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ON SOME APPLICATIONS OF THE SINGULARITY METHOD

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Запрян Запрянов, Николай Марков. О НЕКОТОРЫХ ПРИМЕНЕНИЯХ МЕТОДА ОСОБЕННОСТЕЙ

Гидродинамика при малых числах Рейнольдса играет важную роль в исследованиях механики суспензий, колоидной химии и физиологии мембран. В статье методом особенностей исследуются некоторые стационарные вязкие течения при наличие твёрдых или жидких частиц. Течения представляются фундаментальными решениями уравнений Стокса в виде стокслетов, вырожденных квадрополов, стоксонов и других мультиполов. Рассмотрены следующие задачи: 1) течение, порожденное трансляцией и ротацией твёрдой сферической частицы; 2) трансляция сферической капли в неподвижной вязкой частице, обтекаемой линейным градиентным потоком. Обсуждаются построение решений этих задач и основные характеристики такого вида решений.

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The hydrodynamics of low Reynolds number flows plays an important role in the study of suspension machanics, colloid science and membrane physiology. In the present paper once again (now via the singularity method) some steady viscous flows in the presence of rigid or fluid particles are examined. The flows are represented in terms of fundamental solutions to the governing Stokes equations, including Stokeslets, degenerated quadrupoles, Stokesons and some other multipoles. The problems considered are: 1) flow due to the translation or rotation of a rigid spherical particle; 2) a translating spherical drop in a viscous quiescent fluid; 3) small deformations of a fluid particle in a general linear flow.

The construction of the solutions of these problems and the salient features of such kind of solutions are discussed.

1. INTRODUCTION

An approximate solution to the Navier — Stokes equations can be obtained for the case in which the Reynolds number, or the ratio of inertial to viscous forces, is very small. Then the inertial effects can be neglected, and the action of viscosity is considered to be controlling. We can imagine that the Reynolds number $\mathrm{Re} = \dfrac{LU\rho}{\mu}$ is small either because the fluid is very viscous ($\mu \to \infty$) or because the inertia, or density, is very small ($\rho \to 0$). These flows are frequently called "creeping" flows. This simplification is justified since many multiparticle systems do involve sufficiently slow motions for this assumption to be valid.

The creeping flow and continuity equations are

$$\nabla.T = \mu\nabla^2\vec{v} - \nabla p + \vec{F}^{(c)}\delta(\vec{r}) = 0, \tag{1.1}$$

$$\nabla.\vec{v} = 0. \tag{1.2}$$

Here $\vec{F}^{(c)}$ denotes a point force applied to the fluid at $\vec{r} = 0$, $\delta(\vec{r})$ is the Dirac delta function, $\vec{v}$ is the velocity, $\vec{p}$ is the pressure and T is the stress tensor. The meaning of the first equation is:

i) For $\vec{r} \neq 0$, $\nabla.T = 0$;

ii) For any volume V that encloses the point $\vec{r} = 0$,

$$\iiint_V \nabla.T d\tau = -\vec{F}^{(c)}.$$

The linearity of the creeping flow equations allows the creation of a class of solution methods that is readily applied to various types of hydrodymanic problems. These methods for solving Stokes equations are based upon fundamental solutions, corresponding to the flow produced by a point force in a fluid space. If the boundary shapes of the problem under consideration are simple, then an analytic solution can be achieved by using the internal distributions of force and force multipole singularities.

In this connection it is useful to write down some of the derivatives of the Ossen — Burger's tensor

$$B_{ij} = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \tag{1.3}$$

and degenerate quadrupole $\nabla^2 B_{ij}$:

i) The first derivative of the Ossen — Burger's tensor (the Stokes dipole)

$$\nabla B_{ij} = \frac{\partial B_{ij}}{\partial x_k} = B_{ij,k} = \frac{1}{r^3}(-\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ki}x_j) - \frac{3}{r^5}x_i x_j x_k; \tag{1.4}$$

ii) The second derivative of the Ossen — Burger's tensor

$$\frac{\partial^2 B_{ij}}{\partial x_k \partial x_k} = B_{ij,kk} = -\frac{\delta_{ij}}{r^3} + \frac{3\delta_{ij}}{r^3} + \frac{2}{r^3}\delta_{ik}\delta_{jk} - \frac{3}{r^5}(\delta_{ik}x_j + \delta_{jk}x_i)x_k \tag{1.5}$$

