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LOWER BOUNDING THE FOLKMAN NUMBERS $F_v(a_1, \dots, a_s; m-1)$

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For a graph G the expression $G \overset{v}{\to} (a_1, \dots, a_s)$ means that for every s-coloring of the vertices of G there exists $i \in \{1, \dots, s\}$ such that there is a monochromatic a_i-clique of color i. The vertex Folkman numbers

$$F_v(a_1, \dots, a_s; m-1) = \min\{|V(G)| : G \overset{v}{\to} (a_1, \dots, a_s) \text{ and } K_{m-1} \not\subseteq G\}.$$

are considered, where $m = \sum_{i=1}^{s}(a_i - 1) + 1$. We know the exact values of all the numbers $F_v(a_1, \dots, a_s; m-1)$ when $\max\{a_1, \dots, a_s\} \leq 6$ and also the number $F_v(2, 2, 7; 8) = 20$. In [1] we present a method for obtaining lower bounds on these numbers. With the help of this method and a new improved algorithm, in the special case when $\max\{a_1, \dots, a_s\} = 7$ we prove that $F_v(a_1, \dots, a_s; m-1) \geq m + 11$ and this bound is exact for all m. The known upper bound for these numbers is $m + 12$. At the end of the paper we also prove the lower bounds $19 \leq F_v(2, 2, 2, 4; 5)$ and $29 \leq F_v(7, 7; 8)$.

Keywords: Folkman number, clique number, independence number, chromatic number.

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1. INTRODUCTION

Only finite, non-oriented graphs without loops and multiple edges are considered in this paper. $G_1 + G_2$ denotes the graph G for which $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x, y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e.

G is obtained by connecting with an edge every vertex of G_1 to every vertex of G_2. All undefined terms can be found in [19].

Let $a_1, \dots, a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, \dots, a_s)$ means that for every coloring of $\mathrm{V}(G)$ in s colors (s-coloring) there exists $i \in \{1, \dots, s\}$ such that there is a monochromatic a_i-clique of color i. In particular, $G \overset{v}{\rightarrow} (a_1)$ means that $\omega(G) \geq a_1$. Further, for convenience, instead of $G \overset{v}{\rightarrow} (\underbrace{2, \dots, 2}_{r})$ we write $G \overset{v}{\rightarrow} (2_r)$ and instead of $G \overset{v}{\rightarrow} (\underbrace{2, \dots, 2}_{r}, a_1, \dots, a_s)$ we write $G \overset{v}{\rightarrow} (2_r, a_1, \dots, a_s)$.

Set

$$\mathcal{H}(a_1, \dots, a_s; q) := \left\{ G : G \overset{v}{\rightarrow} (a_1, \dots, a_s) \text{ and } \omega(G) < q \right\};$$
$$\mathcal{H}(a_1, \dots, a_s; q; n) := \{ G : G \in \mathcal{H}(a_1, \dots, a_s; q) \text{ and } |\mathrm{V}(G)| = n \}.$$

The vertex Folkman number $F_v(a_1, \dots, a_s; q)$ is defined by the equality:

$$F_v(a_1, \dots, a_s; q) = \min \{ |\mathrm{V}(G)| : G \in \mathcal{H}(a_1, \dots, a_s; q) \}.$$

The graph G is called an extremal graph in $\mathcal{H}(a_1, \dots, a_s; q)$ if $G \in \mathcal{H}(a_1, \dots, a_s; q)$ and $|\mathrm{V}(G)| = F_v(a_1, \dots, a_s; q)$. We denote by $\mathcal{H}_{extr}(a_1, \dots, a_s; q)$ the set of all extremal graphs in $\mathcal{H}(a_1, \dots, a_s; q)$.

Folkman proved in [6] that:

$$F_v(a_1, \dots, a_s; q) \text{ exists } \Leftrightarrow q > \max \{a_1, \dots, a_s\}. \tag{1.1}$$

Other proofs of (1.1) are given in [5] and [8]. In the special case $s = 2$, a very simple proof of this result is given in [12] with the help of corona product of graphs.

Obviously, $F_v(a_1, \dots, a_s; q)$ is a symmetric function of $a_1, \dots, a_s$, and if $a_i = 1$, then

$$F_v(a_1, \dots, a_s; q) = F_v(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_s; q).$$

Therefore, it suffices to consider only such Folkman numbers $F_v(a_1, \dots, a_s; q)$ for which

$$2 \leq a_1 \leq \dots \leq a_s. \tag{1.2}$$

We call the numbers $F_v(a_1, \dots, a_s; q)$ for which inequalities (1.2) hold canonical vertex Folkman numbers.

In [9] for arbitrary positive integers $a_1, \dots, a_s$ the following terms are defined

$$m(a_1, \dots, a_s) = m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad p = \max \{a_1, \dots, a_s\}. \tag{1.3}$$

It is easy to see that $K_m \overset{v}{\rightarrow} (a_1, \dots, a_s)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1, \dots, a_s)$. Therefore

$$F_v(a_1, \dots, a_s; q) = m, \quad q \geq m + 1.$$

The following theorem for the numbers $F_v(a_1, \dots, a_s; m)$ is true:

Theorem 1.1. *Let $a_1, \ldots, a_s$ be positive integers and let m and p be defined by the equalities* (1.3)*. If $m \geq p+1$, then:*

(a) $F_v(a_1, \ldots, a_s; m) = m + p$, ([9, 8])*;*

(b) $K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$ *is the only extremal graph in* $\mathcal{H}(a_1, \ldots, a_s; m)$*,* ([8])*.*

The condition $m \geq p+1$ is necessary according to (1.1). Other proofs of Theorem 1.1 are given in [13] and [14].

