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A FIRST-ORDER IN THICKNESS MODEL FOR FLEXURAL DEFORMATIONS OF GEOMETRICALLY NON-LINEAR SHELLS

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The shallow shells, characterized by deflections of the order of unity, small deformations and still smaller curvatures, have most thoroughly been studied in the literature. However, the momentum terms, due to which the shell differs essentially from a membrane, are not negligible only for the short-wave-length deformations, when the deflections are small, the deformations — of the order of unity and the curvatures — of the order of the inverse of the small parameter. In order to treat consistently the case of *momentum supporting shells*, the formulas for covariant differentiation in the shell space are revisited. It is shown that the geometrical non-linearity contributes terms of the same order of magnitude as the momentum stresses. For the flexural deformations an equation of Boussinesq type is derived containing fourth-order dispersion and cubic non-linearity.

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1. INTRODUCTION

Since the turning of the century and especially in the late forties the theory of thin shells attracted much attention and many papers were devoted to its mechanical and mathematical aspects. Yet, it is far from completion. It goes beyond the framework of the present paper to give the historical account and the perspective of the numerous shell theories. We generally accept the attitudes of the comprehensive review [9] and the monographs [6, 8, 10] in assessing the vast body of the existing literature.

The theoretical approaches for modelling shells fall generally into two main groups. *To the first belong the theories in which the governing equations are* derived as averaged properties of a very thin curved 2D elastic layer in the 3D space. The second approach originates in [14, 5] and consists in direct application of the mechanical laws to the 2D continuum representing the middle surface of the shell. The Cosserat concept was applied in [7]. For the problems arising in the asymptotic analysis of thin shells we refer the reader to the works of P. Ciarlet, E. Sanchez-Palencia and co-workers (see the recent works [4, 11] and the literature cited there).

When deriving the shell equations from the 3D elasticity, the deflections are assumed to be finite while the strains are small. This implies long wave length of the deformations, resulting in even smaller curvatures. This is the so-called "shallow shell" model. Strictly speaking, the shallow-shell approach is not generic for shells but it is rather adequate for membranes, because the momentum stresses that are supposed to make the difference between a shell and a membrane are proportional to the curvature of the deflections. Hence, in a consistent small-strains/smaller-curvatures approach, the moments are to be neglected to the first order of thickness unless the stiffness coefficient is extremely large. However, large values of the stiffness are very unlikely since the stiffness is proportional to bulk Young modulus and the square of the thickness, the latter being very small. Hence, the short length scale of the deformations is the case where the moment stresses are really important.

The difference between shells and membranes becomes really important when the strains are much larger than deflections, and curvatures — much larger than strains. It is clear that such a structure must be geometrically highly non-linear. We derive here a *consistent* first-order approximation in the shell thickness for the said case.

The assumptions of the present work are:

1. The thickness h of the shell is much smaller in comparison with the length scale L of the flexural deformations of the middle surface, i.e. $h \ll L$ or $\varepsilon \equiv h/L \ll 1$. No restrictions on L are imposed, e.g., $L \ll L_D$ is also an admissible case, where L_D is the length scale of the structure itself.

2. The thickness of the shell is constant within the adopted asymptotic order. Hence the derivatives of the thickness scaled by the thickness itself should not be large values, i.e. $\|h^{-1}(\nabla h)\| \approx O(1)$. The latter means that the length scale of changing the thickness is of order of magnitude larger than the length-scale of the deformations.

3. The loads, e.g. the normal pressure and the tractions on the shell faces, are compatible with the above assumptions, i.e. they possess the necessary asymptotic in order to secure 2D strain and stress states.

4. If the deformations created by the boundary conditions at the rim of the shell structure (the contour-line of the middle surface) are not compatible with (1) and (2), then only the portion of the shell is considered, which is far from the rim, i.e. the 3D effects of the said boundary conditions can be neglected.

5. For the sake of simplicity, no tractions are exerted on the shell faces.

It should also be mentioned that when the thickness of a shell is very small, then the contributions from the physical non-linearity of the material are negligible and geometry is the *only* source of non-linearity. For this reason, in the present work we consider only the linear constitutive relations for elastic continuum (the so-called St-Venan–Kirchhoff materials [3]).

2. GEOMETRY OF THE SHELL SPACE

In this section we develop further the derivations of [12] and [6] incorporating the dependence on the transverse co-ordinate in the shell space. As it will turn out, this is essential, because after averaging some of the terms, neglected in the mentioned works, they become commensurable with those that had been left into the considerations.

Consider an N-dimensional Euclidean space and a structure immersed in it, defined as a thin layer of virtually constant thickness h (in the sense of requirement (1)). It is approximately equipartitioned (in the same sense) by the middle hypersurface of dimension $(N-1)$.

