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# ESTIMATES FOR THE BEST CONSTANT IN A MARKOV $L_2$–INEQUALITY WITH THE ASSISTANCE OF COMPUTER ALGEBRA

GENO NIKOLOV, RUMEN ULUCHEV

We prove two-sided estimates for the best (i.e., the smallest possible) constant $c_n(\alpha)$ in the Markov inequality

$$\|p_n'\|_{w_\alpha} \le c_n(\alpha)\|p_n\|_{w_\alpha}\,, \qquad p_n \in \mathcal{P}_n\,.$$

Here, $\mathcal{P}_n$ stands for the set of algebraic polynomials of degree $\le n$, $w_\alpha(x) := x^\alpha\, e^{-x}$, $\alpha > -1$, is the Laguerre weight function, and $\|\cdot\|_{w_\alpha}$ is the associated $L_2$-norm,

$$\|f\|_{w_\alpha} = \left(\int_0^\infty |f(x)|^2 w_\alpha(x)\,dx\right)^{1/2}\,.$$

Our approach is based on the fact that $c_n^{-2}(\alpha)$ equals the smallest zero of a polynomial $Q_n$, orthogonal with respect to a measure supported on the positive axis and defined by an explicit three-term recurrence relation. We employ computer algebra to evaluate the seven lowest degree coefficients of $Q_n$ and to obtain thereby bounds for $c_n(\alpha)$. This work is a continuation of a recent paper [5], where estimates for $c_n(\alpha)$ were proven on the basis of the four lowest degree coefficients of $Q_n$.

**Keywords:** Markov type inequalities, Laguerre polynomials, three-term recurrence relation, Newton identities, computer algebra.

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Throughout this paper $\mathcal{P}_n$ will stand for the set of algebraic polynomials of degree at most $n$, assumed, without loss of generality, with real coefficients. Let

$w_\alpha(x) := x^\alpha\, e^{-x}$, where $\alpha > -1$, be the Laguerre weight function, and $\|\cdot\|_{w_\alpha}$ be the associated $L_2$-norm,

$$\|f\|_{w_\alpha} = \Big(\int_0^\infty |f(x)|^2 w_\alpha(x)\, dx\Big)^{1/2}.$$

We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\alpha} \le c_n(\alpha) \|p_n\|_{w_\alpha}, \qquad p_n \in \mathcal{P}_n, \tag{1.1}$$

namely the constant

$$c_n(\alpha) := \sup_{p_n\in\mathcal{P}_n} \frac{\|p_n'\|_{w_\alpha}}{\|p_n\|_{w_\alpha}}.$$

Before formulating our results, let us give a brief account on the results known so far.

It is only the case $\alpha = 0$ where the best Markov constant is known, namely, Turán [9] proved that

$$c_n(0) = \Big(2\sin\frac{\pi}{4n+2}\Big)^{-1}.$$

Dörfler [2] showed that $c_n(\alpha) = \mathcal{O}(n)$ for every fixed $\alpha > -1$ by proving the estimates

$$c_n^2(\alpha) \ge \frac{n^2}{(\alpha+1)(\alpha+3)} + \frac{(2\alpha^2+5\alpha+6)\, n}{3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{\alpha+6}{3(\alpha+2)(\alpha+3)}, \tag{1.2}$$

$$c_n^2(\alpha) \le \frac{n(n+1)}{2(\alpha+1)}, \tag{1.3}$$

see [3] for a more accessible source. In the same paper, [3], Dörfler proved for the asymptotic constant

$$c(\alpha) := \lim_{n\to\infty} \frac{c_n(\alpha)}{n}, \tag{1.4}$$

that

$$c(\alpha) = \frac{1}{j_{(\alpha-1)/2,1}}, \tag{1.5}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

Nikolov and Shadrin obtained in [5] the following result:

**Theorem A ([5, Theorem 1]).** *For all $\alpha > -1$ and $n \in \mathbb{N}$, $n \ge 3$, the best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *admits the estimates*

$$\frac{2\big(n+\frac{2\alpha}{3}\big)\big(n-\frac{\alpha+1}{6}\big)}{(\alpha+1)(\alpha+5)} < c_n^2(\alpha) < \frac{\big(n+1\big)\big(n+\frac{2(\alpha+1)}{5}\big)}{(\alpha+1)\big[(\alpha+3)(\alpha+5)\big]^{1/3}}, \tag{1.6}$$

*where for the left-hand inequality it is additionally assumed that $n > (\alpha+1)/6$.*

Theorem A implies some inequalities for the asymptotic Markov constant $c(\alpha)$ and, through (1.5), inequalities for $j_{\nu,1}$, the first positive zero of the Bessel function $J_\nu$ (see [5, Corollaries 1, 3]). It was also shown in [5, Theorem 2] that $c(\alpha) = \mathcal{O}(\alpha^{-1})$, which indicates that the upper estimate for $c_n(\alpha)$ in Theorem A, though rather good for moderate $\alpha$, is not optimal.

In a recent paper [7] Nikolov and Shadrin proved an upper bound for $c_n(\alpha)$ which is of the correct order with respect to both $n$ and $\alpha$ as they tend to infinity.

**Theorem B ([7, Theorem 1.1]).** *For all $n \in \mathbb{N}$, $n \geq 3$, the best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies the inequality*

$$c_n^2(\alpha) \leq \frac{4n\big(n+2+\frac{3(\alpha+1)}{4}\big)}{\alpha^2+10\alpha+8}\,, \qquad \alpha \geq 2\,. \tag{1.7}$$

As a consequence of Theorem B and Dörfler's lower bound (1.2) for $c_n(\alpha)$ Nikolov and Shadrin showed that

$$c_n^2(\alpha) \asymp \frac{n(n+\alpha+3)}{(\alpha+1)(\alpha+8)}\,, \qquad n \geq 3,\ \alpha \geq 2\,.$$

**Corollary C ([7, Corollary 1.1]).** *For all $\alpha \geq 2$ and $n \geq 3$ the best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies*

$$\frac{2n(n+\alpha+3)}{3(\alpha+1)(\alpha+8)} \leq c_n^2(\alpha) \leq \frac{4n(n+\alpha+3)}{(\alpha+1)(\alpha+8)}\,. \tag{1.8}$$

In addition, Nikolov and Shadrin found the limit value of $(\alpha+1)c_n^2(\alpha)$ as $\alpha \to -1$, and proved asymptotic inequalities for $\alpha\, c_n^2(\alpha)$ as $\alpha \to \infty$.

**Corollary D ([7, Corollary 1.2]).** *The best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies:*

*(i)* $\displaystyle \lim_{\alpha\to-1}(\alpha+1)c_n^2(\alpha) = \frac{n(n+1)}{2}$;

*(ii)* $\displaystyle \frac{2n}{3} \leq \lim_{\alpha\to\infty} \alpha\, c_n^2(\alpha) \leq 3n\,.$

A combination of Theorems A and B implies bounds for $c(\alpha)$ defined in (1.4):

**Corollary E ([7, Corollary 1.3]).** *The asymptotic Markov constant $c(\alpha)$ satisfies*

$$\frac{2}{(\alpha+1)(\alpha+5)} < c^2(\alpha) < \begin{cases} \dfrac{1}{(\alpha+1)\sqrt[3]{(\alpha+3)(\alpha+5)}}\,, & -1 < \alpha \leq \alpha^*\,, \\ \dfrac{4}{\alpha^2+10\alpha+8}\,, & \alpha > \alpha^*\,, \end{cases}$$

*where* $\alpha^* \approx 43.4$ .

The ratio of the upper and the lower bound for $c(\alpha)$ in Corollary E is less than $\sqrt{2}$ for all $\alpha > -1$ .

In this paper we investigate the best Markov constant $c_n(\alpha)$ following the approach from [5]. It is known (see Proposition 1 below) that $c_n^{-2}(\alpha)$ is equal to the smallest zero of a polynomial $Q_n$ , which is orthogonal with respect to a measure supported on $\mathbb{R}_+$ . Since $\{Q_n\}_{n\in\mathbb{N}}$ are defined by an explicit three-term recurrence relation, one can evaluate (at least theoretically) as many coefficients of $Q_n$ as necessary. With the assistance of Wolfram's *Mathematica* we find the seven lowest degree coefficients of the polynomial $Q_n$ , and thereby the six highest degree coefficients of $R_n$ , the monic polynomial reciprocal to $Q_n$ . Then we apply a simple technique for estimating the largest zero $x_n$ of $R_n$ on the basis of its $k$ highest degree coefficients, $3 \le k \le 6$ , thus obtaining lower and upper bounds for $c_n^2(\alpha)$ . Our main result in this paper is:

**Theorem 1.** *For* $3 \le k \le 6$ *and for all* $n \ge k$ *, the best constant* $c_n(\alpha)$ *in the Markov inequality* (1.1) *admits the estimates*

$$\underline{c}_{n,k}(\alpha) \le c_n(\alpha) \le \overline{c}_{n,k}(\alpha) , \qquad \alpha > -1 , \tag{1.9}$$

*where*

$$\underline{c}_{n,3}^2(\alpha) = \frac{2\,n\big(n + \frac{3(\alpha+1)}{8}\big)}{(\alpha+1)(\alpha+5)} , \tag{1.10}$$

$$\overline{c}_{n,3}^2(\alpha) = \frac{(n+1)\big(n + \frac{2(\alpha+1)}{5}\big)}{(\alpha+1)\big[(\alpha+3)(\alpha+5)\big]^{1/3}} , \tag{1.11}$$

$$\underline{c}_{n,4}^2(\alpha) = \frac{(5\alpha+17)\,n\big(n + \frac{8(\alpha+1)}{25}\big)}{2(\alpha+1)(\alpha+3)(\alpha+7)} , \tag{1.12}$$

$$\overline{c}_{n,4}^2(\alpha) = \frac{(5\alpha+17)^{1/4}(n+1)\big(n + \frac{3(\alpha+1)}{7}\big)}{(\alpha+1)(\alpha+3)^{1/2}\big[2(\alpha+5)(\alpha+7)\big]^{1/4}} , \tag{1.13}$$

$$\underline{c}_{n,5}^2(\alpha) = \frac{2(7\alpha+31)n\big(n + \frac{25(\alpha+1)}{84}\big)}{(\alpha+1)(\alpha+9)(5\alpha+17)} , \tag{1.14}$$

$$\overline{c}_{n,5}^2(\alpha) = \frac{(7\alpha+31)^{1/5}(n+1)\big(n + \frac{4(\alpha+1)}{9}\big)}{(\alpha+1)(\alpha+3)^{2/5}\big[(\alpha+5)(\alpha+7)(\alpha+9)\big]^{1/5}} , \tag{1.15}$$

$$\underline{c}_{n,6}^2(\alpha) = \frac{\big(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073\big)n\big(n + \frac{2(\alpha+1)}{7}\big)}{(\alpha+1)(\alpha+3)(\alpha+5)(\alpha+11)(7\alpha+31)} , \tag{1.16}$$

$$\overline{c}_{n,6}^2(\alpha) = \frac{\big(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073\big)^{1/6}(n+1)\big(n + \frac{5(\alpha+1)}{11}\big)}{(\alpha+1)(\alpha+3)^{1/2}(\alpha+5)^{1/3}\big[(\alpha+7)(\alpha+9)(\alpha+11)\big]^{1/6}} . \tag{1.17}$$

**Remark 1.** For $3 \le k \le 6$, the pair $\big(\underline{c}_{n,k}(\alpha), \overline{c}_{n,k}(\alpha)\big)$ of bounds for $c_n(\alpha)$ is deduced with the use of the $k$ highest degree coefficients of the polynomial $R_n$ (and (1.11) is also proved in [5]). Generally, the bounds for $c_n(\alpha)$ obtained with larger $k$ are better, though some exceptions are observed for small $n$ and $\alpha$.