Very little is known about the numbers $F_v(a_1, \ldots, a_s; m-1)$. According to (1.1) we have

$$F_v(a_1, \ldots, a_s; m-1) \text{ exists } \Leftrightarrow m \geq p+2. \tag{1.4}$$

The following general bounds are known:

$$m + p + 2 \leq F_v(a_1, \ldots, a_s; m-1) \leq m + 3p, \tag{1.5}$$

where the lower bound is true if $p \geq 2$ and the upper bound is true if $p \geq 3$. The lower bound is obtained in [13] and the upper bound is obtained in [7]. In the border case $m = p+2$ the upper bounds in (1.5) are significantly improved in [18].

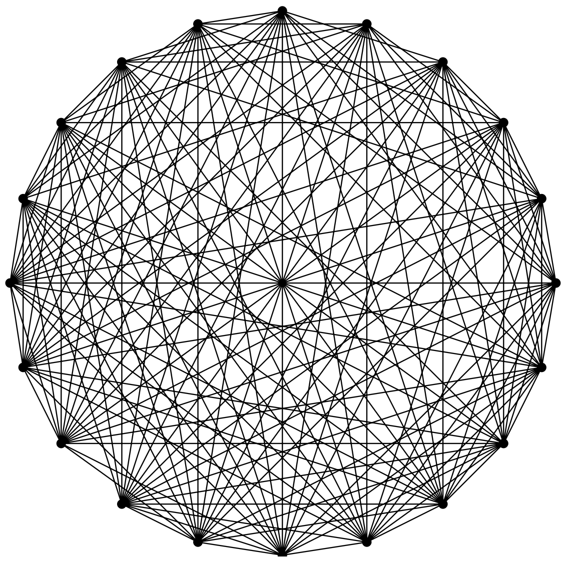

Figure 1: 20-vertex graph in $\mathcal{H}(2, 2, 7; 8)$

We know all the numbers $F_v(a_1, \ldots, a_s; m-1)$ when $\max\{a_1, \ldots, a_s\} \leq 6$, see [4] for details. When $\max\{a_1, \ldots, a_s\} = 7$ it is known that $F_v(2, 2, 7; 8) = 20$ and

$$m + 10 \leq F_v(a_1, \ldots, a_s; m-1) \leq m + 12.$$

The lower bound $F_v(2,2,7;8) \geq 20$ is obtained with the help of Algorithm 3.5, and the upper bound is obtained by constructing 20-vertex graphs in $\mathcal{H}(2,2,7;8)$. An example for such a graph is given in Figure 1.

In this paper we present an algorithm (Algorithm 3.9), with the help of which we can obtain lower bounds on the numbers $F_v(a_1,\ldots,a_s;m-1)$. Using Algorithm 3.9 and $F_v(2,2,7;8) = 20$, we improve the lower bound on the numbers $F_v(a_1,\ldots,a_s;m-1)$ when $\max\{a_1,\ldots,a_s\} = 7$ by proving the following:

Main Theorem. *Assume that $a_1,\ldots,a_s$ are positive integers such that $\max\{a_1,\ldots,a_s\} = 7$ and $m = \sum\limits_{i=1}^{s}(a_i-1)+1 \geq 9$. Then*

$$F_v(a_1,\ldots,a_s;m-1) \geq m+11.$$

Remark 1.2. As is seen from (1.4), the condition $m \geq 9$ in the Main Theorem is necessary.

2. BOUNDS ON THE NUMBERS $F_v(a_1,\ldots,a_s;q)$

Let m and p be positive integers. Denote by $\mathcal{S}(m,p)$ the set of all $(b_1,\ldots,b_r)$ (r is not fixed), where b_i are positive integers such that $\max\{b_1,\ldots,b_r\} = p$ and $\sum_{i=1}^{r}(b_i-1)+1 = m$. Let $(a_1,\ldots,a_s) \in \mathcal{S}(m,p)$. Then obviously

$$\min_{(b_1,\ldots,b_r)\in\mathcal{S}(m,p)} F_v(b_1,\ldots,b_r;q) \leq F_v(a_1,\ldots,a_s;q) \leq \max_{(b_1,\ldots,b_r)\in\mathcal{S}(m,p)} F_v(b_1,\ldots,b_r;q).$$

Note that $(2_{m-p},p) \in \mathcal{S}(m,p)$, $p \geq 2$ and it is easy to prove that

$$\min_{(b_1,\ldots,b_r)\in\mathcal{S}(m,p)} F_v(b_1,\ldots,b_r;q) = F_v(2_{m-p},p;q) \quad \text{(see [1])}.$$

We see that the lower bounding of the vertex Folkman numbers can be achieved by computing or lower bounding the numbers $F_v(2_{m-p},p;q)$. In general, this is a hard problem. However, in the case $q = m-1$, in [1] we presented a method for the computation of these numbers, which is based on the following:

Theorem 2.1 ([1]). *Let $r_0 = r_0(p)$ be the smallest positive integer for which*

$$\min_{r\geq 2}\{F_v(2_r,p;r+p-1)-r\} = F_v(2_{r_0},p;r_0+p-1)-r_0.$$

Then:

(a) $F_v(2_r,p;r+p-1) = F(2_{r_0},p;r_0+p-1)+r-r_0,\ r \geq r_0$.

(b) If $r_0 = 2$, then $F_v(2_r,p;r+p-1) = F_v(2,2,p;p+1)+r-2,\ r \geq 2$.