Assume that the middle surface is parameterized by the curvilinear co-ordinates ξ^α, $\alpha = 1, \ldots, N-1$. The N-th co-ordinate ξ^N is defined as the normal line to the particular point of the middle surface. As far as the shell does not intersect itself, the so defined set of curvilinear co-ordinates is not ambiguous. In addition, it is orthogonal and, within the adopted asymptotic order, it coincides with the material co-ordinates. When the shell thickness is not constant, then it is convenient to scale the normal co-ordinate by it, in order to transform the mathematical problem into one for which the shell faces are co-ordinate surfaces. Then the co-ordinate system is not strictly orthogonal but only to the order $O(\varepsilon^2)$, which is fully compatible with the attempted here theory of approximation $O(\varepsilon)$. We resort here to the case of equidistant surfaces of the shell and the words "equipartitioned by the middle surface" mean that the middle surface is drawn inside the shell, so that the condition $h_{\rm lo}(\xi^1, \ldots, \xi^{N-1}) = -h_{\rm up}(\xi^1, \ldots, \xi^{N-1})$, and hence $h \equiv h_{\rm up} - h_{\rm lo}$, always holds.

The curvilinear co-ordinates ξ^α, $\alpha = 1, \ldots, N-1$, are in fact material (Lagrangian) co-ordinates. They are connected to the geometrical Cartesian co-ordinates (originated somewhere in the ND-space) through the following functional dependences:

$$x^i = x^i(\xi^1, \ldots, \xi^N; t) \quad \text{for} \quad i = 1, \ldots, N\,, \tag{2.1}$$

where t stands for the time. Here and henceforth the Greek indices range from 1 to $N-1$ and serve to mark the variables in the shell. Italics are used for indices when the space quantities are concerned.

Let us assume for definiteness that the initial state of the shell is physically admissible (see, e.g., [13] for the definition). Then the initial state can be parameterized by the same transformation (2.1) but for the specific value of time $t = t_0$. Without loss of generality we set $t_0 = 0$.

The middle surface is characterized by the first and second fundamental forms

$$g_{\alpha\beta}(\xi^1, \ldots, \xi^N; t) \stackrel{\rm def}{=} \sum_{i=1}^{N} \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^i}{\partial \xi^\beta}\,, \quad b_{\alpha\beta}(\xi^1, \ldots, \xi^N; t) \stackrel{\rm def}{=} -\sum_{i=1}^{N} \frac{\partial n^i}{\partial \xi^\alpha}\frac{\partial n^i}{\partial \xi^\beta}\,.$$

In the last formula n^i denote the Cartesian co-ordinates of the normal to the middle surface vector (say, $\boldsymbol{n}$). The outward normal is defined arbitrarily. When the co-ordinates are the lengths of the arcs, then the second fundamental form adopts the specially simple form $b_{\alpha\beta} = \nabla_\alpha \nabla_\beta \zeta$.

The orts of the curvilinear co-ordinate system are expressed as follows

$$g_\alpha \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\partial x^i}{\partial \xi^\alpha} e_\alpha ,$$

where e_α are the orts of the Cartesian co-ordinate system. In order to avoid confusion, we do not use throughout the present work the convention of summation with respect to "dummy" indices when Cartesian co-ordinates are involved. In such a case we put explicit sign Σ. For the sake of completeness we also add the relation

$$g_N \equiv n ,$$

which is true by the definition of the normal co-ordinate. According to this definition the radius vector r of a point inside the ND-space enclosed in the shell can be expressed as

$$r = \vec{r} + s g_N , \tag{2.2}$$

where $\vec{r}$ is the radius-vector of the normal projection of the said point on the shell middle surface. Here we introduce the notation

$$s \equiv \xi^N h(\xi^1, \ldots, \xi^{N-1}) \tag{2.3}$$

as a measure of the length alongside the normal co-ordinate.

From Eqs. (2.2) and (2.3) one obtains for the fundamental tensor of the space enclosed in the shell (see [12, 6])

$$G_{\alpha\beta} = \left(\frac{\partial r}{\partial \xi^\alpha} + s\frac{\partial n}{\partial \xi^\alpha}\right) \cdot \left(\frac{\partial r}{\partial \xi^\beta} + s\frac{\partial n}{\partial \xi^\beta}\right) = \sum_{i=1}^{N} \left(\frac{\partial r^i}{\partial \xi^\alpha} + s\frac{\partial n^i}{\partial \xi^\alpha}\right)\left(\frac{\partial r^i}{\partial \xi^\beta} + s\frac{\partial n^i}{\partial \xi^\beta}\right)$$

$$\equiv g_{\alpha\beta}(\xi^1, \ldots, \xi^{N-1}) - 2sb_{\alpha\beta}(\xi^1, \ldots, \xi^{N-1}) + s^2 c_{\alpha\beta}(\xi^1, \ldots, \xi^{N-1}) , \tag{2.4}$$

$$G_{NN} = 1, \quad G_{\alpha N} = 0 . \tag{2.5}$$

Here $c_{\alpha\beta} = b_{\alpha\delta} b^\delta_\beta$ is the third fundamental form of the middle surface.

