [-\tau, 0]$, and

$$\|z - y^0\|_{L^1} \le \delta, \quad \|\rho\|_{L^1} \le \delta.$$

Here $\rho(t) = D_H(F(t, x^0, y^0, x_t^0, y_t^0), F(t, x^0, z, x_t^0, z_t))$ and $D_H(\cdot, \cdot)$ is the Hausdorff distance between sets. Therefore

$$d((\dot{x}^0(t), \varepsilon \dot{z}(t)), F(t, x^0(t), z(t), x_t^0, z_t)) \le \varepsilon K_\delta + \rho(t). \tag{3}$$

Introduce the following conditions:

$$\begin{aligned}&\langle x^0(t)-u,\dot{x}^0(t)-f\rangle\\ &< 2A(|x^0(t)-u|^2+|z(t)-v|^2+\|z_t-\beta\|^2_{L^1})+\varepsilon K_\delta+\rho(t)+\delta,\end{aligned} \tag{4}$$

$$\begin{aligned}&\langle z(t)-v,\varepsilon\dot{z}(t)-g\rangle\\ &< 2B(|x^0(t)-u|^2+\|x^0_t-\alpha\|^2_C+\|z_t-\beta\|^2_{L^1})-\mu|z(t)-v|^2+\varepsilon K_\delta+\rho(t)+\delta.\end{aligned} \tag{5}$$

Consider the map $\Gamma_\delta(t,u,v,\alpha,\beta)$, which we define only for continuous β, with values as follows:

a) $cl\{(f,g)\in F(t,u,v,\alpha,\beta): g$ satisfies (5)$\}$ for $\alpha-x^0_t\notin\Omega_1, u=\alpha(0)$ and $v=\beta(0)$;

b) $cl\{(f,g)\in F(t,u,v,\alpha,\beta):(f,g)$ satisfies (4) and (5)$\}$ for $\alpha-x^0_t\in\Omega_1, u=\alpha(0)$ and $v=\beta(0)$;

c) $\Gamma_\delta(t,u,v,\alpha,\beta)=F(t,u,v,\alpha,\beta)$ when $u\neq\alpha(0)$ or $v\neq\beta(0)$.

Note that F is almost continuous on $I\times\mathbf{R}^{n+m}\times C(I,\mathbf{R}^{n+m})$. We claim that $\Gamma_\delta(\cdot)$ is almost LSC with nonempty and compact values. To prove that we first note that $\Gamma_\delta(\cdot)$ is compact valued by its definition, Lemma 2.3 and A1. We will show the nonemptiness of $\Gamma_\delta(\cdot)$ only in case b).

By (3) there is $(f^0(t),g^0(t))\in F(t,x^0(t),z(t),x^0_t,z_t)$ such that for a.e. $t\in I$

$$|(\dot{x}^0(t),\varepsilon\dot{z}(t))-(f^0(t),g^0(t))|\le\varepsilon K_\delta+\rho(t).$$

So, there exists $(f,g)\in F(t,u,v,\alpha,\beta)$ such that for $x_1=x^0(t)$, $x_2=u$, $y_1=z(t)$, $y_2=v$, $f_1=f^0$, $f_2=f$, $g_1=g^0$ and $g_2=g$ the inequalities of A2 hold, i.e.

$$\begin{aligned}&\langle x^0(t)-u,f^0-f\rangle<A(|x^0(t)-u|^2+|z(t)-v|^2+\|z_t-\beta\|^2_{L^1}),\\ &\langle z(t)-v,g^0-g\rangle<B(|x^0(t)-u|^2+\|x^0_t-\alpha\|^2_C+\|z_t-\beta\|^2_{L^1})-\mu|z(t)-v|^2.\end{aligned}$$

Therefore the inequalities (4) and (5) are fulfilled.

The fact that $\Gamma_\delta(\cdot)$ is almost LSC has a standard proof (see [1]), which is omitted.