(c) If $r_0 > 2$ and G is an extremal graph in $\mathcal{H}(2_{r_0}, p; r_0 + p - 1)$, then $G \overset{v}{\to} (2, r_0 + p - 2)$.

(d) $r_0 < F_v(2, 2, p; p + 1) - 2p$.

From this theorem it becomes clear that, for a fixed p, the computation of the members of the infinite sequence $F_v(2_{m-p}, p; m - 1)$, $m \geq p + 2$, is reduced to the computation of its first r_0 members, where $r_0 < F_v(2, 2, p; p + 1) - 2p$. We conjecture that it is enough to know only its first member $F_v(2, 2, p; p + 1)$.

Conjecture 2.2 ([1]). *If $p \geq 4$, then*

$$\min_{r \geq 2} \{F_v(2_r, p; r + p - 1) - r\} = F_v(2, 2, p; p + 1) - 2,$$

i.e., $r_0(p) = 2$, and therefore

$$F_v(2_r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2.$$

This conjecture is proved for $p = 4, 5$ and 6 in [16], [1] and [4], respectively. In [4] it is also proved that the conjecture is true when $F_v(2, 2, p; p + 1) \leq 2p + 5$. In this paper we will prove that Conjecture 2.2 is also true when $p = 7$:

Theorem 2.3. $F_v(2_{m-7}, 7; m - 1) = m + 11$.

The Main Theorem follows easily from Theorem 2.3.

Remark 2.4. This method is not suitable for obtaining upper bounds for the vertex Folkman numbers, as it is not clear how $\max_{(b_1,\dots,b_r) \in \mathcal{S}(m,p)} F_v(b_1, \dots, b_r; q)$ can be computed or bounded. In [2] we present another method for upper bounding of the vertex Folkman numbers (see also [1] and [4]).

3. ALGORITHMS

Finding all graphs in $\mathcal{H}(a_1, \dots, a_s; q; n)$ using a brute force approach is practically impossible for $n > 13$. In this section we present algorithms for obtaining these graphs.

We say that G is a maximal graph in $\mathcal{H}(a_1, \dots, a_s; q)$ if $G \in \mathcal{H}(a_1, \dots, a_s; q)$ but $G + e \notin \mathcal{H}(a_1, \dots, a_s; q), \forall e \in \mathrm{E}(\overline{G})$, i.e. $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$. The graphs in $\mathcal{H}(a_1, \dots, a_s; q)$ can be obtained by removing edges from the maximal graphs in this set.

For convenience, we also define the following term:

Definition 3.1. The graph G is called a $(+K_t)$-graph if $G + e$ contains a new t-clique for all $e \in \mathrm{E}(\overline{G})$.

Obviously, $G \in \mathcal{H}(a_1, \ldots, a_s; q)$ is a maximal graph in $\mathcal{H}(a_1, \ldots, a_s; q)$ if and only if G is a $(+K_q)$-graph. We shall denote by $\mathcal{H}_{+K_t}(a_1, \ldots, a_s; q)$ the set of all $(+K_t)$-graphs in $\mathcal{H}(a_1, \ldots, a_s; q)$, and by $\mathcal{H}_{max}(a_1, \ldots, a_s; q)$ all maximal K_q-free graphs in this set. The sets $\mathcal{H}_{max}(a_1, \ldots, a_s; q; n)$ and $\mathcal{H}_{+K_t}(a_1, \ldots, a_s; q; n)$ are defined in the same way as $\mathcal{H}(a_1, \ldots, a_s; q; n)$.

We shall denote by $\mathcal{H}^t_{max}(a_1, \ldots, a_s; q; n)$ and $\mathcal{H}^t_{+K_t}(a_1, \ldots, a_s; q; n)$ the subsets of all graphs with independence number not greater than t in the sets $\mathcal{H}_{max}(a_1, \ldots, a_s; q; n)$ and $\mathcal{H}_{+K_t}(a_1, \ldots, a_s; q; n)$, respectively.

Remark 3.2. In the special case $s = 1$ we have
$\mathcal{H}(a_1; q; n) = \{G : a_1 \leq \omega(G) < q \text{ and } |\,\mathrm{V}(G)| = n\}$.
Obviously, if $a_1 \leq n \leq q - 1$, then $\mathcal{H}_{max}(a_1; q; n) = \{K_n\}$.
If $a_1 \leq q - 1 \leq n$, then $\mathcal{H}_{max}(a_1; q; n) = \mathcal{H}_{max}(q - 1; q; n)$.

Further, we shall use the following propositions, which are easy to prove:

Proposition 3.3 ([4])**.** *Let G be a graph, $G \overset{v}{\to} (a_1, ..., a_s)$ and $a_i \geq 2$. Then for every independent set A in G*

$$G - A \overset{v}{\to} (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_s).$$

Proposition 3.4 ([4])**.** *Let $G \in \mathcal{H}_{max}(a_1, \ldots, a_s; q; n)$ and A be an independent set of vertices of G. Then $G - A \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, \ldots, a_s; q; n - |A|)$.*

The following algorithm for finding all graphs $G \in \mathcal{H}_{max}(a_1, \ldots, a_s; q; n)$ with $r \leq \alpha(G) \leq t$ is given in [4]:

Algorithm 3.5 ([4])**.** The set $\mathcal{A} = \mathcal{H}^t_{max}(a_1 - 1, \ldots, a_s; q; n - r)$ is the input of the algorithm. The output of the algorithm is the set $\mathcal{B}$ of all graphs $G \in \mathcal{H}^t_{max}(a_1, \ldots, a_s; q; n)$ with $\alpha(G) \geq r$.