It is clear now that the fundamental tensor of the space filling the shell is defined both by the fundamental tensor of the middle hyper-surface (the first fundamental form) and by the tensor of curvature (the second fundamental form). For further convenience we cite here the formulas for the contravariant components of the fundamental tensor. Since our aim is a first order approximation with respect to thickness, it fully suffices to retain here only the terms up to $O(s^2)$.

Within the adopted order of approximation $o(s^2)$ the contravariant components of the fundamental tensor are given by

$$G^{\alpha\beta} = g^{\alpha\beta}(\xi^1, \ldots, \xi^{N-1}) + 2sb^{\alpha\beta}(\xi^1, \ldots, \xi^{N-1})$$

$$+3s^2 c^{\alpha\beta}(\xi^1, \ldots, \xi^{N-1}) + o(s^2) , \tag{2.6}$$

$$G^{NN} = 1, \quad G^{\alpha N} = 0 . \tag{2.7}$$

The proof of (2.7) is trivial and is a straightforward corollary of the definition of the matrix of contravariant components as an inverse matrix of the matrix of contravariant components. To prove (2.6), we simply multiply it by (2.4) to obtain

$$G_{\alpha\beta}G^{\alpha\gamma} = g_{\alpha\beta}g^{\alpha\gamma}+2s(b_{\alpha\beta}g^{\alpha\gamma}-g_{\alpha\beta}b^{\alpha\gamma})+s^2(c_{\alpha\beta}g^{\alpha\gamma}-4b_{\alpha\beta}b^{\alpha\gamma}+3g_{\alpha\beta}c^{\alpha\gamma})+O(s^3)$$

$$= \delta^\gamma_\beta + 2s(b^\gamma_\beta - b^\gamma_\beta) + s^2(c^\gamma_\beta - 4c^\gamma_\beta + 3c^\gamma_\beta) + O(s^3) = \delta^\gamma_\beta + O(s^3)\,.$$

3. COVARIANT DIFFERENTIATION IN THE SHELL SPACE

This section uses extensively the results of [12] and [6], but it is not possible to omit it because not all of the necessary formulas are presented there. In addition, the terms proportional to s^2, which are essential for our derivations, are absent in the cited works. In order to make the present paper self-contained, on the one hand, and to fulfill the gaps in the cited works, on the other, we compile here the necessary formulas, deriving those that are not present in the literature.

The covariant derivatives of a vector and of a second-rank tensor are given by

$$A^n \Big\|_i = \frac{\partial A^n}{\partial \xi^i} + \Gamma^m_{ik}A^n\,, \quad A^{nm} \Big\|_i = \frac{\partial A^{mn}}{\partial \xi^i} + \Gamma^m_{ik}A^{kn} + \Gamma^n_{ik}A^{mk}\,. \tag{3.1}$$

The covariant Christofell symbol in N dimensions is given by

$$\Gamma_{ij,l} = \frac{1}{2}\left(\frac{\partial G_{jl}}{\partial x^i} + \frac{\partial G_{il}}{\partial x^j} - \frac{\partial G_{ij}}{\partial x^l}\right)\,, \quad \Gamma^k_{ij} = G^{kl}\Gamma_{ij,l}\,.$$

The contravariant symbols are obtained from the covariant ones through the procedure of "elevation" ("contraction") of indices. It is easily shown now that a Christofell symbol is trivially equal to zero if it contains the index N at least in two positions, i.e.

$$\Gamma_{\alpha N,N} = \Gamma_{NN,\alpha} = \Gamma_{NN,N} = 0\,, \quad \Gamma^N_{\alpha N} = \Gamma^\alpha_{NN} = \Gamma^N_{NN} = 0 \quad \text{for} \quad \alpha = 1,\ldots,N-1\,.$$

Let us treat separately also the symbols containing the index N only in one position, namely:

$$\Gamma_{\alpha\beta,N} \equiv -\Gamma_{\beta N,\alpha} = -\frac{1}{2}\frac{\partial G_{\alpha\beta}}{\partial s} = b_{\alpha\beta} - sc_{\alpha\beta}\,.$$