Now, from [6] we know that the inclusion

$$\begin{pmatrix}\dot{x}(t)\\ \varepsilon\dot{y}(t)\end{pmatrix}\in\Gamma_\delta(t,x(t),y(t),x_t,y_t),\quad x_0=\phi,y_0=\psi,\quad t\in I=[0,1], \tag{6}$$

has a solution $(x^\varepsilon,y^\varepsilon)$ in this case as well. On the other hand, $|x^\varepsilon(t)-x^0(t)|^2\le 2h(t)$ and $|y^\varepsilon(t)-z(t)|^2\le 2r(t)$, where:

$$\begin{aligned}&\dot{h}(t)=2A(h(t)+r(t)+\|r_t\|_{L^1})+\rho(t)+\delta+\varepsilon K_\delta,\quad h(0)=0,\\ &\varepsilon\dot{r}(t)=2Bh(t)-\mu r(t)+2B(\|h_t\|_C+\|r_t\|_{L^1})+\rho(t)+\delta+\varepsilon K_\delta,\quad r(0)=r_0.\end{aligned}$$

We do not indicate the dependence on ε of the solution of the system for the sake of simplicity of notations. Let k be a sufficiently large natural number. We divide $[0,1]$ on k parts with equal lengths. Obviously, by the first equation above we have that $h(\cdot)$ increases, i.e one can suppose without loss of generality that $h(t)=\|h_t\|_C$.

Then solving the second equation on $[0,1/k]$ and integrating by parts one obtains

$$r(t) \le \exp(-\mu t/\varepsilon) r_0 + (1/\varepsilon)\int_0^t \exp(-\mu(t-s)/\varepsilon)(4Bh(s) + \rho(s)$$
$$+ 2B\|r_s\|_{L^1} + \delta + \varepsilon K_\delta)\, ds$$
$$\le \exp(-\mu t/\varepsilon) r_0 + (1/\varepsilon)\int_0^t \exp(-\mu(t-s)/\varepsilon)(\rho(s) + 2B\|r_s\|_{L^1})\, ds$$
$$+ (1/\mu)(4Bh(t) + \delta + \varepsilon K_\delta).$$

Denoting further with C an arbitrary positive constant dependent only on A, B and μ (in the following inequality for example $C = 2A + 8AB/\mu$), we derive that

$$h(t) \le \int_0^t \exp(C(t-s))(\rho(s) + 2A\|r_s\|_{L^1} + (1+2A/\mu)(\delta + \varepsilon K_\delta) + 2A\exp(-\mu s/\varepsilon) r_0)\, ds$$

$$+(2A/\varepsilon)\int_0^t \int_0^s \exp(C(t-s))\exp(-\mu(s-\lambda)/\varepsilon)(\rho(\lambda) + \|r_\lambda\|_{L^1})\, d\lambda\, ds.$$

Thus changing the order of integration we get $h(t) \le C(2\varepsilon K_\delta + \delta + 1/k)$ for $t \in [0, 1/k]$. Consequently,

$$\int_0^t |r(s)|\, ds \le C\left(2\varepsilon K_\delta + \delta + \frac{1}{k}\right) \quad \text{for } t \in \left[0, \frac{1}{k}\right].$$

By induction one can show that

$$h(t) \le C\left(2\varepsilon K_\delta + \delta + \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^i}\right),$$
$$\|r\|_{L^1[0,t]} \le C\left(2\varepsilon K_\delta + \delta + \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^i}\right), \quad t \in \left[0, \frac{i}{k}\right].$$

Finally, one obtains

$$h(t) \le C\left(2\varepsilon K_\delta + \delta + \frac{1}{k-1}\right), \quad \|r(\cdot)\|_{L^1} \le C\left(2\varepsilon K_\delta + \delta + \frac{1}{k-1}\right).$$

Since k is arbitrarily large, we get that there exists a solution $(x^\varepsilon, y^\varepsilon)$ of (1) such that

$$\|x^\varepsilon - x^0\|_C^2 \le C(\varepsilon K_\delta + \delta), \quad \|y^\varepsilon - y^0\|_{L^1}^2 \le C(\varepsilon K_\delta + \delta).$$

Since δ is arbitrary and K_δ depends on δ but not on ε, the LSC in the considered topology is established. ■

Remark. A preliminary version of this theorem is reported in [9].