1. By removing edges from the graphs in $\mathcal{A}$ obtain the set
$\mathcal{A}' = \mathcal{H}^t_{+K_{q-1}}(a_1 - 1, \ldots, a_s; q; n - r)$.

2. For each graph $H \in \mathcal{A}'$:

2.1. Find the family $\mathcal{M}(H) = \{M_1, \ldots, M_l\}$ of all maximal K_{q-1}-free subsets of $\mathrm{V}(H)$.

2.2. Find all r-element multisets $N = \{M_{i_1}, M_{i_2}, \ldots, M_{i_r}\}$ of elements of $\mathcal{M}(H)$, which fulfill the conditions:

(a) $K_{q-2} \subseteq M_{i_j} \cap M_{i_k}$ for every $M_{i_j}, M_{i_k} \in N$.

(b) $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq t - |N'|$ for every subset N' of N.

2.3. For each r-element multiset $N = \{M_{i_1}, M_{i_2}, \ldots, M_{i_r}\}$ of elements of $\mathcal{M}(H)$ found in step 2.2 construct the graph $G = G(N)$ by adding new independent vertices $v_1, v_2, \ldots, v_r$ to $\mathrm{V}(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \ldots, r$. If $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$, then add G to $\mathcal{B}$.

3. Remove the isomorphic copies of graphs from $\mathcal{B}$.

4. Remove from the obtained in step 3 set $\mathcal{B}$ all graphs G for which $G \overset{v}{\not\to} (a_1, \dots, a_s)$.

Theorem 3.6 ([4])**.** *After the execution of Algorithm 3.5, the obtained set $\mathcal{B}$ coincides with the set of all graphs $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ with $\alpha(G) \geq r$.*

Algorithm 3.5 is based on a very similar algorithm that we used in [3] to prove the lower bound $F_e(3, 3; 4) > 19$. It is possible to prove the Main Theorem using Algorithm 3.5, but it would take months of computational time. For this reason, we will present an algorithm which is a modification of Algorithm 3.5 and helped us prove the Main Theorem in less than a week work of a personal computer.

Further we shall use the following term:

Definition 3.7. We say that v is a cone vertex in the graph G if v is adjacent to all other vertices in G.

Suppose that $G \in \mathcal{H}_{max}(a_1, \dots, a_s; q; n)$ and G has a cone vertex, i.e. $G = K_1 + H$. According to Proposition 3.3, $H \in \mathcal{H}_{max}(a_1 - 1, \dots, a_s; q - 1; n - 1)$. Therefore, if we know all the graphs in $\mathcal{H}_{max}(a_1 - 1, \dots, a_s; q - 1; n - 1)$, we can easily obtain the graphs in $\mathcal{H}_{max}(a_1, \dots, a_s; q; n)$, which have a cone vertex. We will use this fact to modify Algorithm 3.5 and make it faster in the case where all graphs in $\mathcal{H}_{max}(a_1 - 1, \dots, a_s; q - 1; n - 1)$ are already known. The new modified algorithm is based on the following:

Proposition 3.8. *Let $G \in \mathcal{H}_{max}(a_1, \dots, a_s; q; n)$ be a graph without cone vertices and A be an independent set in G such that $G - A$ has a cone vertex, i.e. $G - A = K_1 + H$. Then $G = \overline{K}_{r+1} + H$, where $r = |A|$, H has no cone vertices and $K_1 + H \in \mathcal{H}_{max}(a_1, \dots, a_s; q; n - r)$.*

Proof. Let $A = \{v_1, \dots, v_r\}$ be an independent set in G and $G - A = K_1 + H = \{u\} + H$. Since G has no cone vertices, there exist $v_i \in A$ such that v_i is not adjacent to u. Then $N_G(v_i) \subseteq N_G(u)$ and since G is a maximal K_q-free graph, we obtain $N_G(v_i) = N_G(u) = \mathrm{V}(H)$. Hence, u is not adjacent to any of the vertices in A, and therefore $N_G(v_j) = N_G(u) = \mathrm{V}(H), \forall v_j \in A$. We derived $G = \overline{K}_{r+1} + H$. The graph H has no cone vertices, since any cone vertex in H would be a cone vertex in G. It is easy to see that if $\overline{K}_{r+1} + H \overset{v}{\to} (a_1, \dots, a_s)$, then $K_1 + H \overset{v}{\to} (a_1, \dots, a_s)$. Therefore $K_1 + H \in \mathcal{H}_{max}(a_1, \dots, a_s; q; n - r)$. □

Now we present the main algorithm used in this paper, which is a modification of Algorithm 3.5.

Algorithm 3.9. The input of the algorithm are the set $\mathcal{A}_1 = \mathcal{H}^t_{max}(a_1 - 1, \dots, a_s; q; n - r)$ and the set $\mathcal{A}_2 = \mathcal{H}^t_{max}(a_1 - 1, \dots, a_s; q - 1; n - 1)$. The output of the algorithm is the set $\mathcal{B}$ of all graphs $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ with $\alpha(G) \geq r$.