Due to the specific properties of the fundamental tensor, namely, that $G^{Nj} = \delta^{Nj}$, one has

$$\Gamma^N_{\alpha\beta} = G^{Nj}\Gamma_{\alpha\beta,j} = \Gamma_{\alpha\beta,N} = b_{\alpha\beta} - sc_{\alpha\beta}\,.$$

Respectively,

$$\Gamma^\alpha_{\beta N} = \frac{1}{2}G^{\alpha\kappa}\frac{\partial G_{\beta\kappa}}{\partial s} = -(g^{\alpha\kappa} + 2sb^{\alpha\kappa} + 3s^2c^{\alpha\kappa})(b_{\beta\kappa} - sc_{\beta\kappa})$$

$$= -b^\alpha_\beta + sc^\alpha_\beta - s^2c^{\alpha\kappa}b_{\beta\kappa}\,.$$

Note that the last term is obtained after the following fairly obvious manipulation is applied $c^{\alpha\kappa}b_{\beta\kappa} = 3c^{\alpha\kappa}b_{\beta\kappa} - 2b^{\alpha\kappa}c_{\beta\kappa}$.

Finally, for the Christofell symbols which do not contain the index N, one derives

$$\Gamma_{\beta\gamma,\alpha} = [\beta\gamma,\alpha]^g - 2s[\beta\gamma,\alpha]^b + \frac{s^2}{2}[\beta\gamma,\alpha]^c\,, \tag{3.2}$$

where

$$\begin{aligned}
[\beta\gamma,\alpha]^g &\stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial g_{\beta\alpha}}{\partial x^\gamma} + \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right),\\
[\beta\gamma,\alpha]^b &\stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial b_{\beta\alpha}}{\partial x^\gamma} + \frac{\partial b_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial b_{\beta\gamma}}{\partial x^\alpha}\right),\\
[\beta\gamma,\alpha]^c &\stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial c_{\beta\alpha}}{\partial x^\gamma} + \frac{\partial c_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial c_{\beta\gamma}}{\partial x^\alpha}\right)
\end{aligned} \tag{3.3}$$

are the connections generated by the tensors $g_{\alpha\beta}, b_{\alpha\beta}$ and $c_{\alpha\beta}$, respectively. One sees that due to the curvature of the middle surface the connections in the shell space are more complicated making its restriction to the $(N-1)$D-surface non–Riemannian. Note that the first term of the connections, namely ${}^g[\beta\gamma,\alpha]$, is nothing else but the Riemannian connection (ND-Christofell symbol) for the $(N-1)$-dimensional space of the middle surface.

The related contravariant Christofell symbol is expressed as usual

$$\Gamma^\alpha_{\beta\gamma} = G^{\alpha\kappa}\Gamma_{\beta\gamma,\kappa} = \left(g^{\alpha\kappa} + 2sb^{\alpha\kappa} + 3s^2c^{\alpha\kappa}\right)\left([\beta\gamma,\kappa]^g - 2s[\beta\gamma,\kappa]^b + \frac{s^2}{2}[\beta\gamma,\kappa]^c\right).$$

Then

$$\Gamma^\alpha_{\beta\gamma} = \left\{{\alpha \atop \beta\gamma}\right\}^g + 2s\left\{{\alpha \atop \beta\gamma}\right\}^b + s^2\left\{{\alpha \atop \beta\gamma}\right\}^c,$$

$$\left\{{\alpha \atop \beta\gamma}\right\}^g = g^{\alpha\kappa}[\beta\gamma,\kappa]^g\,,\quad \left\{{\alpha \atop \beta\gamma}\right\}^b = b^{\alpha\kappa}[\beta\gamma,\kappa]^g - g^{\alpha\kappa}[\beta\gamma,\kappa]^b,$$

$$\left\{{\alpha \atop \beta\gamma}\right\}^c = g^{\alpha\kappa}[\beta\gamma,\kappa]^c - 4g^{\alpha\kappa}[\beta\gamma,\kappa]^b + 3c^{\alpha\kappa}[\beta\gamma,\kappa]^g.$$

Now we are equipped to derive the expressions for the ND-covariant derivatives $\|_i$ for the space inside the shell. By definition we have

$$A^m \Big\|_i = \frac{\partial A^m}{\partial \xi^i} + \Gamma^m_{in}A^n\,. \tag{3.4}$$

Let us also introduce the notation

$$A^\mu \Big|_\alpha = \frac{\partial A^\mu}{\partial \xi^\alpha} + \left\{{\mu \atop \alpha\nu}\right\}^g A^\nu\,, \tag{3.5}$$

which will be called "restriction of the covariant derivative." For $s=0$ it is nothing else but the covariant derivative in the $(N-1)$D-space of the middle surface of the shell.