Consider the following special case of (1):

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in F(t, x(t), y(t), x(t-\tau(t)), y(t-\tau(t))), \qquad (7)$$
$$x(t) = \phi(t),\ y(t) = \psi(t), \quad t \in [-\lambda, 0],$$

where $\tau(t) \in [0, \lambda]$ is a monotone non-increasing function on I. Suppose that:

B1. The map F is Caratheodory's and bounded on the bounded sets. Moreover, there exist constants $a, b, \mu > 0$ such that for every $t \in I$, $(x(t), y(t)) \in \mathbf{R}^{n+m}$

$$\sigma(x(t), \hat{F}(t, x(t), y(t), x(t-\tau(t)), y(t-\tau(t)))) \le a(1 + |x(t)|^2 + |y(t)|^2 + |x(t-\tau(t))|^2 + |y(t-\tau(t))|^2),$$
$$\sigma(y(t), \bar{F}(t, x(t), y(t), x(t-\tau(t)), y(t-\tau(t)))) \le b(1 + |x(t)|^2 + |x(t-\tau(t))|^2 + |y(t-\tau(t))|^2) - \mu|y(t)|^2.$$

B2 (one-side Lipschitz condition). There exist positive constants A, B and μ such that for every $(f_1, g_1) \in F(t, x_1(t), y_1(t), x_1(t-\tau(t)), y_1(t-\tau(t)))$ there is $(f_2, g_2) \in F(t, x_2(t), y_2(t), x_2(t-\tau(t)), y_2(t-\tau(t)))$ such that

$$\langle x_1 - x_2, f_1 - f_2 \rangle \le A(|x_1 - x_2|^2 + |y_1 - y_2|^2 + |\alpha_1 - \alpha_2|^2 + |\beta_1 - \beta_2|^2),$$
$$\langle y_1 - y_2, g_1 - g_2 \rangle \le B(|x_1 - x_2|^2 + |\alpha_1 - \alpha_2|^2 + |\beta_1 - \beta_2|^2) - \mu|y_1 - y_2|^2$$

for a.e. $t \in I$. Here $\alpha_i(t) = x_i(t - \tau(t))$, $\beta_i(t) = y_i(t - \tau(t))$, $i = 1, 2$.

B3. If $\inf_{t \in I} \tau(t) = 0$, then $\mu > B$.

Theorem 2.5. *Under the assumptions* B1–B3, *the solution set* $Z(.)$ *is lower semicontinuous in* $C(I, R^n) \times L^1(I, R^m)$ *topology.*

Proof. Define the sequence $t_{i+1} = \sup\{t \in I | t_{i-1} \le t - \tau(t) \le t_i\}$, where $t_0 = -\lambda$, $t_1 = 0$. There are two cases. If $t_k = 1$ for some k, one can easily complete the proof exploiting the same fashion as in a fixed time lag, see Theorem 3.2 from [3]. In the opposite case there exists obviously $\nu \le 1$ with $\nu = \lim_{i \to \infty} t_i$. Then **B3** holds. Moreover, $\tau(t) = 0$ for $t \ge \nu$, i.e. the inclusion (7) becomes an ordinary differential one. Let $\delta > 0$ be given and (x^0, y^0) be the solution of (7) for $\varepsilon = 0$. Then for every $t < \nu$ again on the base of [3] one can find $\varepsilon(t, \delta)$ such that there exists $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$ whenever $0 \le \varepsilon \le \varepsilon(t, \delta)$ with

$$\|x^0 - x^\varepsilon\|_{C[0,t]} + \|y^0 - y^\varepsilon\|_{L^1[0,t]} < \delta/3.$$

Note that the norms above are evaluated on $[0, t]$. Moreover, $Z(\varepsilon)$ is LSC on $[\nu, 1]$ with respect to $C([\nu, 1], \mathbf{R}^n) \times L^1([\nu, 1], \mathbf{R}^m)$, see [11]. So without loss of generality one can suppose that

$$\|x^0 - x^\varepsilon\|_{C[\nu,1]} + \|y^0 - y^\varepsilon\|_{L^1[\nu,1]} < \delta/3.$$