1. By removing edges from the graphs in $\mathcal{A}_1$ obtain the set

$$\mathcal{A}'_1 = \left\{ H \in \mathcal{H}^t_{+K_{q-1}}(a_1 - 1, \dots, a_s; q; n - r) : \ H \text{ has no cone vertices} \right\}.$$

2. Repeat step 2 of Algorithm 3.5.

3. Repeat step 3 of Algorithm 3.5.

4. Repeat step 4 of Algorithm 3.5.

5. If $t > r$, find the subset $\mathcal{A}_1''$ of $\mathcal{A}_1$ containing all graphs with exactly one cone vertex. For each graph $H \in \mathcal{A}_1''$, if $K_1 + H \overset{v}{\rightarrow} (a_1, ..., a_s)$, then add $\overline{K}_{r+1} + H$ to $\mathcal{B}$.

6. For each graph H in $\mathcal{A}_2$ such that $\alpha(H) \geq r$, if $K_1 + H \overset{v}{\rightarrow} (a_1, \dots, a_s)$, then add $K_1 + H$ to $\mathcal{B}$.

Theorem 3.10. *After the execution of Algorithm 3.9, the obtained set $\mathcal{B}$ coincides with the set of all graphs $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ with $\alpha(G) \geq r$.*

Proof. Suppose that after the execution of Algorithm 3.9, $G \in \mathcal{B}$. If after step 4 $G \in \mathcal{B}$, then according to Theorem 3.6, $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ and $\alpha(G) \geq r$. If G is added to $\mathcal{B}$ in step 5 or step 6, then clearly $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ and $\alpha(G) \geq r$.

Now let $G \in \mathcal{H}^t_{max}(a_1, \dots, a_s; q; n)$ and $\alpha(G) \geq r$. If $G = K_1 + H$ for some graph H, then, according to Proposition 3.3, $H \in \mathcal{A}_2$ and in step 6 G is added to $\mathcal{B}$. Suppose that G has no cone vertices and G has an independent set A such that $|A| = r$ and $G - A$ has a cone vertex, i.e. $G - A = K_1 + H$. Then, according to Proposition 3.8, $G = \overline{K}_{r+1} + H$, $K_1 + H$ has exactly one cone vertex and $K_1 + H \overset{v}{\rightarrow} (a_1, \dots, a_s)$. It is clear that $t > r$ and hence in step 5 G is added to $\mathcal{B}$. Finally, if $G - A$ has no cone vertices, then according to Proposition 3.4, $G - A \in \mathcal{A}_1'$ and it follows from Theorem 3.6 that after the execution of step 4, $G \in \mathcal{B}$. ☐

Remark 3.11. Note that if $n \geq q$ and $r = 2$, then Algorithms 3.5 and 3.9 obtain all graphs in $G \in \mathcal{H}^t_{max}(a_1, ..., a_s; q; n)$.

The *nauty* programs [10] have an important role in this paper. We use them for fast generation of non-isomorphic graphs and isomorphic rejection.

4. PROOF OF THE MAIN THEOREM AND THEOREM 2.3

We will first prove Theorem 2.3 by proving Conjecture 2.2 in the case $p = 7$. Since $F_v(2, 2, 7; 8) = 20$ [4], in view of Theorem 2.1(d), to prove the conjecture in this case we need to prove the inequalities $F_v(2, 2, 2, 7; 9) > 20$, $F_v(2, 2, 2, 2, 7; 10) > 21$ and $F_v(2, 2, 2, 2, 2, 7; 11) > 22$. It is easy to see that it is enough to prove only the last of the three inequalities (see [4] for details). Using Algorithm 3.5 it can be proved that $F_v(2, 2, 2, 2, 2, 7; 11) > 22$, but it would require a lot of computational time. Instead, we will prove the three inequalities successively using Algorithm 3.9. Only the proof of the first inequality is presented in details, since the proofs of the others are very similar. We will show that $\mathcal{H}(2, 2, 7; 8; 19) = \emptyset$. The proof uses the graphs $\mathcal{H}^3_{max}(4; 8; 8)$, $\mathcal{H}^3_{max}(5; 8; 10)$, $\mathcal{H}^3_{max}(6; 8; 12)$, $\mathcal{H}^3_{max}(7; 8; 14)$,

$\mathcal{H}^3_{max}(2,7;8;16)$, $\mathcal{H}^3_{max}(2,2,7;8;19)$, $\mathcal{H}^2_{max}(4;8;9)$, $\mathcal{H}^2_{max}(5;8;11)$, $\mathcal{H}^2_{max}(6;8;13)$, $\mathcal{H}^2_{max}(7;8;15)$, $\mathcal{H}^2_{max}(2,7;8;17)$, $\mathcal{H}^2_{max}(2,2,7;8;19)$ obtained in [4] in the proof of the lower bound $F_v(2,2,7;8) \geq 20$ (see Table 1).

For positive integers $a_1,\dots,a_s$ and m and p defined by (1.3), Nenov proved in [15] that if $G \in \mathcal{H}(a_1,\dots,a_s;m-1;n)$ and $n < m+3p$, then $\alpha(G) < n-m-p+1$. Suppose that $G \in \mathcal{H}(2,2,2,7;9;20)$. It follows that $\alpha(G) \leq 3$ and it is clear that $\alpha(G) \geq 2$. Therefore, it is enough to prove that there are no graphs with independence number 2 or 3 in $\mathcal{H}_{max}(2,2,2,7;9;20)$.