Clearly, inequalities (1.9) imply bounds for the asymptotic Markov constant $c(\alpha)$. Here, it is not difficult to prove that the larger $k$, the better the implied lower and upper bounds for $c(\alpha)$, hence the best bounds for $c(\alpha)$ are obtained from (1.9) with $k = 6$.

Thus, Theorem 1 yields an improvement of the estimates for the asymptotic Markov constant $c(\alpha)$ in Corollary E.

**Corollary 1.** *The asymptotic Markov constant* $c(\alpha) = \lim_{n\to\infty} n^{-1} c_n(\alpha)$ *satisfies the inequalities*

$$\underline{c}(\alpha) < c(\alpha) < \overline{c}(\alpha)\,,$$

*where*

$$\underline{c}^2(\alpha) := \frac{21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073}{(\alpha+1)(\alpha+3)(\alpha+5)(\alpha+11)(7\alpha+31)}$$

*and*

$$\overline{c}^{\,2}(\alpha) := \begin{cases} \dfrac{\big(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073\big)^{1/6}}{(\alpha+1)(\alpha+3)^{1/2}(\alpha+5)^{1/3}\big[(\alpha+7)(\alpha+9)(\alpha+11)\big]^{1/6}}\,, & -1 < \alpha \le \alpha^\star\,, \\ \dfrac{4}{\alpha^2 + 10\alpha + 8}\,, & \alpha > \alpha^\star, \end{cases}$$

*with* $\alpha^\star \approx 172$.

It is worth noticing that the ratio of the upper and the lower bound for $c(\alpha)$ in Corollary 1 does no exceed $\frac{2\sqrt{3}}{3} \approx 1.1547$ for all $\alpha > -1$.

Theorem 1, in particular inequality (1.16), implies an improvement of the lower bound in Corollary D(ii).

**Corollary 2.** *The best constant* $c_n(\alpha)$ *in the Markov inequality* (1.1) *satisfies:*

$$\frac{6n}{7} \le \lim_{\alpha\to\infty} \alpha\, c_n^2(\alpha) \le 3n\,.$$

The rest of the paper is organized as follows. Section 2 contains some preliminaries. In Section 2.1 we characterize the squared best Markov constant as the largest zero of an $n$-th degree monic polynomial $R_n$ with positive roots, and propose a recursive procedure for the evaluation of its coefficients (Proposition 2). Two-sided estimates for the largest zero of polynomials with only positive roots in terms of few of their coefficients are proposed in Sect. 2.2 (Proposition 2.3). The assisted by Wolfram's *Mathematica* proof of our results is given in Section 3.

In Section 4 we give some final remarks and conclusions, and formulate two conjectures concerning the asymptotic behavior of the best Markov constant and the coefficients of the characteristic polynomial $R_n$ .

## 2. PRELIMINARIES

### 2.1. AN ORTHOGONAL POLYNOMIAL RELATED TO $c_n(\alpha)$

It is well-known that the squared best constant in a Markov-type inequality in $L_2$-norm is equal to the largest eigenvalue of a related positive definite $n \times n$ matrix $\mathbf{A}_n$ , thus the problem of finding the best Markov constant is equivalent to evaluating the largest eigenvalue of $\mathbf{A}_n$. Perhaps, a less known fact is that for a wide class of $L_2$-norms, the inverse matrix $\mathbf{A}_n^{-1}$ is tri-diagonal, see [1, Sect. 2]. In the particular case of the $L_2$-norm induced by the Laguerre weight function $w_\alpha$ this connection is given by the following proposition:

**Proposition 1** (**[3, p. 85]**). *The quantity $c_n^{-2}(\alpha)$ is equal to the smallest zero of the polynomial $Q_n(x) = Q_n(x, \alpha)$ , which is defined recursively by*

$$
\begin{aligned}
&Q_{n+1}(x) = (x - d_n)Q_n(x) - \lambda_n^2 Q_{n-1}(x), \quad n \ge 0\,; \\
&Q_{-1}(x) := 0, \quad Q_0(x) := 1\,; \\
&d_0 := 1 + \alpha, \quad d_n := 2 + \frac{\alpha}{n+1}\,, \quad n \ge 1\,; \\
&\lambda_0 > 0 \ \text{ arbitrary, } \lambda_n^2 := 1 + \frac{\alpha}{n}\,, \quad n \ge 1\,.
\end{aligned}
$$

By Favard's theorem, for any $\alpha > -1$ , $\{Q_n(x, \alpha)\}_{n=0}^{\infty}$ form a system of monic orthogonal polynomials. Since $Q_n$ is the characteristic polynomial of the inverse of a positive definite matrix (which is also positive definite), it follows that all the zeros of $Q_n$ are positive (and distinct). Consequently, $\{Q_n\}_{n=0}^{\infty}$ are orthogonal with respect to a measure supported on $\mathbb{R}_+$.

By Proposition 1, we have

$$
Q_{n+1}(x) = \Big(x - 2 - \frac{\alpha}{n+1}\Big)Q_n(x) - \Big(1 + \frac{\alpha}{n}\Big)Q_{n-1}(x)\,, \quad n \ge 1\,, \tag{2.1}
$$

$$
Q_0(x) = 1\,, \quad Q_1(x) = x - \alpha - 1\,. \tag{2.2}
$$

If we write $Q_n$ in the form

$$
Q_n(x) = x^n - a_{n-1,n}\, x^{n-1} + a_{n-2,n}\, x^{n-2} - \cdots + (-1)^n\, a_{0,n}\,,
$$

then

$$
a_{0,n} = \binom{n+\alpha}{n}, \qquad n \in \mathbb{N}_0\,, \tag{2.3}
$$

with the convention that the right-hand side is equal to 1 for $n = 0$. The proof is by induction with respect to $n$. For $n = 0, 1$, (2.3) follows from (2.2). Assuming (2.3) is true for all $m \le n$, we verify it for $m = n + 1$ by putting $x = 0$ in (2.1) and using the induction hypothesis:

$$\begin{aligned}(-1)^{n+1} a_{0,n+1} &= \Big(2 + \frac{\alpha}{n+1}\Big)(-1)^{n+1}\binom{n+\alpha}{n} + \Big(1 + \frac{\alpha}{n}\Big)(-1)^n \binom{n-1+\alpha}{n-1} \\ &= (-1)^{n+1}\binom{n+1+\alpha}{n}.\end{aligned}$$

Now, instead of $\{Q_n\}_{n=0}^{\infty}$, we consider the sequence of orthogonal polynomials $\{\widetilde{Q}_n\}_{n=0}^{\infty}$ normalized so that $\widetilde{Q}_n(0) = 1$, $n \in \mathbb{N}_0$, i.e.,

$$Q_n(x) = (-1)^n \binom{n+\alpha}{n} \widetilde{Q}_n(x), \qquad n \in \mathbb{N}_0 .$$

It follows from (2.1) and (2.2) that $\{\widetilde{Q}_n\}_{n\in\mathbb{N}_0}$ are determined by

$$\Big(1 + \frac{\alpha}{n+1}\Big)\widetilde{Q}_{n+1}(x) = \Big(2 + \frac{\alpha}{n+1} - x\Big)\widetilde{Q}_n(x) - \widetilde{Q}_{n-1}(x), \quad n \ge 1, \tag{2.4}$$

$$\widetilde{Q}_0(x) = 1, \quad \widetilde{Q}_1(x) = 1 - \frac{x}{\alpha+1}. \tag{2.5}$$

Writing $\widetilde{Q}_n$ in the form

$$\widetilde{Q}_n(x) = 1 - A_{1,n}\, x + A_{2,n}\, x^2 - \cdots + (-1)^n A_{n,n}\, x^n$$

and rewriting (2.4) as

$$\widetilde{Q}_{n+1}(x) - \widetilde{Q}_n(x) = \frac{n+1}{n+\alpha+1}\big(\widetilde{Q}_n(x) - \widetilde{Q}_{n-1}(x)\big) + \frac{n+1}{n+\alpha+1}\, x\, \widetilde{Q}_n(x), \quad n \in \mathbb{N},$$

we deduce the following recurrence relation for the evaluation of the coefficients $\{A_{i,m}\}$:

$$\begin{aligned} &A_{i,n+1} - A_{i,n} = \frac{n+1}{n+\alpha+1}\big(A_{i,n} - A_{i,n-1}\big) + \frac{n+1}{n+\alpha+1} A_{i-1,n}, \quad n \ge k \ge 1, \\ &\text{with } A_{0,n} = 1 \text{ and } A_{1,1} = \frac{1}{\alpha+1}. \end{aligned} \tag{2.6}$$

Since, by Proposition 1, $c_n^{-2}(\alpha)$ is equal to the smallest zero of $\widetilde{Q}_n$, it follows that $c_n^2(\alpha)$ equals the largest zero of the reciprocal polynomial of $\widetilde{Q}_n$,

$$R_n(x) = x^n\, \widetilde{Q}_n(1/x). \tag{2.7}$$

The above observations allow us to reformulate Proposition 1 in the following equivalent form:

**Proposition 2.** *The squared best Markov constant* $c_n^2(\alpha)$ *is equal to the largest zero of the polynomial*

$$R_n(x) = x^n - A_{1,n}\, x^{n-1} + A_{2,n}\, x^{n-2} - \cdots + (-1)^n A_{n,n}\,. \tag{2.8}$$

*The coefficients of* $R_n$ *are evaluated recursively by the following procedure:*