Since Eq. (3.5) is valid for the whole space inside the shell, it can only loosely be called "restriction of the covariant derivative". We shall return to this issue later on. For the time being it is enough to be noted that the only variables (3.5) that depend on the normal co-ordinate s are the components of the vector A^μ.

Combining Eqs. (3.5) and (3.4) and using the formulas for the Christofell symbols, one derives the following expressions for the covariant derivative $\|_i$:

$$A^\mu \Big\|_\alpha = A^\mu \Big|_\alpha + \left(2s\left\{ {\mu \atop \nu\alpha} \right\}^b + s^2\left\{ {\mu \atop \nu\alpha} \right\}^c\right) A^\nu - (b^\mu_\alpha - sc^\mu_\alpha + s^2 c^{\mu\kappa} b_{\alpha\kappa})A^N .$$

It is a generalization of the respective formula of Neuber because of the dependence on s of the components of the differentiated vector. Further on we have

$$A^N \Big\|_\alpha = A^N \Big|_\alpha + (b_{\nu\alpha} - sc_{\nu\alpha})A^\nu = \frac{\partial A^N}{\partial \xi^\alpha} + (b_{\nu\alpha} - sc_{\nu\alpha})A^\nu ,$$

because as far as the subspace of the middle surface is concerned, the component A^N behaves as a scalar, which means that

$$A^N \Big|_\alpha \equiv \frac{\partial A^N}{\partial \xi^\alpha} .$$

In the same manner we obtain

$$A^\alpha \Big\|_N = \frac{\partial A^\alpha}{\partial s} - (b^\alpha_\mu - sc^\alpha_\mu + s^2 c^{\alpha\kappa} b_{\nu\kappa})A^\nu \quad \text{and} \quad A^N \Big\|_N = \frac{\partial A^N}{\partial s} .$$

Following the same line of reasoning, we obtain the formulas for the covariant differentiation of tensors:

$$A^{\alpha\beta} \Big\|_\gamma = A^{\alpha\beta} \Big|_\gamma + (2s[\nu\gamma,\alpha]^b + s^2[\nu\gamma,\alpha]^c)A^{\nu\beta} + (2s[\nu\gamma,\beta]^b + s^2[\nu\gamma,\beta]^c)A^{\alpha\nu}$$

$$- (b^\alpha_\gamma - sc^\alpha_\gamma + s^2 c^{\alpha\kappa} b_{\gamma\kappa})A^{N\beta} - (b^\beta_\gamma - sc^\beta_\gamma + s^2 c^{\beta\kappa} b_{\gamma\kappa})A^{\alpha N} ,$$

$$A^{\alpha N} \Big\|_\gamma = A^{\alpha N} \Big|_\gamma + (2s[\nu\gamma,\alpha]^b + s^2[\nu\gamma,\alpha]^c)A^{\nu N}$$

$$+ (b_{\nu\gamma} - sc_{\nu\gamma})A^{\alpha\nu} - (b^\alpha_\gamma - sc^\alpha_\gamma + s^2 c^{\alpha\kappa} b_{\gamma\kappa})A^{NN} ,$$

$$A^{N\beta} \Big\|_\gamma = A^{N\beta} \Big|_\gamma + (2s[\nu\gamma,\beta]^b + s^2[\nu\gamma,\beta]^c)A^{N\nu}$$

$$+ (b_{\nu\gamma} - sc_{\nu\gamma})A^{\nu\beta} - (b^\beta_\gamma - sc^\beta_\gamma + s^2 c^{\beta\kappa} b_{\gamma\kappa})A^{NN} ,$$

$$A^{NN} \Big\|_\gamma = A^{NN} \Big|_\gamma + (b_{\nu\gamma} - sc_{\nu\gamma})A^{N\nu} + (b_{\nu\gamma} - sc_{\nu\gamma})A^{\nu N} .$$

In the end we consider $A^{N\beta}$ and $A^{\alpha N}$, which are in fact components of a vector as far as differentiation in the middle surface of the shell is concerned:

$$A^{\alpha N} \Big\|_N = \frac{\partial A^{\alpha N}}{\partial s} - (b^\alpha_\nu - sc^\alpha_\nu + s^2 c^{\alpha\kappa} b_{\nu\kappa})A^{\nu N} ,$$

$$A^{N\beta}\Big\|_N = \frac{\partial A^{N\beta}}{\partial s} - (b^\beta_\mu - sc^\beta_\mu + s^2 c^{\beta\kappa} b_{\nu\kappa}) A^{N\mu}\,, \quad A^{NN}\Big\|_N = \frac{\partial A^{NN}}{\partial s}\,.$$

Let us note again that our derivations are not restricted (as it is the case with [12] and [6]) to the middle surface but are valid for the entire shell space.