$$-\frac{3}{r^5}(\delta_{ik}x_jx_k+\delta_{jk}x_kx_i+x_ix_j)+\frac{15}{r^5}x_ix_j;$$

iii) The degenerate quadrupole

$$\nabla^2 B_{ij}=\nabla.(\nabla B_{ij})=B_{ij,ii}=\frac{2\delta_{ij}}{r^3}-\frac{6}{r^5}x_ix_j; \tag{1.6}$$

iv) The derivative of the degenerate quadrupole (the degenerate octupole)

$$\nabla(\nabla^2 B_{ij})=\nabla^2 B_{ij,k}=-\frac{6}{r^5}(\delta_{ij}x_k+\delta_{jk}x_i+\delta_{ki}x_j)+\frac{30}{r^7}x_ix_jx_k. \tag{1.7}$$

(Here we shall note that in order to obtain (1.3)–(1.7) one has to use the following formulas:

$$\delta_{ij}x_i=x_j,\quad \delta_{ii}=3,\quad \frac{\partial x_i}{\partial x_j}=\delta_{ij},\quad x_ix_i=r^2,\quad \frac{\partial r}{\partial x_i}=\frac{x_i}{r}.)$$

For the general particle shape the multipole expansion representation [1–3] requires an infinite number of terms. If the particle shape is simple the multipole expansion representation may contain only a finite number of terms. For the case of a spherical particle, for example, the multipole expansion contains (as we shall see in section 2) only two terms. For the other simple shapes like ellipsoid a truncated expansion in just the lower order singularities is possible, provided that these singularities are distributed over a region. The other example is the solution for very elongated, slender particles where the integral representation can approximately be reduced to a line distribution of point forces along the centrē line of the particle.

For interior flows (like the drop inside a flow) the velocity field of the Stokeson

$$v_i=H_{ij}U_j=(2r^2\delta_{ij}-x_ix_j)U_j \tag{1.8}$$

is used. Since the Stokeson is a linear with respect to a constant vector $\vec{U}$ (as we shall see in section 3), it enters into the solution for a translating drop. In view of the fact that the Stokeson is quadratic in r, its gradient is linear in r.

Other interior solutions are roton and stresson which are equal to the symmetric and antisymmetric derivatives of the Stokeson, respectively. Since the roton and stresson correspond to a rigid body rotation and a constant rate of strain field (which are not typical for fluid flows) they are used rarely.

A knowledge of the flow in and around a droplet submerged in an unbounded or bounded fluid is of considerable practicle interest. The submerged of the basic equation subject to the boundary conditions for such type of problems has to yield explicitly the flow fields interior to the droplet and exterior to it, and the general equation of the interface. However, the mathematical treatment of solving simultaneously the flow fields and the equation of the interface is excessively difficult. That is why an iterative procedure was adopted by Taylor [4].

First, the drop is postulated to be spherical and the flow fields are determined using the boundary conditions of continuity of the tangential velocity vectors, vanishing of the normal component of the velocity vectors, and continuity of the tangential components of the stress vectors inside and outside of the spherical drop.

Later, the function describing the deviation of the droplet from sphericity is determined using the relation between the outside and inside values of the normal components of the stress vectors. The newly determined interface may then be used for calculating the flow fields of the second iteration and so on.

2. FLOW DUE TO THE TRANSLATION OR ROTATION OF A RIGID SPHERICAL PARTICLE

The slow translation of a rigid spherical particle of radius a through a quiescent viscous fluid induces a flow which can be found by means of the singularity method. Since this flow produces a net force on the spherical particle in order to construct a solution via internal singularities, we require a Stokeslet $B.\vec{F}^{(c)}$ located at the sphere centre. However, the Stokeslet is most often accompanied by the degenerate quadrupole. Thus, we suggest trying to construct a solution that is a superposition of a Stokeslet and a degenerate quadrupole both located at the centre of the spherical particle, i. e.

$$\vec{v}(\vec{r}) = \vec{p}.B(\vec{r}) + \vec{q}\nabla^2 B(\vec{r}). \tag{2.1}$$

Since each term in this expression satisfies the creeping flow equations (1.1) and (1.2), further we shall try to determine the unknown vectors $\vec{p}$ and $\vec{q}$ from the following boundary conditions:

$$\vec{v} = \vec{U} \quad \text{at} \quad r = a, \tag{2.2.}$$

$$\vec{v} \to 0 \quad \text{as} \quad r \to \infty, \tag{2.3}$$

where $\vec{U}$ is the velocity of the particle.