- $A_{1,1} = \frac{1}{\alpha+1}$ *;*
- *Set* $A_{0,m} = 1\,,\ m = 0, \ldots, n\,$*;*
- *For* $i = 1$ *to* $n$*:*
  1. *Find the sequence* $\{D_{i,m}\}_{m=i-1}^{n}$ *as solution of the recurrence equation*
  $$D_{i,m+1} = \frac{m+1}{m+\alpha+1}\, D_{i,m} + \frac{m+1}{m+\alpha+1}\, A_{i-1,m} \tag{2.9}$$
  *with the initial condition* $D_{i,i-1} = 0\,$*;*
  2. *Evaluate*
  $$A_{i,n} = \sum_{m=i}^{n} D_{i,m}\,. \tag{2.10}$$

### 2.2. POLYNOMIALS WITH POSITIVE ROOTS: BOUNDS FOR THE LARGEST ZERO

Let $P$ be a monic polynomial of degree $n$ with zeros $\{x_i\}_{i=1}^n$,

$$P(x) = \prod_{i=1}^{n}(x - x_i) = x^n - b_1\, x^{n-1} + b_2\, x^{n-2} - \cdots + (-1)^n b_n\,.$$

The coefficients $b_r = b_r(P)$, $r = 1, \ldots, n$, are given by the elementary symmetric functions of $\{x_i\}_{i=1}^n$,

$$b_r = s_r = s_r(P) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}\,, \qquad r = 1, \ldots, n\,.$$

It is well known that the elementary symmetric functions $\{s_r\}$ and the Newton functions (sums of powers of $x_i$)

$$p_r = p_r(P) = \sum_{i=1}^{n} x_i^r\,, \qquad r = 1, 2, 3, \ldots\,,$$

are connected by the Newton identities:

$$p_r + \sum_{i=1}^{r-1} (-1)^i p_{r-i}\, s_i + (-1)^r r\, s_r = 0\,, \qquad \text{if } 1 \le r \le n\,, \tag{2.11}$$

$$p_r + \sum_{i=1}^{n} (-1)^i p_{r-i}\, s_i = 0\,, \qquad \text{if } r > n\,. \tag{2.12}$$

For a proof, see e.g. [10] or [4].

Our interest in the Newton functions is motivated by the fact that they provide tight bounds for the largest zero of a polynomial whose roots are all positive. For any such polynomial $P$, we set

$$\ell_k(P) := \frac{p_k(P)}{p_{k-1}(P)}, \qquad u_k(P) := \big[p_k(P)\big]^{1/k}, \qquad k \in \mathbb{N},$$

with the convention that $p_0(P) := \deg(P)$.

**Proposition 3.** *Let* $P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^{n-1} b_{n-1} x + (-1)^n b_n$ *be a polynomial with positive zeros* $x_1 \le x_2 \le \cdots \le x_n$.

*Then the largest zero* $x_n$ *of* $P$ *satisfies the inequalities*

$$\ell_k(P) \le x_n < u_k(P), \qquad k \in \mathbb{N}. \tag{2.13}$$

*Moreover, the sequence* $\{\ell_k(P)\}_{k=1}^{\infty}$ *is monotonically increasing while the sequence* $\{u_k(P)\}_{k=1}^{\infty}$ *is monotonically decreasing, and*

$$\lim_{k\to\infty} \ell_k(P) = \lim_{k\to\infty} u_k(P) = x_n. \tag{2.14}$$

*Proof.* For $i = 1, \dots, n-1$, we set $a_i := \frac{x_i}{x_n}$, then $0 < a_i \le 1$. Now both inequalities (2.13) and the limit relations (2.14) readily follow from the representations

$$\ell_k(P) = \frac{a_1^k + \cdots + a_{n-1}^k + 1}{a_1^{k-1} + \cdots + a_{n-1}^{k-1} + 1}\, x_n, \qquad u_k(P) = \big(a_1^k + \cdots + a_{n-1}^k + 1\big)^{1/k} x_n.$$

The monotonicity of the sequence $\{\ell_k(P)\}_{k=1}^{\infty}$ follows easily from Cauchy-Bouniakowsky's inequality. Indeed, we have

$$\Big(\sum_{i=1}^n x_i^k\Big)^2 = \Big(\sum_{i=1}^n x_i^{\frac{k-1}{2}} x_i^{\frac{k+1}{2}}\Big)^2 \le \Big(\sum_{i=1}^n x_i^{k-1}\Big)\Big(\sum_{i=1}^n x_i^{k+1}\Big),$$

whence $p_k^2(P) \le p_{k-1}(P)\, p_{k+1}(P)$, and consequently

$$\ell_k(P) = \frac{p_k(P)}{p_{k-1}(P)} \le \frac{p_{k+1}(P)}{p_k(P)} = \ell_{k+1}(P).$$

To prove monotonicity of the sequence $\{u_k(P)\}_{k=1}^{\infty}$, we recall that $0 < a_i \le 1$ and therefore $a_i^{k+1} \le a_i^k$. We have

$$\big(a_1^{k+1} + \cdots + a_{n-1}^{k+1} + 1\big)^{1/(k+1)} < \big(a_1^{k+1} + \cdots + a_{n-1}^{k+1} + 1\big)^{1/k} \le \big(a_1^k + \cdots + a_{n-1}^k + 1\big)^{1/k},$$

which yields

$$u_{k+1}(P) < u_k(P).$$

□

## 3. COMPUTER ALGEBRA ASSISTED PROOF OF THE RESULTS

Here we give the algorithms, the source code and the results of the computer algebra assisted proof of estimates (1.10)-(1.17) in Theorem 1. While the case $k = 3$ and to a certain extent $k = 4$ could be studied by hand, it seems impossible to provide similar calculations for larger $k$. We implement the idea from [5] for estimating $c_n(\alpha)$ using $k = 3$ highest degree coefficients of the polynomial $R_n(x)$ and with the assistance of Wolfram's *Mathematica* v. 10 software we investigate the cases $k = 4, 5, 6$, as well. Software based on the algorithms described below failed with calculations for $k > 6$.

Henceforth, we write the polynomial $R_n$ from (2.7) and (2.8) in the form

$$R_n(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} + \cdots + (-1)^n b_n .$$

### 3.1. LOWER BOUNDS FOR $c_n(\alpha)$

We apply Proposition 3 to estimate the largest zero $x_n = c_n^2(\alpha)$ of the polynomial $R_n(x)$ from below,

$$x_n \geq \ell_k(R_n) = \frac{p_k(R_n)}{p_{k-1}(R_n)} , \qquad k = 3,\ 4,\ 5,\ 6 ,$$

and then with the help of computer algebra obtain a further estimation of the form

$$\ell_k(R_n) \geq c\, n(n + \sigma(\alpha+1)),$$

with the optimal (i.e., the largest possible) constants $c = c(k)$ and $\sigma = \sigma(k)$.

**Algorithm 1** Estimating $c_n(\alpha)$ from below

| | |
|---|---|
| *Input:* | $k \in \{3, 4, 5, 6\}$ – the number of the highest degree coefficients of $R_n(x)$ |
| *Step 1.* | Express the power sums $p_{k-1}(R_n)$ and $p_k(R_n)$ in terms of $\{b_i\}_{i=1}^k$ |
| *Step 2.* | Find coefficients $\{b_i\}_{i=1}^k$ in terms of $n$ and $\alpha$ using Proposition 2 |
| *Step 3.* | Find a proper value $\sigma$ for parameter $s$ in $p_k - c\,n(n + s(\alpha+1))p_{k-1}$, where $c$ is the coefficient of $n^2$ in the quotient $p_k/p_{k-1}$ |
| *Step 4.* | Represent the numerator of $f = p_k - c\,n(n + \sigma(\alpha+1))p_{k-1}$ in powers of $n$ and $(\alpha+1)$ |
| *Step 5.* | Estimate from below the expression $f$ to prove that $f \geq 0$ |

*Step 1:* Let $\{x_i\}_{i=1}^n$ be all the zeros of the polynomial $R_n(x)$ from (2.7). In order to express a power sum $p_r = \sum_{i=1}^r x_i^r$, $1 \leq r \leq n$, by $\{b_i\}_{i=1}^r$, we apply the direct formula

$$p_r = \begin{vmatrix} b_1 & 1 & 0 & \cdots & 0 \\ 2b_2 & b_1 & 1 & \cdots & 0 \\ 3b_3 & b_2 & b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ rb_r & b_{r-1} & b_{r-2} & \cdots & b_1 \end{vmatrix}$$

which easily follows from the Newton identities (2.11).

Below is the code of the programme and the results for $k = 1, \dots, 6$:

```
k = 6;
Do[pκ = Det[Table[Which[j == 1, i bi, 1 < j ≤ i, bi+1−j, j == i + 1, 1, j > i + 1, 0], {i, κ}, {j, κ}]];
    Print[Subscript["p", κ], "=", TraditionalForm[pκ]], {κ, k}]
```

$$p_1=b_1$$
$$p_2=b_1^2-2\,b_2$$
$$p_3=b_1^3-3\,b_2\,b_1+3\,b_3$$
$$p_4=b_1^4-4\,b_2\,b_1^2+4\,b_3\,b_1+2\,b_2^2-4\,b_4$$
$$p_5=b_1^5-5\,b_2\,b_1^3+5\,b_3\,b_1^2+5\,b_2^2\,b_1-5\,b_4\,b_1-5\,b_2\,b_3+5\,b_5$$
$$p_6=b_1^6-6\,b_2\,b_1^4+6\,b_3\,b_1^3+9\,b_2^2\,b_1^2-6\,b_4\,b_1^2-12\,b_2\,b_3\,b_1+6\,b_5\,b_1-2\,b_2^3+3\,b_3^2+6\,b_2\,b_4-6\,b_6$$