First we prove that there are no graphs in $\mathcal{H}_{max}(2,2,2,7;9;20)$ with independence number 3. It is clear that K_7 is the only graph in $\mathcal{H}_{max}(4;9;7)$. By applying Algorithm 3.9($r=2;t=3$) with $\mathcal{A}_1 = \mathcal{H}^3_{max}(4;9;7) = \{K_7\}$ and $\mathcal{A}_2 = \mathcal{H}^3_{max}(4;8;8)$ were obtained all graphs in $\mathcal{H}^3_{max}(5;9;9)$ (see Remark 3.11). In the same way, we successively obtained all graphs in $\mathcal{H}^3_{max}(6;9;11)$, $\mathcal{H}^3_{max}(7;9;13)$, $\mathcal{H}^3_{max}(2,7;9;15)$ and $\mathcal{H}^3_{max}(2,2,7;9;17)$ (see Remark 3.11). In the end, by applying Algorithm 3.9($r=3;t=3$) with $\mathcal{A}_1 = \mathcal{H}^3_{max}(2,2,7;9;17)$ and $\mathcal{A}_2 = \mathcal{H}^3_{max}(2,2,7;8;19) = \emptyset$, no graphs with independence number 3 in $\mathcal{H}_{max}(2,2,2,7;9;20)$ were obtained.

Next we prove that there are no graphs in $\mathcal{H}_{max}(2,2,2,7;9;20)$ with independence number 2. Clearly, K_8 is the only graph in $\mathcal{H}_{max}(4;9;8)$. By applying Algorithm 3.9($r=2;t=2$) with $\mathcal{A}_1 = \mathcal{H}^2_{max}(4;9;8) = \{K_8\}$ and $\mathcal{A}_2 = \mathcal{H}^2_{max}(4;8;9)$ were obtained all graphs in $\mathcal{H}^2_{max}(5;9;10)$ (see Remark 3.11). In the same way, we successively obtained all graphs in $\mathcal{H}^2_{max}(6;9;12)$, $\mathcal{H}^2_{max}(7;9;14)$, $\mathcal{H}^2_{max}(2,7;9;16)$ and $\mathcal{H}^2_{max}(2,2,7;9;18)$ (see Remark 3.11). In the end, by applying Algorithm 3.9($r=2;t=2$) with $\mathcal{A}_1 = \mathcal{H}^2_{max}(2,2,7;9;18)$ and $\mathcal{A}_2 = \mathcal{H}^2_{max}(2,2,7;8;19) = \emptyset$, no graphs with independence number 2 in $\mathcal{H}_{max}(2,2,2,7;9;20)$ were obtained.

We proved that $\mathcal{H}_{max}(2,2,2,7;9;20) = \emptyset$ and $F_v(2,2,2,7;9) > 20$.

Similarly, the graphs obtained in the proof of the inequality $F_v(2,2,2,7;9) > 20$ are used to prove $F_v(2,2,2,2,7;10) > 21$ and the graphs obtained in the proof of the inequality $F_v(2,2,2,2,7;10) > 21$ are used to prove $F_v(2,2,2,2,2,7;11) > 22$. The number of graphs obtained in each step of the proofs is shown in Table 2, Table 3 and Table 4. Notice that the number of graphs without cone vertices is relatively small, which reduces the computation time significantly.

Thus, $r_0(7) = 2$ and

$$F_v(2_{m-7};7;m-1) = F_v(2,2,7;8) + m - 9 = m + 11,$$

which completes the proof of Theorem 2.3. The Main Theorem now follows easily. Indeed, let $a_1,\dots,a_s$ be positive integers such that $\max\{a_1,\dots,a_s\} = 7$ and $m = \sum_{i=1}^{s}(a_i - 1) + 1$. Then

$$F_v(a_1,\dots,a_s;m-1) \geq F_v(2_{m-7};7;m-1) = m + 11. \qquad \square$$

5. CONCLUDING REMARKS

The proposed method for obtaining of lower bounds for $F_v(a_1,\dots,a_s;q)$ produces good and accurate results when $q = m-1$. However, when $q < m-1$, the bounds are not exact. We will consider the most interesting case $q = p+1$, where $p = \max\{a_1,\dots,a_s\}$. In [1] we prove the inequality

$$F_v(a_1,\dots,a_s;p+1) \geq F_v(2,2,p;p+1) + \sum_{i=3}^{m-p} \alpha(i,p), \tag{5.1}$$

where $\alpha(i,p) = \max\{\alpha(G) : G \in \mathcal{H}_{extr}(2_i,p;p+1)\}$. Since $\alpha(i,p) \geq 2$, from (5.1) it follows that

$$F_v(a_1,...,a_s;p+1) \geq F_v(2,2,p;p+1) + 2(m-p-2).$$

In the special case $p = 7$, since $F_v(2,2,7;8) = 20$, we obtain

$$F_v(a_1,\dots,a_s;8) \geq 2m+2. \tag{5.2}$$

In particular, when $m = 13$ we have $F_v(a_1,\dots,a_s;8) \geq 28$. Since the Ramsey number $R(3,8) = 28$, it follows that $\alpha(i,7) \geq 3$, when $i \geq 6$. Now from (5.1) we obtain easily the following result:

Theorem 5.1. *If* $m \geq 13$, *and* $\max\{a_1,...,a_s\} = 7$, *then*

$$F_v(a_1,\dots,a_s;8) \geq 3m-10.$$

It is clear that when $3m-10 \geq R(4,8)$, these bounds for $F_v(a_1,\dots,a_s;8)$ can be improved significantly.

In [21] is proved the inequality $F_v(p,p;p+1) \geq 4p-1$. From this result it follows that $F_v(7,7;8) \geq 27$. From (5.2) we deduce that $F_v(7,7;8) \geq 28$, and from Theorem 5.1 we obtain $F_v(7,7;8) \geq 29$.