Using the boundedness of the solution set and thus of the right hand side of (7), we can manage also on the interval $[t, \nu]$. Namely, if $\nu - t$ is small enough, then

$$\|x^0 - x^\varepsilon\|_{C[t,\nu]} + \|y^0 - y^\varepsilon\|_{L^1[t,\nu]} < \delta/3.$$

Consequently, there exists $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$ such that

$$\|x^0 - x^\varepsilon\|_C + \|y^0 - y^\varepsilon\|_{L^1} < \delta$$

for sufficiently small ε. ∎

3. TIKHONOV TYPE THEOREM FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

Consider now the following singularly perturbed system of functional-differential equations:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t), x_t, y_t), \quad x_0 = \phi, \\ \varepsilon\dot{y}(t) &= g(t, x(t), y(t), x_t, y_t), \quad y_0 = \psi, \end{aligned} \tag{8}$$

derived from (1) when F is single valued. Here $f(\cdot)$ and $g(\cdot)$ are Caratheodory's functions, satisfying A1 and A2.

First we shall show that the reduced system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t), x_t, y_t), \quad x_0 = \phi, \\ 0 &= g(t, x(t), y(t), x_t, y_t), \quad y_0 = \psi, \end{aligned} \tag{9}$$

admits $C(I, \mathbf{R}^n) \times L^1(I, \mathbf{R}^n)$ solution, i.e. the next lemma is true.

Lemma 3.1. *Under the assumptions* A1 *and* A2 *the degenerate system* (9) *has a solution.*

Proof. First we shall consider the case when f and g are jointly continuous, i.e. continuous in all arguments.

By Lemma 2.3 for $0 < \delta < \mu$ there is a constant M_δ such that for all $t \in I$

$$\begin{aligned} &|x(t)| + |y(t)| \le M_\delta, \text{ when} \\ &|\dot{x}(t) - f(t, x(t), y(t), x_t, y_t)| \le \delta, \quad |g(t, x(t), y(t), x_t, y_t)| \le \delta. \end{aligned}$$

Choose a sequence $\delta_i \to 0^+$ and construct the corresponding sequence of approximate solutions (x^i, y^i) as follows. By the well-known theorem of Minty-Browder there exists $\beta_0 \in \mathbf{R}^m$ such that

$$0 = g(t, \phi(0), \beta_0, \phi, \psi). \tag{10}$$

Let

$$x^i(t) = \phi(0) + t f(0, \phi(0), \beta_0, \phi, \psi), \quad y^i(t) = \beta_0,$$

for $t \in [0, \nu_1]$. Here ν_1 is the maximal ν for which (10) and

$$|\dot{x}^i(t) - f(t, x^i(t), y^i(t), x^i_t, y^i_t)| \le \delta_i, \quad |g(t, x^i(t), y^i(t), x^i_t, y^i_t)| \le \delta_i$$

hold on $[0, \nu]$. Using continuity of f, g and Zorn's lemma, it is not difficult to show the existence of such (x^i, y^i) on the whole I. By the Arzela-Ascoli's theorem $\{x^i(\cdot)\}_{i=1}^\infty$ is $C(I, \mathbf{R}^n)$ precompact and passing to subsequences if needed, there exists a cluster point $x^0(\cdot) \in C(I, \mathbf{R}^n)$. We shall show that $\{y^i(\cdot)\}_{i=1}^\infty$ is a Cauchy sequence in $L^1(I,\mathbf{R}^m)$. Denote $r(t) \equiv r_{ij}(t) = |y^i(t)-y^j(t)|$, $\delta_{ij} = \|x^i(\cdot)-x^j(\cdot)\|_C$. Then, of course, $\|x_t^i - x_t^j\|_{L^1} \le \delta_{ij}$ and by A2 we obtain

$$\mu r^2(t) \le B(\delta_{ij}^2 + \|r_t\|_{L^1}^2) + C(\delta_i + \delta_j).$$

For the sake of simplicity of notations here and further we denote with C an arbitrary constant and with δ_{ij} an expression tending to zero with $i,j \to \infty$. Hence

$$r(t) \le C(\delta_{ij} + \|r_t\|_{L^1}),\ \ t \in I \text{ and } \ r(t) = 0,\ t \in [-\tau, 0].$$