*Step 2:* We find coefficients $\{b_i\}_{i=1}^k$ of the polynomial $R_n(x)$ using Proposition 2. The source and the results for $k = 1, \dots, 6$ follow below:

```
k = 6;
fb[κ_, n_] :=
  If[κ == 1, Sum[FullSimplify[RSolveValue[{ru[q + 1] == (ru[q] + 1) (q + 1) / (q + 1 + α), ru[1] == 1 / (α + 1)}, ru[q], q]], {q, 1, n}],
      Sum[Simplify[RSolveValue[{rv[q + 1] == (rv[q] + fb[κ − 1, q]) (q + 1) / (q + 1 + α), rv[1] == 0}, rv[q], q]], {q, 1, n}]]
Do[If[κ == 1, bκ = fb[κ, n],
      bκ = Factor[Part[FactorTermsList[Numerator[fb[κ, n]], α], 2]] *
          Collect[Part[FactorTermsList[Numerator[fb[κ, n]], α], 3], n, FullSimplify] / Denominator[fb[κ, n]]];
   Print[Subscript["b", κ], "=", TraditionalForm[bκ]], {κ, 1, k}]
```

$$b_1=\frac{n\,(n+1)}{2\,(\alpha+1)}$$
$$b_2=\frac{(n-1)\,n\,(n+1)\,(3\,n\,(\alpha+2)+2\,(\alpha+6))}{24\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)}$$
$$b_3=\frac{(n-2)\,(n-1)\,n\,(n+1)\,\big(5\,(\alpha+2)\,(\alpha+4)\,n^2+(\alpha\,(5\,\alpha+86)+200)\,n+12\,(\alpha+20)\big)}{240\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(\alpha+5)}$$
$$\begin{aligned}b_4=\Big(&(n-3)\,(n-2)\,(n-1)\,n\,(n+1)\,\big(105\,(\alpha+2)\,(\alpha+4)\,(\alpha+6)\,n^3+3\,(\alpha\,(7\,\alpha\,(5\,\alpha+204)+9316)+15\,120)\,n^2\\&+(131\,040-2\,\alpha\,(7\,\alpha\,(5\,\alpha+44)-17\,244))\,n-8\,(\alpha\,(7\,\alpha\,(\alpha+28)+2244)-15\,120)\big)\Big)\Big/\\&(40\,320\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(\alpha+5)\,(\alpha+6)\,(\alpha+7))\end{aligned}$$
$$\begin{aligned}b_5=\Big(&(n-4)\,(n-3)\,(n-2)\,(n-1)\,n\,(n+1)\,\big(21\,(\alpha+2)\,(\alpha+4)\,(\alpha+6)\,(\alpha+8)\,n^4\\&+2\,(\alpha\,(\alpha\,(7\,\alpha\,(\alpha+108)+9956)+42\,928)+56\,448)\,n^3+(\alpha\,(\alpha\,(17\,988-7\,\alpha\,(7\,\alpha+212))+248\,496)+572\,544)\,n^2\\&+(1\,241\,856-2\,\alpha\,(\alpha\,(\alpha\,(21\,\alpha+1096)+26\,468)-34\,832))\,n-240\,(\alpha\,(\alpha\,(\alpha+38)+1528)-4032)\big)\Big)\Big/\\&(80\,640\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(\alpha+5)\,(\alpha+6)\,(\alpha+7)\,(\alpha+8)\,(\alpha+9))\end{aligned}$$
$$\begin{aligned}b_6=\Big(&(n-5)\,(n-4)\,(n-3)\,(n-2)\,(n-1)\,n\,(n+1)\\&\big(3465\,(\alpha+2)\,(\alpha+4)\,(\alpha+6)\,(\alpha+8)\,(\alpha+10)\,n^5+360\,(13\,\alpha\,(11\,\alpha\,(\alpha\,(7\,\alpha+164)+1348)+49\,936)+739\,200)\,n^4\\&+9\,(\alpha\,(131\,884\,640-11\,\alpha\,(\alpha\,(35\,\alpha\,(5\,\alpha+278)-10\,644)-1\,805\,704))+229\,152\,000)\,n^3\\&-8\,(\alpha\,(11\,\alpha\,(\alpha\,(5\,\alpha\,(28\,\alpha+2685)+620\,812)+2\,759\,292)-220\,067\,280)-964\,656\,000)\,n^2\\&+44\,(\alpha\,(\alpha\,(\alpha\,(5\,\alpha\,(35\,\alpha+1014)-37\,756)-20\,283\,336)-53\,575\,200)+315\,705\,600)\,n\\&+96\,(\alpha\,(11\,\alpha\,(\alpha\,(5\,\alpha\,(\alpha+66)+7714)+237\,564)-62\,191\,440)+99\,792\,000)\big)\Big)\Big/\\&(159\,667\,200\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(\alpha+5)\,(\alpha+6)\,(\alpha+7)\,(\alpha+8)\,(\alpha+9)\,(\alpha+10)\,(\alpha+11))\end{aligned}$$

*Step 3:* The quotient $p_k/p_{k-1}$ is a quadratic polynomial in $n$, and we denote by $c$ its leading coefficient.

The goal of this step is to find a proper value (say $\sigma$) for parameter $s$ in the expression

$$f_s = p_k - c\,n(n + s(\alpha+1))p_{k-1}\,,$$

such that $f_\sigma \geq 0$ for all admissible $\alpha$ and $n$. For a fixed $k$ quantity $f_s$ depends on $\alpha$, $n$ and $s$. It is a polynomial of degree $2k-1$ in $n$ and a rational function in $\alpha$. Let us write the numerator of $f_s$ in the form

$$\sum_{i=1}^{2k-1} \sum_{j=0}^{d} \mu_{i,j}(s)(\alpha+1)^{d-j} n^{2k-i}\,.$$

The highest order coefficients in $\sum_j \mu_{i,j}(s)(\alpha+1)^{d-j}$ are linear functions in $s$ of the form $A_i - B_i s$, with $A_i > 0$ and $B_i > 0$. We denote their zeros by $s_i$ for each $i$ and set $\sigma = \min_i s_i$. Since we seek estimates valid for all $\alpha > -1$, our choice of $\sigma$ guarantee that for $\alpha$ sufficiently large the inequality $\sum_j \mu_{i,j}(s)(\alpha+1)^{d-j} > 0$ holds true.

The code is as follows:

```
r = PolynomialQuotient[pk, pk−1, n];
c = Factor[Coefficient[r, n, 2]];
fs = pk − c n (n + s (α + 1)) pk−1;
numfs = Numerator[Together[Apart[fs, α]]]
Do[gs = Factor[Coefficient[numfs, n, i]];
    num = Normal[Series[ gs , {α, −1, Exponent[gs, α]}]];
    sols = Solve[Coefficient[num, α, Exponent[gs, α]] == 0, s, Reals];
    ss[i] = s /. Flatten[sols], {i, 2 k − 1, 1, −1}]
σ = Min[Table[ss[i], {i, 2, 2 k − 1}]];
```

Table 1 gives results for the optimal values of $c$ and $\sigma$ for $k = 3, 4, 5, 6$.

Table 1: The optimal values of $c$ and $\sigma$ in the lower bounds for $c_n^2(\alpha)$.

| $k$ | $c$ | $\sigma$ |
|---|---|---|
| 3 | $\dfrac{2}{(\alpha+1)(\alpha+5)}$ | $\dfrac{3}{8}$ |
| 4 | $\dfrac{5\alpha+17}{2(\alpha+1)(\alpha+3)(\alpha+7)}$ | $\dfrac{8}{25}$ |
| 5 | $\dfrac{2(7\alpha+31)}{(\alpha+1)(\alpha+9)(5\alpha+17)}$ | $\dfrac{25}{84}$ |
| 6 | $\dfrac{21\alpha^3+299\alpha^2+1391\alpha+2073}{(\alpha+1)(\alpha+3)(\alpha+5)(\alpha+11)(7\alpha+31)}$ | $\dfrac{2}{7}$ |

*Step 4:* We set

$$f = p_k - c\,n(n+\sigma(\alpha+1))p_{k-1} =: \frac{\varphi(n,\alpha)}{\psi(\alpha)}$$

with $c$ and $\sigma$ determined in Step 3. Here, $\varphi(n,\alpha)$ is a bivariate polynomial in $n$ and $\alpha$, and $\psi(\alpha)$ is a polynomial in $\alpha$. More precisely, $\varphi(n,\alpha)$ has degree $2k-1$ in $n$, and degree $d$ in $\alpha$ which our programme calculates for each fixed $k$.

Note that $\psi(\alpha) > 0$ for $\alpha > -1$ since it is a product of powers of $\alpha + j$, $j \geq 1$ and multipliers $A\alpha + B$, $0 < A < B$. Therefore, $\operatorname{sign} f = \operatorname{sign} \varphi$.

We expand $\varphi(n,\alpha)$ in the form

$$\varphi(n,\alpha) = \sum_{i=1}^{2k-1} \sum_{j=0}^{d} \mu_{i,j} (\alpha+1)^{d-j} n^{2k-i} = \begin{pmatrix} n^{2k-1} \\ n^{2k-2} \\ \vdots \\ n \end{pmatrix}^{\top} \mathbf{M} \begin{pmatrix} (\alpha+1)^{d} \\ (\alpha+1)^{d-1} \\ \vdots \\ 1 \end{pmatrix},$$

where $\mathbf{M} = \big(\mu_{i,j}\big)_{i=1,j=0}^{2k-1,d}$ and all entries $\mu_{i,j}$ are integer numbers.

The source for computation of the matrix $\mathbf{M}$ is listed below.

```
f = p_k − c n (n + σ (α + 1)) p_{k−1};
φ = Numerator[Together[Apart[f, α]]];
ψ = Denominator[Together[Apart[f, α]]];
Do[g = Factor[Coefficient[φ, n, i]]; dag[i] = Exponent[g, α] , {i, 2 k − 1, 1, −1}]
d = Max[Table[dag[i], {i, 1, 2 k − 1}]] + 1;
μ = ConstantArray[0, {2 k − 1, d}];
Do[g = Factor[Coefficient[φ, n, i]];
    Table[μ[[2 k − i, d − j]] = SeriesCoefficient[Series[g, {α, −1, dag[i]}], j], {j, 0, dag[i]}],
    {i, 2 k − 1, 1, −1}];
```

If $\mu_{i,j} \geq 0$ for all $i,j$, then $\varphi(n,\alpha) \geq 0$ and $f \geq 0$ for all $\alpha > -1$ and $n \geq k$. In a case some of coefficients $\mu_{i,j} < 0$ we apply the next step of the algorithm.

The results for $k = 3, 4, 5, 6$ are given together with the estimates from Step 5.

Step 5: If there are coefficients $\mu_{i,j} < 0$ we need additional arguments to verify that $f \geq 0$ for all $\alpha > -1$ and $n \geq k$. We bring into use a new $(2k-1) \times (d+1)$ matrix $\mathbf{\Lambda}$ which elements we put initially $\lambda_{i,j} := \mu_{i,j}$, for $i = 1, \ldots, 2k-1$ and $j = 0, \ldots, d$.