The numbers $F_v(p,p;p+1)$ are of significant interest, but so far we know very little about them. Only two of these numbers are known, $F_v(2,2;3) = 5$ (obvious), and $F_v(3,3;4) = 14$ ([11, 17]). It is also known that $17 \leq F_v(4,4;5) \leq 23$, [20], $F_v(5,5;6) \geq 23$, [1], $28 \leq F_v(6,6;7) \leq 70$, [4], and $F_v(7,7;8) \geq 29$ from this paper. Using Algorithm 3.5, we managed to improve the known lower bound $F_v(2,2,2,4;5) \geq 17$ and thus improved the lower bound on $F_v(4,4;5)$ as well:

Theorem 5.2. $F_v(4,4;5) \geq F_v(2,3,4;5) \geq F_v(2,2,2,4;5) \geq 19$.

Proof. The inequalities $F_v(4,4;5) \geq F_v(2,3,4;5) \geq F_v(2,2,2,4;5)$ are easy to prove (see eq. (4.1) in [1]). It remains to prove that $F_v(2,2,2,4;5) \geq 19$. Suppose that $\mathcal{H}_{max}(2,2,2,4;5;18) \neq \emptyset$ and let $G \in \mathcal{H}_{max}(2,2,2,4;5;18)$. Since the Ramsey number $R(3,5) = 14$, $\alpha(G) \geq 3$. In [20] it is proved that $F_v(2,2,4;5) = 13$ and

$\mathcal{H}(2,2,4;5;13) = \{Q\}$, where Q is the unique 13-vertex K_5-free graph with independence number 2. From Proposition 3.3 and the equality $F_v(2,2,4;5) = 13$ it follows that $\alpha(G) \leq 5$. By applying Algorithm 3.5 to the graph Q it follows that there are no graphs in $\mathcal{H}_{max}(2,2,2,4;5;18)$ with independence number 5. It remains to prove that there are no graphs in $\mathcal{H}_{max}(2,2,2,4;5;18)$ with independence number 3 or 4. Using *nauty* it is easy to obtain the sets $\mathcal{H}^4_{max}(3;5;8)$ and $\mathcal{H}^3_{max}(3;5;9)$. By applying Algorithm 3.5 ($r = 2, t = 4$) starting from the set $\mathcal{H}^4_{max}(3;5;8)$ were successively obtained all graphs in the sets $\mathcal{H}^4_{max}(4;5;10)$, $\mathcal{H}^4_{max}(2,4;5;12)$, $\mathcal{H}^4_{max}(2,2,4;5;14)$ (see Remark 3.11), and by applying Algorithm 3.5 ($r = 4, t = 4$) were found no graphs in $\mathcal{H}_{max}(2,2,2,4;5;18)$ with independence number 4. Next, we applied Algorithm 3.5 ($r = 2, t = 3$) starting from the set $\mathcal{H}^3_{max}(3;5;9)$ to successively obtain all graphs in the sets $\mathcal{H}^3_{max}(4;5;11)$, $\mathcal{H}^3_{max}(2,4;5;13)$, $\mathcal{H}^3_{max}(2,2,4;5;15)$ (see Remark 3.11), and by applying Algorithm 3.5 ($r = 3, t = 3$) were found no graphs in $\mathcal{H}_{max}(2,2,2,4;5;18)$ with independence number 3. The number of graphs obtained in each of the steps is shown in Table 5. We obtained $\mathcal{H}_{max}(2,2,2,4;5;18) = \emptyset$ and therefore $F_v(2,2,2,4;5) \geq 19$. □

The upper bound $F_v(4,4;5) \leq 23$ is proved in [20] with the help of a 23-vertex transitive graph. We were not able to obtain any other graphs in $\mathcal{H}(4,4;5;23)$, which leads us to believe that this bound may be exact. We did find a large number of 23-vertex graphs in $\mathcal{H}(2,2,2,4;5)$, but so far we have not obtained smaller graphs in this set.

Concluding this section, let us pose the following question:

Question 5.1. Is it true that the sequence $F_v(p,p;p+1), p \geq 2$, is increasing?

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A. RESULTS OF COMPUTATIONS

set	ind. number	maximal graphs	$(+K_7)$-graphs
$\mathcal{H}(2,7;8;15)$	≤ 4	1	1
$\mathcal{H}(2,2,7;8;19)$	$= 4$	0	
$\mathcal{H}(3;8;6)$	≤ 3	1	1
$\mathcal{H}(4;8;8)$	≤ 3	1	4
$\mathcal{H}(5;8;10)$	≤ 3	3	45
$\mathcal{H}(6;8;12)$	≤ 3	12	3 104
$\mathcal{H}(7;8;14)$	≤ 3	169	4 776 518
$\mathcal{H}(2,7;8;16)$	≤ 3	34	22 896
$\mathcal{H}(2,2,7;8;19)$	$= 3$	0	
$\mathcal{H}(3;8;7)$	≤ 2	1	1
$\mathcal{H}(4;8;9)$	≤ 2	1	8
$\mathcal{H}(5;8;11)$	≤ 2	3	84
$\mathcal{H}(6;8;13)$	≤ 2	10	5 394
$\mathcal{H}(7;8;15)$	≤ 2	102	4 984 994
$\mathcal{H}(2,7;8;17)$	≤ 2	2760	380 361 736
$\mathcal{H}(2,2,7;8;19)$	$= 2$	0	
$\mathcal{H}(2,2,7;8;19)$		0	