4. GOVERNING EQUATIONS IN CAUCHY FORM

We prefer to derive in the beginning the averaged Cauchy form and only after that to turn to constitutive relations, because even when considering stress balance, the role of geometrical non-linearity is conspicuous. The Cauchy form of the balance laws for a continuous media reads

$$\left[\rho_* a^j - P^{ij}\,\|_i - F^j\right] \boldsymbol{g}_j = 0, \quad i,j = 1,\ldots,N\,, \tag{4.1}$$

where ρ_* is the ND-density of the elastic medium filling the shell; $\boldsymbol{g}_j$ are the above defined orts of the curvilinear co-ordinate system; P^{ij} are the components of stress tensor; a^j are the components of the acceleration vector and F^j — the components of the N-dimensional body forces. Respectively, $\|_i$ stands for the covariant derivative in $(N-1)$-dimensional space.

Upon substituting into Eq. (4.1) the above defined connection of $\|_i$ to the $(N-1)$D-covariant derivatives $|_\alpha$, the Cauchy law (4.1) is recast into a system for the "surface" (laminar) components and a scalar equation for the N-th component, namely

$$\rho_* a^\alpha - P^{\beta\alpha}\Big|_\beta = \frac{\partial P^{N\alpha}}{\partial s} - (b^\beta_\beta - sc^\beta_\beta + s^2 c^{\beta\kappa} b_{\beta\kappa}) P^{N\alpha}$$

$$- 2(b^\alpha_\nu - sc^\alpha_\nu + s^2 c^{\alpha\kappa} b_{\nu\kappa}) P^{N\nu} + 2\left(2s\left\{ {\alpha \atop \beta\nu} \right\}^b + s^2 \left\{ {\alpha \atop \beta\nu} \right\}^c\right) P^{\nu\beta} + o(s^2)\,, \tag{4.2}$$

$$\rho_* a^N - P^{\beta N}\Big|_\beta = \frac{\partial P^{NN}}{\partial s} + (b_{\beta\nu} - sc_{\beta\nu}) P^{\beta\nu}$$

$$- (b^\beta_\beta - sc^\beta_\beta + s^2 c^{\beta\kappa} b_{\beta\kappa}) P^{NN} + \left(2s\left\{ {\beta \atop \beta\nu} \right\}^b + s^2 \left\{ {\beta \atop \beta\nu} \right\}^c\right) P^{\nu N} + F^N + o(s^2)\,. \tag{4.3}$$

We simplify the above system by taking into account the main assumptions of the present derivations, namely that the shell is a thin layer $h \ll 1$ and that the length-scale of the deformations in the middle surface is $L \gg h$, then we have the small parameter $\varepsilon = h/L$. Dimensionless variables are introduced as follows:

$$s = hs'\,, \quad |_\alpha \simeq L^{-1}\,, \quad b_{\alpha\beta} = L^{-1} b'_{\alpha\beta}\,, \quad c_{\alpha\beta} = L^{-2} c'_{\alpha\beta}\,, \quad P_{ij} = \mu P'_{ij}\,,$$

$$\left\{ {\beta \atop \alpha\nu} \right\}^b = L^{-1} \left\{ {\beta \atop \alpha\nu} \right\}^b, \quad \left\{ {\beta \atop \alpha\nu} \right\}^c = L^{-2} \left\{ {\beta \atop \alpha\nu} \right\}^c,$$

$$t = \frac{L}{c\sqrt{\delta}} t'\,, \quad c = \sqrt{\frac{\mu}{\rho_*}} \quad \Rightarrow \quad a^\alpha = \delta \frac{c^2}{L} a'^\alpha\,, \quad a^N = \delta \frac{c^2}{L} a'^N\,; \tag{4.4}$$

here μ is the shear elastic modulus and c is the speed of shear waves. Note the special scaling for the time involving the square root of the parameter δ, which will be identified later on. In a sense we consider motions of the shell that are of certain characteristic time. Omitting the primes without fear of confusion, the governing equations (4.2) and (4.3) read

$$\delta a^\alpha - P^{\beta\alpha}\Big|_\beta = \frac{1}{\varepsilon}\frac{\partial P^{N\alpha}}{\partial s} - (b^\beta_\beta - s\varepsilon c^\beta_\beta + s^2\varepsilon^2 c^{\beta\kappa} b_{\beta\kappa})P^{N\alpha}$$

$$-2(b^\alpha_\nu - s\varepsilon c^\alpha_\nu + s^2\varepsilon^2 c^{\alpha\kappa} b_{\nu\kappa})P^{N\nu} + 2\left(2s\varepsilon\left\{ {\alpha \atop \beta\nu} \right\}^b + s^2\varepsilon^2\left\{ {\alpha \atop \beta\nu} \right\}^c\right) P^{\nu\beta} + o(\varepsilon^2), \quad (4.5)$$