The procedure described below checks recursively all coefficients $\lambda_{i,j}$ and makes the corresponding estimations. We need not introduce a new matrix after each iteration, but only replace a pair of elements in a column of $\mathbf{\Lambda}$ with new entries in such a manner that the value of the function

$$\Phi(\mathbf{\Lambda}) = \sum_{i=1}^{2k-1} \sum_{j=0}^{d} \lambda_{i,j} (\alpha+1)^{d-j} n^{2k-i} = \begin{pmatrix} n^{2k-1} \\ n^{2k-2} \\ \vdots \\ n \end{pmatrix}^{\top} \mathbf{\Lambda} \begin{pmatrix} (\alpha+1)^{d} \\ (\alpha+1)^{d-1} \\ \vdots \\ 1 \end{pmatrix}$$

decreases. At the end of the procedure we get a matrix $\mathbf{\Lambda}$ satisfying $\mathbf{0} \leq \mathbf{\Lambda} \leq \mathbf{M}$ (in the sense that $0 \leq \lambda_{i,j} \leq \mu_{i,j}$ for all $i,j$) and therefore

$$0 \leq \Phi(\mathbf{\Lambda}) \leq \Phi(\mathbf{M}) = \varphi(n,\alpha).$$

Suppose that $\lambda_{i,j} < 0$ for some pair of indices $i, j$. Then we set

$$h := \min\{i - \eta : \lambda_{\eta,j} > 0,\ 1 \le \eta \le i-1\} \quad \text{and} \quad \delta := \frac{\lambda_{i,j}}{k^{i-h}} \quad (\delta < 0)\,.$$

If $\lambda_{h,j} + \delta \ge 0$, for $n \ge k$ we have

$$\begin{aligned}(\lambda_{h,j} + \delta) n^{2k-h} + 0\, n^{2k-i} &= \Big(\lambda_{h,j} + \frac{\lambda_{i,j}}{k^{i-h}}\Big) n^{2k-h} = \lambda_{h,j} n^{2k-h} + \lambda_{i,j} \frac{n^{2k-h}}{k^{i-h}} \\ &\le \lambda_{h,j} n^{2k-h} + \lambda_{i,j} \frac{n^{2k-h}}{n^{i-h}} = \lambda_{h,j} n^{2k-h} + \lambda_{i,j} n^{2k-i}\,.\end{aligned}$$

Otherwise, if $\lambda_{h,j} + \delta < 0$, for $n \ge k$ we have

$$\begin{aligned}0\, n^{2k-h} + \big(\lambda_{h,j} k^{i-h} + \lambda_{i,j}\big) n^{2k-i} &= \lambda_{h,j} n^{2k-i} k^{i-h} + \lambda_{i,j} n^{2k-i} \\ &\le \lambda_{h,j} n^{2k-i} n^{i-h} + \lambda_{i,j} n^{2k-i} \\ &\le \lambda_{h,j} n^{2k-h} + \lambda_{i,j} n^{2k-i}\,.\end{aligned}$$

So, replacing only two elements in $\mathbf{\Lambda}$,

$$\begin{cases} \lambda_{h,j} := \lambda_{h,j} + \lfloor \delta \rfloor & \text{and} \ \ \lambda_{i,j} := 0\,, \quad \text{if } \lambda_{h,j} + \delta \ge 0, \\ \lambda_{i,j} := \lambda_{h,j}\, k^{i-h} + \lambda_{i,j} & \text{and} \ \ \lambda_{h,j} := 0\,, \quad \text{otherwise}\,, \end{cases}$$

we obtain that

$$\lambda_{h,j} (\alpha + 1)^{d+1-j} n^{2k-h} + \lambda_{i,j} (\alpha+1)^{d+1-j} n^{2k-i}$$

decreases for the new values of $\lambda_{h,j}$ and $\lambda_{i,j}$, and hence $\Phi(\mathbf{\Lambda})$ also decreases.

Applying recursively the above iteration process for $i = 2k-1, 2k-2, \dots, 1$ and $j = 0, 1, \dots, d$ we finally obtain a matrix $\mathbf{\Lambda}$ satisfying $\mathbf{0} \le \mathbf{\Lambda} \le \mathbf{M}$. Then $\varphi(n, \alpha) \ge 0$, $f \ge 0$ and therefore

$$c_n^2(\alpha) \ge \frac{p_k}{p_{k-1}} \ge c\, n(n + \sigma(\alpha+1))$$

for the optimal $c$ and $\sigma$ evaluated in Step 3. For $k = 3, 4, 5, 6$ we obtain estimates (1.10), (1.12), (1.14), and (1.16), respectively.

The following source implements the procedure described in Step 5.

```
λ = μ;
For[i = 2 k − 1, i > 1, i−−,
    For[j = 1, j ≤ d, j++, If[λ[[i, j]] ≥ 0, Continue[]];
        h = i − First[FirstPosition[Positive[λ[[i − 1 ;; 1 ;; −1, j]]], True]];
        δ = λ[[i, j]] / (k^(i − h));
        If[λ[[h, j]] + δ ≥ 0, λ[[h, j]] = λ[[h, j]] + Floor[δ]; λ[[i, j]] = 0,
                    λ[[i, j]] = λ[[h, j]] ∗ k^(i − h) + λ[[i, j]]; λ[[h, j]] = 0; i = i + 1]]]
Print["Λ = ", MatrixForm[λ]]
Print["M = ", MatrixForm[μ]]
```

Next, we give matrices $\mathbf{M}$ from Step 4 and $\mathbf{\Lambda}$ from Step 5 obtained with *Mathematica*.

Case $k = 3$:

This partial case needs a special attention as we have to assume strict inequality $n > k$, i.e., $n \geq 4$, to obtain estimate (1.10). This causes a minor modification in Step 5 of Algorithm 1, namely, replacement of $k^{i-h}$ with $(k+1)^{i-h}$. Namely, we determine $\delta := \lambda_{i,j}/(k+1)^{i-h}$ and set

$$\begin{cases} \lambda_{h,j} := \lambda_{h,j} + \lfloor \delta \rfloor & \text{and} \quad \lambda_{i,j} := 0\,, \quad \text{if } \lambda_{h,j} + \delta \geq 0, \\ \lambda_{i,j} := \lambda_{h,j}\,(k+1)^{i-h} + \lambda_{i,j} & \text{and} \quad \lambda_{h,j} := 0\,, \quad \text{otherwise}\,. \end{cases}$$

Matrices $\mathbf{M}$ and $\mathbf{\Lambda}$ in this case are

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 4 & -4 & 225 & 360 \\ 0 & 0 & 390 & 510 & 720 \\ 15 & 155 & 205 & 1185 & 360 \\ 15 & 270 & 495 & 900 & 0 \\ 0 & 36 & 684 & 0 & 0 \end{pmatrix} \qquad \mathrm{M} = \begin{pmatrix} 0 & 19 & -4 & 225 & 360 \\ 0 & -60 & 390 & 510 & 720 \\ 15 & 155 & 205 & 1185 & 360 \\ 15 & 270 & 495 & 900 & 0 \\ 0 & 36 & 684 & 0 & 0 \end{pmatrix}.$$

Although there is a negative element of $\mathbf{\Lambda}$, from $4(\alpha+1)^2 - 4(\alpha+1) + 225 \geq 0$ for all $\alpha > -1$ we conclude that $4(\alpha+1)^3 - 4(\alpha+1)^2 + 225(\alpha+1) + 360 > 0$ and consequently $\Phi(\mathbf{\Lambda}) \geq 0$ for $n \geq 4$.

By a direct verification one can see that inequality (1.10) holds also in the case $n = k = 3$.

Case $k = 4$:

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & 10200 & 72480 & 323700 & 1413060 & 3602340 & 4340700 & 1890000 \\ 0 & 4882 & 30891 & 359695 & 2625259 & 7966210 & 13275570 & 12707100 & 5670000 \\ 0 & 0 & 229110 & 1642830 & 6282570 & 16699200 & 24837120 & 18692100 & 5670000 \\ 2100 & 46515 & 120645 & 2404465 & 10159765 & 20026720 & 25810890 & 16625700 & 1890000 \\ 2756 & 106120 & 876330 & 2582090 & 7616630 & 17567550 & 18060000 & 6300000 & 0 \\ 0 & 11060 & 662604 & 2653840 & 6215776 & 11121880 & 7413000 & 0 & 0 \\ 0 & 0 & 0 & 1120600 & 4777900 & 3435000 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{M} = \begin{pmatrix} 0 & 0 & 10200 & 72480 & 323700 & 1413060 & 3602340 & 4340700 & 1890000 \\ 0 & 8715 & 30891 & 359695 & 2625259 & 7966210 & 13275570 & 12707100 & 5670000 \\ 0 & -15330 & 229110 & 1642830 & 6282570 & 16699200 & 24837120 & 18692100 & 5670000 \\ 2100 & 46515 & 120645 & 2404465 & 10159765 & 20026720 & 25810890 & 16625700 & 1890000 \\ 2800 & 106120 & 876330 & 2582090 & 7616630 & 17567550 & 18060000 & 6300000 & 0 \\ 0 & 15960 & 722904 & 2653840 & 6215776 & 11121880 & 7413000 & 0 & 0 \\ -700 & -19600 & -241200 & 1120600 & 4777900 & 3435000 & 0 & 0 & 0 \end{pmatrix}$$

Case $k = 5$:

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 64925 & 1064665 & 8138830 & 43256150 & 172898565 & 474925185 & 805850640 & 734423760 & 266716800 \\ 0 & 0 & 91665 & 1204470 & 9699090 & 71280390 & 373661895 & 1241223900 & 2610599670 & 3473555400 & 2804336640 & 1066867200 \\ 0 & 19824 & 130578 & 3408188 & 48487642 & 313463920 & 1271550350 & 3522779568 & 6544523790 & 7686433440 & 5117787360 & 1600300800 \\ 0 & 0 & 1451982 & 16288020 & 114900450 & 672910770 & 2546690160 & 6152610870 & 9859721760 & 10218685680 & 5871579840 & 1066867200 \\ 3675 & 128835 & 0 & 24490445 & 226233910 & 991504675 & 3153540110 & 7169071245 & 10438959825 & 9013742640 & 3935025360 & 266716800 \\ 6027 & 381850 & 6416795 & 22404550 & 169885205 & 1005110890 & 2985302145 & 5744010510 & 7716554370 & 5584488840 & 1111320000 & 0 \\ 0 & 52297 & 5062484 & 58263912 & 213196158 & 589342950 & 1804792500 & 3787471002 & 4038237000 & 1770703200 & 0 & 0 \\ 0 & 0 & 0 & 15084950 & 144208510 & 409403975 & 1057769610 & 1931913900 & 1309770000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 256255650 & 690284700 & 417538800 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{M} = \begin{pmatrix} 0 & 0 & 0 & 64925 & 1064665 & 8138830 & 43256150 & 172898565 & 474925185 & 805850640 & 734423760 & 266716800 \\ 0 & 0 & 91665 & 1204470 & 9699090 & 71280390 & 373661895 & 1241223900 & 2610599670 & 3473555400 & 2804336640 & 1066867200 \\ 0 & 27804 & 130578 & 3408188 & 48487642 & 313463920 & 1271550350 & 3522779568 & 6544523790 & 7686433440 & 5117787360 & 1600300800 \\ 0 & -39900 & 1500030 & 16288020 & 114900450 & 672910770 & 2546690160 & 6152610870 & 9859721760 & 10218685680 & 5871579840 & 1066867200 \\ 3675 & 128835 & -240240 & 24490445 & 226233910 & 991504675 & 3153540110 & 7169071245 & 10438959825 & 9013742640 & 3935025360 & 266716800 \\ 6125 & 381850 & 6416795 & 22404550 & 169885205 & 1005110890 & 2985302145 & 5744010510 & 7716554370 & 5584488840 & 1111320000 & 0 \\ 0 & 77616 & 5699022 & 58263912 & 213196158 & 589342950 & 1804792500 & 3787471002 & 4038237000 & 1770703200 & 0 & 0 \\ -2450 & -123445 & -3055430 & 20292530 & 152590030 & 409403975 & 1057769610 & 1931913900 & 1309770000 & 0 & 0 & 0 \\ 0 & -15750 & -636300 & -26037900 & -41907600 & 256255650 & 690284700 & 417538800 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case $k = 6$:

$$\Lambda^T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 48510 & 95223 & 0 & 0 & 0 & 16170 \\
0 & 0 & 0 & 425810 & 0 & 2817045 & 9741270 & 1348462 & 0 & 0 & 1252020 \\
0 & 0 & 3476550 & 6110115 & 48434732 & 0 & 336258384 & 218861747 & 0 & 0 & 38848656 \\
0 & 6055665 & 95465370 & 190273710 & 1221447150 & 1171139970 & 2726237052 & 7298343195 & 0 & 0 & 1158647028 \\
3128160 & 204553195 & 1480047030 & 5336244870 & 15771654360 & 32618391960 & 21628131756 & 73442566505 & 29020437724 & 0 & 0 \\
116263280 & 3318028175 & 18873326010 & 77596724865 & 174458095350 & 356484794820 & 298526146072 & 392659895320 & 419332019003 & 0 & 0 \\
1988081620 & 35746404925 & 197029544250 & 726747795015 & 1551387171180 & 2562636437130 & 2754345379016 & 1907270574440 & 2403530867430 & 384166681454 & 0 \\
21102099620 & 290306961329 & 1558206940290 & 4960832042100 & 10157416978170 & 14397653320512 & 15830016304564 & 10335346465675 & 8454107203080 & 3721378398370 & 0 \\
157521933940 & 1842856573327 & 9151953918030 & 25828662738780 & 49078270584420 & 64463871381756 & 64790433176084 & 46215397662665 & 25413887653770 & 13866272170542 & 3624993260826 \\
879576036500 & 9097329993521 & 40326294432270 & 103410904320900 & 179681190528840 & 223720508502183 & 206003553429058 & 148200002432020 & 74515561079190 & 38329760467746 & 21352330210512 \\
3768921407020 & 34425402760287 & 134937782918400 & 317406347163180 & 506548245985320 & 592781349468231 & 516092578758680 & 349390783269990 & 182484813042840 & 86130722275092 & 36302262824520 \\
12408373123020 & 98585568531450 & 344523626901300 & 742720332283380 & 1098864688687920 & 1195101212250330 & 988299659161584 & 622626376181040 & 315159824447160 & 127817022168000 & 27594339093216 \\
30888195414300 & 211221314490186 & 667161153364860 & 1314049020225480 & 1805851509808500 & 1815258345062778 & 1389893720693940 & 808485195464100 & 350770360746960 & 103510976206176 & 7959911420160 \\
56418683248620 & 333956069661060 & 961433219937960 & 1730658737031840 & 2184547015159740 & 2011470759046980 & 1371614133582000 & 691021013177880 & 227021138467200 & 33927611477760 & 0 \\
72303376012560 & 380597496158880 & 996948009953280 & 1640782277386560 & 1861980233153040 & 1520944502505120 & 876545356468320 & 330199345808640 & 64746646272000 & 0 & 0 \\
60744201708960 & 297669661581600 & 704885552078400 & 1045548136987200 & 1036553459911200 & 695810390758560 & 300533746867200 & 63688771238400 & 0 & 0 & 0 \\
29689237670400 & 143165195712000 & 307454361984000 & 391950244224000 & 317437952832000 & 151152068390400 & 31685955840000 & 0 & 0 & 0 & 0 \\
6337191168000 & 31685955840000 & 63371911680000 & 63371911680000 & 31685955840000 & 6337191168000 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$M^T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 48510 & 97020 & 0 & -64680 & 0 & 16170 \\
0 & 0 & 0 & 544005 & -709170 & 2817045 & 9741270 & 2279970 & -5453910 & -810810 & 1252020 \\
0 & 0 & 3476550 & 6110115 & 51415980 & -17887485 & 336258384 & 259149660 & -233284590 & -50657310 & 38848656 \\
0 & 6055665 & 95465370 & 190273710 & 1221447150 & 1171139970 & 2726237052 & 7522825695 & -627553080 & -4316051520 & 1158647028 \\
3128160 & 204553195 & 1480047030 & 5336244870 & 15771654360 & 32618391960 & 21628131756 & 73442566505 & 36182631870 & -42053593230 & -5517429876 \\
116263280 & 3318028175 & 18873326010 & 77596724865 & 174458095350 & 356484794820 & 298526146072 & 392659895320 & 439184120760 & -87653879280 & -188752387572 \\
1988081620 & 35746404925 & 197029544250 & 726747795015 & 1551387171180 & 2562636437130 & 2754345379016 & 1907270574440 & 2403530867430 & 568697131530 & -1107182700456 \\
21102099620 & 290306961329 & 1558206940290 & 4960832042100 & 10157416978170 & 14397653320512 & 15830016304564 & 10335346465675 & 8454107203080 & 4052140160904 & -1984570575204 \\
157521933940 & 1842856573327 & 9151953918030 & 25828662738780 & 49078270584420 & 64463871381756 & 64790433176084 & 46215397662665 & 25413887653770 & 13866272170542 & 3624993260826 \\
879576036500 & 9097329993521 & 40326294432270 & 103410904320900 & 179681190528840 & 223720508502183 & 206003553429058 & 148200002432020 & 74515561079190 & 38329760467746 & 21352330210512 \\
3768921407020 & 34425402760287 & 134937782918400 & 317406347163180 & 506548245985320 & 592781349468231 & 516092578758680 & 349390783269990 & 182484813042840 & 86130722275092 & 36302262824520 \\
12408373123020 & 98585568531450 & 344523626901300 & 742720332283380 & 1098864688687920 & 1195101212250330 & 988299659161584 & 622626376181040 & 315159824447160 & 127817022168000 & 27594339093216 \\
30888195414300 & 211221314490186 & 667161153364860 & 1314049020225480 & 1805851509808500 & 1815258345062778 & 1389893720693940 & 808485195464100 & 350770360746960 & 103510976206176 & 7959911420160 \\
56418683248620 & 333956069661060 & 961433219937960 & 1730658737031840 & 2184547015159740 & 2011470759046980 & 1371614133582000 & 691021013177880 & 227021138467200 & 33927611477760 & 0 \\
72303376012560 & 380597496158880 & 996948009953280 & 1640782277386560 & 1861980233153040 & 1520944502505120 & 876545356468320 & 330199345808640 & 64746646272000 & 0 & 0 \\
60744201708960 & 297669661581600 & 704885552078400 & 1045548136987200 & 1036553459911200 & 695810390758560 & 300533746867200 & 63688771238400 & 0 & 0 & 0 \\
29689237670400 & 143165195712000 & 307454361984000 & 391950244224000 & 317437952832000 & 151152068390400 & 31685955840000 & 0 & 0 & 0 & 0 \\
6337191168000 & 31685955840000 & 63371911680000 & 63371911680000 & 31685955840000 & 6337191168000 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

### 3.2. UPPER BOUNDS FOR $c_n(\alpha)$

We apply Proposition 3 to estimate the largest zero $x_n = c_n^2(\alpha)$ of the polynomial $R_n(x)$ from above,

$$x_n \le u_k(R_n) = p_k(R_n)^{1/k}, \qquad k = 3,\,4,\,5,\,6\,.$$

Then with the assistance of computer algebra we obtain a further estimation of the form

$$u_k(R_n) \le c^{1/k}\,(n+1)(n+\sigma(\alpha+1)),$$

with the optimal (i.e., the smallest possible) constants $c = c(k)$ and $\sigma = \sigma(k)$.

The algorithm is analogous to Algorithm 1, and the code has only a few differences which are specified later.

| **Algorithm 2** | Estimating $c_n(\alpha)$ from above |
|---|---|
| *Input:* | $k \in \{3,4,5,6\}$ – the number of the highest degree coefficients of $R_n(x)$ |
| *Step 1.* | Express the power sum $p_k(R_n)$ in terms of $\{b_i\}_{i=1}^k$ |
| *Step 2.* | Find $\{b_i\}_{i=1}^k$ in terms of $n$ and $\alpha$ using Proposition 2 |
| *Step 3.* | Find a proper value $\sigma$ for parameter $s$ in the expression $c\,(n+1)^k(n+s(\alpha+1))^k - p_k$, where $c$ is the coefficient of $n^{2k}$ in $p_k$ |
| *Step 4.* | Represent the numerator of $f = c\,(n+1)^k(n+\sigma(\alpha+1))^k - p_k$ in powers of $n$ and $(\alpha+1)$ |
| *Step 5.* | Estimate from below the expression $f$ to prove that $f \ge 0$ |

*Step 1:* The same as in Algorithm 1.

*Step 2:* Identical to that in Algorithm 1.

*Step 3:* The only differences with Algorithm 1 are that we set $c$ to be the coefficient of $n^{2k}$ in $p_k$ and

$$f_s = c\,(n+1)^k(n+s(\alpha+1))^k - p_k\,.$$

The highest order coefficients in $\sum_j \mu_{i,j}(s)(\alpha+1)^{d-j}$ are functions in $s$ of the form $A_i s^{\nu} - B_i$, with $A_i > 0$ and $B_i \geq 0$. We denote their non-negative zeros by $s_i$ for each $i$ and choose $\sigma = \max_i s_i$.

The results for $k = 3, 4, 5, 6$ obtained by symbolic computations are given in Table 2.