Table 1: Steps in finding all maximal graphs in $\mathcal{H}(2,2,7;8;19)$

set	ind. number	max. graphs	max. graphs no cone v.	$(+K_8)$-graphs	$(+K_8)$-graphs no cone v.
$\mathcal{H}(2,2,7;9;16)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,7;9;20)$	$= 4$	0	0		
$\mathcal{H}(4;9;7)$	≤ 3	1	0	1	0
$\mathcal{H}(5;9;9)$	≤ 3	1	0	4	0
$\mathcal{H}(6;9;11)$	≤ 3	3	0	45	0
$\mathcal{H}(7;9;13)$	≤ 3	12	0	3 113	9
$\mathcal{H}(2,7;9;15)$	≤ 3	169	0	4 783 615	7 097
$\mathcal{H}(2,2,7;9;17)$	≤ 3	36	2	22 918	22
$\mathcal{H}(2,2,2,7;9;20)$	$= 3$	0	0		
$\mathcal{H}(4;9;8)$	≤ 2	1	0	1	0
$\mathcal{H}(5;9;10)$	≤ 2	1	0	8	0
$\mathcal{H}(6;9;12)$	≤ 2	3	0	85	1
$\mathcal{H}(7;9;14)$	≤ 2	10	0	5 474	80
$\mathcal{H}(2,7;9;16)$	≤ 2	103	1	5 346 982	361 988
$\mathcal{H}(2,2,7;9;18)$	≤ 2	2845	85	387 948 338	7 586 602
$\mathcal{H}(2,2,2,7;9;20)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,7;9;20)$		0	0		

Table 2: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,7;9;20)$

set	ind. number	max. graphs	max. graphs no cone v.	$(+K_9)$-graphs	$(+K_9)$-graphs no cone v.
$\mathcal{H}(2,2,2,7;10;17)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 4$	0	0		
$\mathcal{H}(5;10;8)$	≤ 3	1	0	1	0
$\mathcal{H}(6;10;10)$	≤ 3	1	0	4	0
$\mathcal{H}(7;10;12)$	≤ 3	3	0	45	0
$\mathcal{H}(2,7;10;14)$	≤ 3	12	0	3 115	2
$\mathcal{H}(2,2,7;10;16)$	≤ 3	169	0	4 784 483	868
$\mathcal{H}(2,2,2,7;10;18)$	≤ 3	36	0	22 919	1
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 3$	0	0		
$\mathcal{H}(5;10;9)$	≤ 2	1	0	1	0
$\mathcal{H}(6;10;11)$	≤ 2	1	0	8	0
$\mathcal{H}(7;10;13)$	≤ 2	3	0	85	0
$\mathcal{H}(2,7;10;15)$	≤ 2	10	0	5 495	21
$\mathcal{H}(2,2,7;10;17)$	≤ 2	103	0	5 371 651	24 669
$\mathcal{H}(2,2,2,7;10;19)$	≤ 2	2848	3	387 968 658	20 320
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,2,7;10;21)$		0	0		

Table 3: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,2,7;10;21)$

set	ind. number	max. graphs	max. graphs no cone v.	$(+K_{10})$-graphs	$(+K_{10})$-graphs no cone v.
$\mathcal{H}(2,2,2,2,7;11;18)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 4$	0	0		
$\mathcal{H}(6;11;9)$	≤ 3	1	0	1	0
$\mathcal{H}(7;11;11)$	≤ 3	1	0	4	0
$\mathcal{H}(2,7;11;13)$	≤ 3	3	0	45	0
$\mathcal{H}(2,2,7;11;15)$	≤ 3	12	0	3 116	1
$\mathcal{H}(2,2,2,7;11;17)$	≤ 3	169	0	4 784 638	155
$\mathcal{H}(2,2,2,2,7;11;19)$	≤ 3	36	0	22 919	0
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 3$	0	0		
$\mathcal{H}(6;11;10)$	≤ 2	1	0	1	0
$\mathcal{H}(7;11;12)$	≤ 2	1	0	8	0
$\mathcal{H}(2,7;11;14)$	≤ 2	3	0	85	0
$\mathcal{H}(2,2,7;11;16)$	≤ 2	10	0	5 502	7
$\mathcal{H}(2,2,2,7;11;18)$	≤ 2	103	0	5 374 143	2 492
$\mathcal{H}(2,2,2,2,7;11;20)$	≤ 2	2848	0	387 968 676	18
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,2,2,7;11;22)$		0	0		

Table 4: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,2,2,7;11;22)$

set	ind. number	maximal graphs	$(+K_4)$-graphs
$\mathcal{H}(2,2,4;5;13)$	≤ 5	1	1
$\mathcal{H}(2,2,2,4;5;18)$	$= 5$	0	
$\mathcal{H}(3;5;8)$	≤ 4	7	274
$\mathcal{H}(4;5;10)$	≤ 4	44	65 422
$\mathcal{H}(2,4;5;12)$	≤ 4	1 059	18 143 174
$\mathcal{H}(2,2,4;5;14)$	≤ 4	13	71
$\mathcal{H}(2,2,2,4;5;18)$	$= 4$	0	
$\mathcal{H}(3;5;9)$	≤ 3	11	2 252
$\mathcal{H}(4;5;11)$	≤ 3	135	1 678 802
$\mathcal{H}(2,4;5;13)$	≤ 3	11 439	2 672 047 607
$\mathcal{H}(2,2,4;5;15)$	≤ 3	1 103	78 117
$\mathcal{H}(2,2,2,4;5;18)$	$= 3$	0	
$\mathcal{H}(2,2,2,4;5;18)$		0	

Table 5: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,4;5;18)$

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