$$\delta a^N - P^{\beta N}\Big|_\beta = \frac{1}{\varepsilon}\frac{\partial P^{NN}}{\partial s} + (b_{\beta\nu} - \varepsilon s c_{\beta\nu})P^{\beta\nu}$$

$$-(b^\beta_\beta - s\varepsilon c^\beta_\beta + s^2\varepsilon^2 c^{\beta\kappa} b_{\beta\kappa})P^{NN} + \left(2s\varepsilon\left\{ {\beta \atop \beta\nu} \right\}^b + s^2\varepsilon^2\left\{ {\beta \atop \beta\nu} \right\}^c\right) P^{\nu N} + o(\varepsilon^2). \quad (4.6)$$

It is too early to make here assumptions about the relative asymptotic order of the different stress components. Yet one can compare the terms containing the same stress component and to neglect those which are of higher asymptotic order. Since we only consider here the flexural deformations, we can neglect the acceleration terms in the equations for the laminar components of motion. Thus we obtain

$$-P^{\beta\alpha}\Big|_\beta = \frac{1}{\varepsilon}\frac{\partial P^{N\alpha}}{\partial s}, \quad (4.7)$$

$$\delta a^N - P^{\beta N}\Big|_\beta = \frac{1}{\varepsilon}\frac{\partial P^{NN}}{\partial s} + (b_{\beta\nu} - \varepsilon s c_{\beta\nu})P^{\beta\nu}. \quad (4.8)$$

The essential component of derivation of any kind of shell theory is the introduction of averaged across the shell variables, namely

$$\sigma^{\alpha\beta} \stackrel{\text{def}}{=} \int P^{\alpha\beta}\,\mathrm{d}s, \quad m^{\alpha\beta} \stackrel{\text{def}}{=} \int sP^{\alpha\beta}\,\mathrm{d}s, \quad q^\alpha \stackrel{\text{def}}{=} \int P^{N\alpha}\,\mathrm{d}s. \quad (4.9)$$

Integrating the asymptotically reduced equation (4.7), we get

$$\sigma^{\alpha\beta}\Big|_\beta = 0, \quad (4.10)$$

where it is acknowledged that there are no tractions on the shell faces. The last equation has an obvious solution

$$\sigma^{\alpha\beta} = \kappa_0 g^{\alpha\beta}, \quad (4.11)$$

which, depending on the sign of κ_0, corresponds to the case of uniform compression/dilation of the middle surface of the shell. Such a stress state is possible without motion in the middle surface. Henceforth we shall consider only the flexural deformations and the most complicated stress state in the middle surface will be given by Eq. (4.11).

Multiplying Eq. (4.7) by s, integrating and discarding the tractions on the faces, we get

$$\varepsilon m^{\alpha\beta}\Big|_\beta = q^\alpha. \quad (4.12)$$

Let us assume now that on the shell faces different normal pressures act with difference of order of $O(\varepsilon)$. Then

$$P^{NN}\Big|_{s=-\frac{1}{2}} = 0\,, \quad P^{NN}\Big|_{s=\frac{1}{2}} = \varepsilon V_g\,,$$

where εV_g stands for the pressure difference. Here it becomes clear that one can have effectively 2D stress and strain fields only when the normal pressure is of the above adopted order in the small parameter.

Integrating Eq. (4.8) with respect to s, taking into account the boundary conditions for P^{NN} and using Eq. (4.12), yields

$$\frac{\delta}{\varepsilon}\int a^N\,\mathrm{d}s = m^{\alpha\beta}\,\big|_\beta\big|_\alpha + \frac{\kappa_0}{\varepsilon} b_{\beta\nu} g^{\beta\nu} - c_{\beta\nu} m^{\beta\nu} + \frac{1}{2} V_g\,. \tag{4.13}$$

Obtaining the last equation has been the primary objective of the present paper, because it gives the opportunity to identify the geometrical non-linearity, namely the terms of type $c_{\beta\nu}m^{\beta\nu}$ containing the third fundamental form of the middle surface. Now it becomes clear that the spatial derivatives of the moment stresses are of the *same order* as the geometrical non-linearity. This is a new result and it is obtained due to the more consistent treatment of the covariant derivatives in the shell space in comparison with [12, 6].