Table 2: The optimal values of $c$ and $\sigma$ in the upper bounds for $c_n^2(\alpha)$.

| $k$ | $c$ | $\sigma$ |
|---|---|---|
| 3 | $\dfrac{1}{(\alpha+1)^3(\alpha+3)(\alpha+5)}$ | $\dfrac{2}{5}$ |
| 4 | $\dfrac{5\alpha+17}{2(\alpha+1)^4(\alpha+3)^2(\alpha+5)(\alpha+7)}$ | $\dfrac{3}{7}$ |
| 5 | $\dfrac{(7\alpha+31)}{(\alpha+1)^5(\alpha+3)^2(\alpha+5)(\alpha+7)(\alpha+9)}$ | $\dfrac{4}{9}$ |
| 6 | $\dfrac{21\alpha^3+299\alpha^2+1391\alpha+2073}{(\alpha+1)^6(\alpha+3)^3(\alpha+5)^2(\alpha+7)(\alpha+9)(\alpha+11)}$ | $\dfrac{5}{11}$ |

*Step 4:* With $c$ and $\sigma$ determined in the previous Step 3 we set

$$f = c\,(n+1)^k(n+\sigma(\alpha+1))^k - p_k =: \frac{\varphi(n,\alpha)}{\psi(\alpha)}\,.$$

The rest of the source has no difference with Step 4 of Algorithm 1.

*Step 5:* The same as in Algorithm 1. Using the same recursive procedure we find a matrix $\mathbf{\Lambda}$ satisfying $\mathbf{0} \leq \mathbf{\Lambda} \leq \mathbf{M}$. Then $\varphi(n,\alpha) \geq 0$, $f \geq 0$ and therefore

$$c_n^{2k}(\alpha) \leq p_k \leq c\,(n+1)^k(n+\sigma(\alpha+1))^k$$

for the corresponding $c$ and $\sigma$ evaluated in Step 3. For $k = 3, 4, 5, 6$ we obtain estimations (1.11), (1.13), (1.15), and (1.17), respectively.

The matrices $\mathbf{M}$ from Step 4 and $\mathbf{\Lambda}$ from Step 5 obtained with *Mathematica* are given below.

Case $k = 3$:

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 1500 & 3300 \\ 0 & 115 & 1885 & 4170 & 4233 \\ 32 & 598 & 3026 & 6360 & 0 \\ 96 & 979 & 2143 & 850 & 0 \\ 96 & 624 & 1098 & 0 & 0 \end{pmatrix} \qquad \mathrm{M} = \begin{pmatrix} 0 & 0 & 0 & 1500 & 3300 \\ 0 & 115 & 1885 & 4170 & 4650 \\ 32 & 598 & 3026 & 6360 & -600 \\ 96 & 979 & 2143 & 1560 & -1950 \\ 96 & 624 & 1098 & -2130 & 0 \end{pmatrix}$$

Case $k=4:$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 905520 & 8808240 & 29717520 & 41571600 & 19756800 \\ 0 & 0 & 54390 & 2038890 & 16676660 & 60285680 & 115770830 & 117031110 & 48774600 \\ 0 & 42294 & 1237572 & 10966494 & 52723608 & 141477042 & 198565500 & 127823850 & 24194362 \\ 6075 & 266115 & 3694950 & 25364010 & 85166735 & 157047575 & 154257320 & 46893642 & 0 \\ 24300 & 617510 & 5700800 & 26734470 & 72437020 & 97039330 & 34815501 & 0 & 0 \\ 36450 & 678780 & 4979940 & 16392810 & 28823750 & 17907835 & 0 & 0 & 0 \\ 24300 & 360421 & 2131108 & 6792156 & 5246162 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 905520 & 8808240 & 29717520 & 41571600 & 19756800 \\ 0 & 0 & 54390 & 2038890 & 16676660 & 60285680 & 115770830 & 117031110 & 48774600 \\ 0 & 42294 & 1237572 & 10966494 & 52723608 & 141477042 & 198565500 & 127823850 & 27783000 \\ 6075 & 266115 & 3694950 & 25364010 & 85166735 & 157047575 & 154257320 & 52558380 & -11730600 \\ 24300 & 617510 & 5700800 & 26734470 & 72437020 & 97039330 & 38636640 & -18088350 & -10495800 \\ 36450 & 678780 & 4979940 & 16392810 & 28823750 & 20280800 & -12849340 & -18282390 & 0 \\ 24300 & 360421 & 2131108 & 6792156 & 5246162 & -9491857 & -9740850 & 0 & 0 \end{pmatrix}$$

Case $k=5:$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 85424220 & 1436596560 & 8988832440 & 26097558480 & 34662943980 & 16203045600 \\ 0 & 0 & 0 & 4261005 & 260814330 & 3617057430 & 22250151630 & 73071107235 & 134891273160 & 134642808090 & 56710659600 \\ 0 & 0 & 5436720 & 241567920 & 3235204800 & 22246774740 & 91003127400 & 223063050420 & 312360753600 & 222393230640 & 64812182400 \\ 0 & 1982358 & 88937982 & 1392482448 & 12340605438 & 63755213760 & 194677526736 & 357163148790 & 375802372260 & 186521488020 & 12638375568 \\ 200704 & 14563010 & 340432890 & 4020858058 & 25446365294 & 99455228208 & 241336266948 & 338611016520 & 235926284580 & 44541786567 & 0 \\ 1003520 & 42390775 & 693405300 & 6004806185 & 31876009900 & 96870254355 & 175080003840 & 176585507595 & 54286938720 & 0 & 0 \\ 2007040 & 63580160 & 829630410 & 5638883530 & 22495811450 & 57112266330 & 77686343280 & 30853075478 & 0 & 0 & 0 \\ 2007040 & 52428341 & 568553244 & 3375204826 & 9950248616 & 17535199185 & 13032227178 & 0 & 0 & 0 & 0 \\ 1003520 & 22758400 & 207566490 & 998218460 & 3486984100 & 3092469120 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 85424220 & 1436596560 & 8988832440 & 26097558480 & 34662943980 & 16203045600 \\ 0 & 0 & 0 & 4261005 & 260814330 & 3617057430 & 22250151630 & 73071107235 & 134891273160 & 134642808090 & 56710659600 \\ 0 & 0 & 5436720 & 241567920 & 3235204800 & 22246774740 & 91003127400 & 223063050420 & 312360753600 & 222393230640 & 64812182400 \\ 0 & 1982358 & 88937982 & 1392482448 & 12340605438 & 63755213760 & 194677526736 & 357163148790 & 375802372260 & 186521488020 & 16203045600 \\ 200704 & 14563010 & 340432890 & 4020858058 & 25446365294 & 99455228208 & 241336266948 & 338611016520 & 235926284580 & 51689001420 & -16203045600 \\ 1003520 & 42390775 & 693405300 & 6004806185 & 31876009900 & 96870254355 & 175080003840 & 176585507595 & 59214803760 & -31849915230 & -8101522800 \\ 2007040 & 63580160 & 829630410 & 5638883530 & 22495811450 & 57112266330 & 77686343280 & 32878980540 & -21278795580 & -19430795160 & 0 \\ 2007040 & 52428341 & 568553244 & 3375204826 & 9950248616 & 17535199185 & 14090589072 & -8987585040 & -16802648100 & 0 & 0 \\ 1003520 & 22758400 & 207566490 & 998218460 & 3486984100 & 3092469120 & -5291809470 & -5709701340 & 0 & 0 & 0 \end{pmatrix}$$