Let $r(t) = M$ on $[0,\tau]$. Thus $\|r_t\|_{L^1} \le \int\limits_0^t M\,ds = Mt$ for $t \in [0,\tau]$. Therefore

$$r(t) \le C\delta_{ij} + CMt,\ \ t \in [0,\tau].$$

Since $\|r_t\|_{L^1} = \int\limits_0^\tau r(t-s)\,ds = \int\limits_0^t r(s)\,ds$, we have

$$\|r_t\|_{L^1} \le C\delta_{ij}t + CM\frac{t^2}{2!},\ \ t \in [0,\tau].$$

Then it follows

$$r(t) \le C\delta_{ij}\left(1 + \frac{Ct}{1!}\right) + CM\frac{t^2}{2!},\ \ t \in [0,\tau].$$

Proceeding in the same way, we find that

$$r(t) \le C\delta_{ij}\left(1 + \frac{Ct}{1!} + \frac{(Ct)^2}{2!} + \cdots\right) + M\lim_{n\to\infty}\frac{(Ct)^n}{n!} \le \delta_{ij}C\exp(Ct),\ \ t \in [0,\tau].$$

Thus $\lim\limits_{i,j\to\infty} r(t) \equiv \lim\limits_{i,j\to\infty} r_{ij}(t) = 0$ and $\{y^i(\cdot)\}_{i=1}^\infty$ is a Cauchy sequence on $[0,\tau]$. Therefore $\lim\limits_{i\to\infty} y^i(t) = y(t), t \in [0,\tau]$ exists. It is easy to show that $(x(t),y(t))$ is a solution of (9) on $[0,\tau]$. Analogously (keeping in mind that $r(t) = 0$, $t \in [0,\tau]$), the solution can be extended on $[\tau, 2\tau]$ and therefore by induction on $[0,1]$.

Now let $f(\cdot)$ and $g(\cdot)$ be Caratheodory's functions. By Scorza-Dragoni's theorem $f(\cdot)$ and $g(\cdot)$ are almost continuous, so we can use the same fashion. Namely, for $\delta_i > 0$ consider $A_i \subset I$ with $meas A_i < \delta_i$, $A_{i+1} \subset A_i$. Also let us have on $I \setminus A_i$ that $f(\cdot)$ and $g(\cdot)$ are continuous and for the approximate solutions (x^i, y^i) the following relations are true:

$$\left|\dot{x}^i(t) - f(t, x^i(t), y^i(t), x_t^i, y_t^i)\right| \le \delta_i, \quad \left|g(t, x^i(t), y^i(t), x_t^i, y_t^i)\right| \le \delta_i.$$

On A_i the above distances are less or equal to L.

Denote again $r(t) = |y^i - y^j|$. One can show that $r(t) \le \delta_{ij}(t)D\exp(t)$, where $\delta_{ij}(t) \le M, t \in A_i$, and $\delta_{ij}(t) \le \delta_{ij}, t \in I \setminus A_i$, where $\lim\limits_{i,j\to\infty}\delta_{ij} = 0$. Therefore $(x^i(\cdot), y^i(\cdot)) \to (x(\cdot), y(\cdot))$, which is a solution of (9) on $[0,1]$. ∎

Now one can easily prove the next variant of the Tikhonov's theorem.

Theorem 3.2. *Under conditions* A1, A2 *for single valued* F *the solution set* $Z(\varepsilon)$ *of* (8) *is continuous in* $C([0,1],\mathbf{R}^n) \times L^1([0,1],\mathbf{R}^m)$ *topology at* $\varepsilon = 0^+$.

Proof. The solution set $Z(0)$ of (9) is non-empty thanks to Lemma 3.1. By A2 it follows (see [8]) that $Z(\varepsilon)$ is single valued. Then by the LSC of $Z(\varepsilon)$ at $\varepsilon = 0^+$ (Theorem 2.4) the proof is completed. ∎

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