5. CONSTITUTIVE RELATIONS. St-VENAN–KIRCHHOFF MATERIALS

We shall not dwell much on the constitutive relations for the shell. The main assumption is that for the very thin shells under consideration the material non-linearity is negligible and that the hypothesis of Kirchhoff–Love holds true. According to the latter, the laminar displacements u_α in the shell space are related to the $(N-1)$D-displacements $\tilde{u}_\alpha$ in the shell middle surface as follows:

$$u^\alpha = \tilde{u}^\alpha - \varepsilon s \nabla^\alpha \zeta\,. \tag{5.1}$$

Being consistent with the limiting case of flexural deformation, we neglect in what follows the laminar components $\tilde{u}_{\alpha\beta}$ of the displacement vector. Respectively, the transverse (flexural) displacement and the acceleration, due to the latter, are given by

$$u^N = \zeta \quad \Longrightarrow \quad \int a^N\,\mathrm{d}s = \varepsilon \frac{\partial^2 \zeta}{\partial t^2}\,.$$

We consider an elastic material (called St-Venan–Kirchhoff material) whose constitutive relations are linear regardless to the presence or absence of geometrical non-linearity (see the thorough treatment of these materials in [2]). Without going into much detail one can derive the following linear constitutive relations for the averaged stresses and momenta in the middle surface:

$$m^{\alpha\beta} = -\bar{D} b^{\alpha\beta} \equiv -\bar{D}\nabla^\alpha\nabla^\beta\zeta\,, \tag{5.2}$$

where

$$\bar{D} = \frac{Dh}{\mu L^2}\,D$$

is the dimensionless stiffness coefficient, while D is the stiffness of shell. Alternatively, under the same assumptions the constitutive relation for the moment stresses can be postulated (see, [7]) and then the hypothesis of Kirchhoff–Love (5.1) is not necessary. Furthermore, the overbar will be omitted without fear of confusion.

Introducing Eq. (5.2) into Cauchy equations we get

$$\frac{\delta}{\varepsilon}\frac{\partial^2\zeta}{\partial t^2} = D\left[-\Delta\Delta\zeta + (\nabla_\beta\nabla_\delta\zeta)(\nabla^\beta\nabla_\mu\zeta)(\nabla^\mu\nabla^\delta\zeta)\right] + \frac{\kappa_0}{\varepsilon}\Delta\zeta + V_g\,, \qquad (5.3)$$

where $\Delta \equiv \nabla_\nu\nabla^\nu$, $\Delta\Delta \equiv \nabla_\nu\nabla^\nu(\nabla_\kappa\nabla^\kappa)$.

Now it is time to assess the length and time scales for which the momentum stresses are important, i.e. when the shell is not essentially a membrane. These scales are the ones for which the different coefficients in Eq. (5.3) are of the same order. For the sake of brevity, let us consider the case $V_g = 0$ when the normal load is absent. In fact, one can think that either the shell is a vast sheet, compressed at its rims, or a sphere subjected to normal pressure. In the second case, part of the membrane stress is balanced by V_g and one can subtract $V_g g_{\alpha,\beta}$ from the term $\kappa_0 b_{\alpha\beta}$. As a result the normal pressure drops off from the equation and its sole role is to create the uniform compression.

Thus the uniform membrane tension must be of order

$$|\kappa_0| = \frac{Dh^2}{\mu L^3} \qquad (5.4)$$

and the dimensionless time scale $\delta = |\kappa_0|$. Conversely, for a shell of given stiffness and shear modulus Eq. (5.4) defines the length scale of the "shell-type" deformations when the uniform compression/dilation κ_0 is selected. The governing equation then reads

$$\frac{\partial^2\zeta}{\partial t^2} = \left[-\Delta\Delta\zeta + (\nabla_\beta\nabla_\delta\zeta)(\nabla^\beta\nabla_\mu\zeta)(\nabla^\mu\nabla^\delta\zeta)\right] + \mathrm{sign}(\kappa_0)\Delta\zeta\,. \qquad (5.5)$$

One sees that Eq. (5.5) contains a very strong non-linearity — the cubic power of the curvature of the deformation. In this way it looks very much like the Boussinesq equation [1], being in fact a Boussinesq equation for the curvature $\Delta\zeta$, if the middle surface is subjected to uniform dilation $\kappa_0 > 0$. For the opposite case $\kappa_0 < 0$, when there is a uniform compression, it is more proper to be called *anti*-Boussinesq equation.

6. CONCLUSIONS

In the present paper a consistent asymptotic treatment of a 3D thin elastic layer is attempted for the purposes of derivation of shell theory. The main small parameter is the ratio between the thickness of the shell and the length scale of the deformation of the middle surface. No additional assumptions, such as "shallowness" of the flexural deformation, are implied. For the "steeper" deflections the geometrical non-linearity is identified and shown to be proportional to the cubic power of the curvature of the middle surface. The equation for flexural deformations turns out to be a Boussinesq-like equation.

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