Case $k=6:$

$$\Lambda^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 137812500 & 826875000 & 2067187500 & 2756250000 & 2067187500 & 826875000 \\ 0 & 0 & 0 & 0 & 1712831340 & 16225847190 & 61013597190 & 121629375000 & 141073384270 & 96161625000 & 35900159390 \\ 0 & 0 & 0 & 7214978925 & 129414811020 & 682431963570 & 1909641192060 & 3176038395495 & 3233829060980 & 1990746645930 & 685515044860 \\ 0 & 0 & 11987252200 & 496609292025 & 3975634217280 & 15545520413640 & 34393961720958 & 48625624140345 & 44026671755710 & 24295459922370 & 7604368197232 \\ 0 & 6757992780 & 889711262440 & 13291579355670 & 71880656791680 & 218617776740580 & 407202449242428 & 488239798537050 & 391807489052700 & 198688598846220 & 53934381373052 \\ 0 & 678222070680 & 23583750724380 & 204179315141340 & 848078318563740 & 2083353346583670 & 3322865459747106 & 3448886831378940 & 2400975627767010 & 1108865878357650 & 280425349855044 \\ 156657528720 & 19589397975840 & 339201489248160 & 2064129233162535 & 6887056367079900 & 14142830057025810 & 19301411805553332 & 17606432763060855 & 10537484120843400 & 4243406630751540 & 1099649486098068 \\ 5737886375760 & 285634042298220 & 3098176511479280 & 14556440238205635 & 39911599747186200 & 69897865311080520 & 81762211401327198 & 65213552886377595 & 33832857309375390 & 11331273200216130 & 2850997375559272 \\ 90962828787600 & 2522414547688728 & 19367097472935520 & 73609280525046420 & 168732748048142520 & 255352902195144824 & 255297208204811052 & 174944618786085540 & 78706656458079300 & 21128915605090608 & 3776309107237628 \\ 821967442647120 & 14663875740671388 & 85966291726204120 & 270402692712289830 & 525000679384868280 & 675267348901207524 & 584508472246139364 & 337073370501014070 & 126378098307624240 & 25176310282435158 & 100310771384846 \\ 4683403445822640 & 58501440227708016 & 275002627721514000 & 724008553578199890 & 1198318687198670400 & 1310711831601226488 & 960008914211603088 & 452338940602722600 & 125459625652811660 & 10624247696162964 & 0 \\ 17541804701337840 & 163077338127380292 & 63513827175290600 & 1404602858405626110 & 1976258187012861840 & 1815905828323340796 & 1083067213944785244 & 386368902093765990 & 50117292233704841 & 0 & 0 \\ 43597315400007600 & 317712831386568156 & 1035759237211329400 & 1939380788446101000 & 2286621788179676640 & 1716811248944619828 & 767878312551389272 0 & 147265658296879789 & 0 & 0 & 0 \\ 70706025735594480 & 424476237961133820 & 1166286653916203820 & 1837004374966081860 & 1763385304898752920 & 1006399215292377060 & 256729802815313776 & 0 & 0 & 0 & 0 \\ 71282695085965440 & 370662526010533680 & 857715825988763280 & 1112754260403129960 & 820925242403999040 & 277604251541734810 & 0 & 0 & 0 & 0 & 0 \\ 40151827863100800 & 190176873072832800 & 371893024262944800 & 374764348674777600 & 176396296906922400 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9572830210944000 & 43261828837920000 & 72716691025440000 & 49617294906564000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{M}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 137812500 & 826875000 & 2067187500 & 2756250000 & 2067187500 & 826875000 \\ 0 & 0 & 0 & 0 & 1712831340 & 16225847190 & 61013597190 & 121629375000 & 141073384270 & 96161625000 & 35900159390 \\ 0 & 0 & 0 & 7214978925 & 129414811020 & 682431963570 & 1909641192060 & 3176038395495 & 3233829060980 & 1990746645930 & 685515044860 \\ 0 & 0 & 11987252200 & 496609292025 & 3975634217280 & 15545520413640 & 34393961720958 & 48625624140345 & 44026671755710 & 24295459922370 & 7604368197232 \\ 0 & 6757992780 & 889711262440 & 13291579355670 & 71880656791680 & 218617776740580 & 407202449242428 & 488239798537050 & 391807489052700 & 198688598846220 & 53934381373052 \\ 0 & 678222070680 & 23583750724380 & 204179315141340 & 848078318563740 & 2083353346583670 & 3322865459747106 & 3448886831378940 & 2400975627767010 & 1108865878357650 & 280425349855044 \\ 156657528720 & 19589397975840 & 339201489248160 & 2064129233162535 & 6887056367079900 & 14142830057025810 & 19301411805553332 & 17606432763060855 & 10537484120843400 & 4243406630751540 & 1099649486098068 \\ 5737886375760 & 285634042298220 & 3098176511479280 & 14556440238205635 & 39911599747186200 & 69897865311080520 & 81762211401327198 & 65213552886377595 & 33832857309375390 & 11331273200216130 & 2850997375559272 \\ 90962828787600 & 2522414547688728 & 19367097472935520 & 73609280525046420 & 168732748048142520 & 255352902195144824 & 255297208204811052 & 174944618786085540 & 78706656458079300 & 21128915605090608 & 3776309107237628 \\ 821967442647120 & 14663875740671388 & 85966291726204120 & 270402692712289830 & 525000679384868280 & 675267348901207524 & 584508472246139364 & 337073370501014070 & 126378098307624240 & 25176310282435158 & 100310771384846 \\ 4683403445822640 & 58501440227708016 & 275002627721514000 & 724008553578199890 & 1198318687198670400 & 1310711831601226488 & 960008914211603088 & 452338940602722600 & 125459625652811660 & 11652636597863226 & -6170333410201568 \\ 17541804701337840 & 163077338127380292 & 635138271752906000 & 1404602858405626110 & 1976258187012861840 & 1815905828323340796 & 1083067213944785244 & 386368902093765990 & 52601940894289780 & -13776737929843668 & -6786924207395784 \\ 43597315400007600 & 317712831386568156 & 1035759237211329400 & 1939380788446101000 & 2286621788179676640 & 1716811248944619828 & 767878312551389272 0 & 152704407947442120 & -28750678275282040 & -22906285043773824 & -2307796348667040 \\ 70706025735594480 & 424476237961133820 & 1166286653916203820 & 1837004374966081860 & 1763385304898752920 & 1006399215292377060 & 264555646816493340 & -39077291676501540 & -45675842674954800 & -9544747851001440 & 0 \\ 71282695085965440 & 370662526010533680 & 857715825988763280 & 1112754260403129960 & 820925242403999040 & 283521195216151200 & -25185070761515280 & -59085225024570360 & -16885936111968000 & 0 & 0 \\ 40151827863100800 & 190176873072832800 & 371893024262944800 & 374764348674777600 & 178350638317790400 & -40686454332210600 & -43598489040136800 & -14075574910689600 & 0 & 0 & 0 \\ 9572830210944000 & 43261828837920000 & 72716691025440000 & 49705079941440000 & 18409288867200000 & -13438780873056000 & -4602322216800000 & 0 & 0 & 0 & 0 \end{pmatrix}$$

## 4. CONCLUDING REMARKS

**1.** In our computer algebra approach for derivation of bounds for the best Markov constant $c_n(\alpha)$ we perform some optimization with respect to parameter $s$.

Our motivation for searching lower bounds for $c_n^2(\alpha)$ with a factor depending on $n$ of the special form $n\big(n+\sigma(\alpha+1)\big)$ is Corollary D(ii).

An interesting observation about the lower bounds $\underline{c}_{n,k}(\alpha)$ in Theorem 1 is that they imply

$$\frac{k\,n}{k+1} = \lim_{\alpha\to\infty} \alpha\, \underline{c}_{n,k}^2(\alpha) \le \lim_{\alpha\to\infty} \alpha\, c_n^2(\alpha)\,, \qquad 3\le k\le 6$$

(the lower bound in Corollary 2 follows from the case $k=6$). This observation and Proposition 3 give rise for the following

**Conjecture 1.** The best Markov constant $c_n(\alpha)$ satisfies:

$$\lim_{\alpha\to\infty} \alpha\, c_n^2(\alpha) = n\,.$$

We also performed a search for lower bounds for $c_n^2(\alpha)$ with a factor depending on $n$ of the form $(n+1)\big(n+\sigma(\alpha+1)\big)$. Such a choice is reasonable, as the resulting lower bounds preserve the limit relation in Corollary D (i). The optimal value then is $\sigma=-1/3$ (the same for all $k$, $3\le k\le 6$), and we obtain lower bounds as in Theorem 1 with $n\big(n+\sigma(\alpha+1)\big)$ replaced by $(n+1)\big(n-(\alpha+1)/3\big)$. These lower bounds make sense only for $n>(\alpha+1)/3$, and are better than those in Theorem 1 only for $\alpha$ close to $-1$.

**2.** The bounds $\big(\underline{c}_{n,k}(\alpha), \overline{c}_{n,k}(\alpha)\big)$ $(3\le k\le 6)$ in Theorem 1 imply bounds $(\ell_k(\alpha), u_k(\alpha))$ (occurring in the middle columns of Tables 1 and 2) for the asymptotic Markov constant $c(\alpha)$, and the bounds deduced with a larger $k$ are superior. While the lower bounds $\ell_k(\alpha)$ are of the correct order $\mathcal{O}(\alpha^{-1})$ as $\alpha\to\infty$, for the upper bound $u_k(\alpha)$ we have $u_k(\alpha)=\mathcal{O}(\alpha^{-1+\frac{1}{2k}})$ as $\alpha\to\infty$, $(3\le k\le 6)$. The ratio

$$\rho_k(\alpha) := \frac{u_k(\alpha)}{\ell_k(\alpha)}\,, \qquad 3\le k\le 6,$$

tends to 1 as $\alpha\to -1$, which indicates that for moderate $\alpha$ the bounds $\ell_k(\alpha)$ and $u_k(\alpha)$ are rather tight. This observation is clearly seen in the particular case $\alpha=0$, where, according to Turán's result, we have $c(0)=\frac{2}{\pi}$. We give the lower and the upper bounds for $c(0)$ and the overestimation factors in Table 3.

**3.** Another interesting observation, concerning the coefficients of $R_n$ inspires the following

**Conjecture 2.** For every fixed $k\in\mathbb{N}$, the coefficient $b_{k,n}$, $n>k$, of the polynomial $R_n(x)=x^n-b_{1,n}\,x^{n-1}+b_{2,n}\,x^{n-2}-\cdots+(-1)^n\,b_{n,n}$, satisfies

$$b_{k,n} = \frac{n^{2k}}{2^k\,k!(\alpha+1)\cdots(\alpha+2k-1)} + \mathcal{O}(n^{2k-1})\,. \tag{4.1}$$

Conjecture 2 is verified with our computer algebra approach for $1\le k\le 6$, but so far we do not have a proof for the general case. Having (4.1) proved, we could try to find the explicit form of $d_k$, the coefficient of $n^{2k}$ in Newton's function $p_k(R_n)$, and consequently to obtain two sequences $\{\ell_k\}$ and $\{u_k\}$ defined by $\ell_k = \sqrt{d_k/d_{k-1}}$ and $u_k = \sqrt[2k]{d_k}$ which converge monotonically from below and from above, respectively, to $c(\alpha)$, the sharp asymptotic Markov constant.

Table 3: The lower and the upper bounds for the asymptotic Markov constant $c(0)$ and the overestimation factors.

| $k$ | $\ell_k(0)$ | $u_k(0)$ | $\dfrac{c(0)}{\ell_k(0)}$ | $\dfrac{u_k(0)}{c(0)}$ |
|---|---|---|---|---|
| 3 | $\sqrt{\frac{2}{5}} \approx 0.63245553$ | $\sqrt[6]{\frac{1}{15}} \approx 0.63677321$ | 1.006584242 | 1.00024103 |
| 4 | $\sqrt{\frac{17}{42}} \approx 0.63620901$ | $\sqrt[8]{\frac{17}{630}} \approx 0.63663212$ | 1.00064564 | 1.00001939 |
| 5 | $\sqrt{\frac{62}{153}} \approx 0.63657580$ | $\sqrt[10]{\frac{31}{2835}} \approx 0.63662085$ | 1.00006906 | 1.00000170 |
| 6 | $\sqrt{\frac{2073}{5115}} \approx 0.63661494$ | $\sqrt[12]{\frac{2073}{467775}} \approx 0.63661987$ | 1.00000757 | 1.00000015 |

Although the ratios $\rho_k$, $3 \le k \le 6$, satisfy $\rho_k(\alpha) \to \infty$ as $\alpha \to \infty$, they grow rather slowly. For instance, $\rho_6(\alpha) < 2$ for $\alpha < 140000$, see Figure 1.

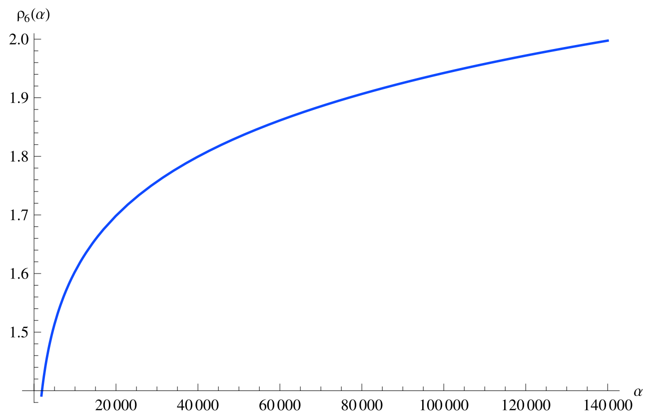


Figure 1: The graph of $\rho_6(\alpha) < 2$.

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Geno Nikolov, Rumen Uluchev

Department of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mails: geno@fmi.uni-sofia.bg
rumenu@fmi.uni-sofia.bg