In fact, since $B(\vec{r})$ and $\nabla^2 B(\vec{r})$ tend to zero as $r \to \infty$, the boundary condition (2.3) is fulfilled automatically. If we succeed to do this then from the uniqueness theorem (cited in section 1) it follows that we have found the solution of the considered problem.

Introducing in (2.1) the explicit forms of the singularities we obtain

$$\vec{v}(\vec{r}) = \vec{p}.\left(\frac{I}{r} + \frac{\vec{r}\vec{r}}{r^3}\right) + \vec{q}.\left(-\frac{2I}{r^3} + \frac{6\vec{r}\vec{r}}{r^5}\right). \tag{2.4}$$

If $\vec{n}$ is the unit normal vector to the spherical particle surface, we have $\vec{r} = a\vec{n}$ at $r = a$ and the boundary condition (2.3) gives

$$\vec{U} = \vec{p}.\left(\frac{I}{a} + \frac{\vec{n}\vec{n}}{a}\right) + \vec{q}.\left(\frac{2I}{a^3} - 6\frac{\vec{n}\vec{n}}{a^3}\right)$$

or

$$a^3\vec{U} = a^2\vec{p} + 2\vec{q} + (a^2\vec{p} - 6\vec{q}).\vec{n}\vec{n}. \tag{2.5}$$

Since the problem considered is linear it is reasonable to assume that the unknown vectors $\vec{p}$ and $\vec{q}$ in the equation (2.1) are expressed linearly via the particle velocity $\vec{U}$, i. e.

$$\vec{p} = C_0\vec{U}, \quad \vec{q} = C_2\vec{U}, \tag{2.6}$$

where C_0 and C_2 are unknown constants. With (2.6) the equation (2.5) becomes

$$(-a^3 + C_0 a^2 + 2C_2)\vec{U} + (a^2 C_0 - 6C_2)\vec{U}.\vec{n}\vec{n} = 0$$

or

$$(-a^3 + C_0 a^2 + 2C_2)\vec{U} + (a^2 C_0 - 6C_2)U_n\vec{n} = 0. \tag{2.7}$$

Taking into account that $\vec{U}$ and $\vec{n}$ are independent vectors we obtain the following system for the constants C_0 and C_2:

$$\left| \begin{array}{l} a^2 C_0 + 2C_2 = a^3 \\ a^2 C_0 - 6C_2 = 0. \end{array} \right.$$

Hence $C_0 = \dfrac{3}{4}a$ and $C_2 = \dfrac{a^3}{8}$, and (2.1) becomes

$$\begin{aligned} \vec{v}(\vec{r}) &= \frac{3a}{4}\vec{U}.B(\vec{r}) + \frac{a^3}{8}\vec{U}.\nabla^2 B(\vec{r}) \\ &= 6\pi\mu a\vec{U}.\left(1 + \frac{a^2}{6}\nabla^2\right)\frac{B(\vec{r})}{8\pi\mu}. \end{aligned} \tag{2.8}$$

It is easy to show that this expression is identical to the standard result in spherical co-ordinates given in elementary books on fluid mechanics. (See for example [5].)

Therefore, indeed the translating spherical particle in a Stokes flow requires a degenerate quadrupole, in addition to a monopole of strength $6\pi\mu aU$. Of special interest is the fact that we have derived the Stokes law, $\vec{F} = -6\pi\mu a\vec{U}$, for the drag on the spherical particle undergoing a steady translation, without an explicit computation of the surface stress vector $\vec{t}_n = T.\vec{n}|_{r=a}$. Here we have used the statement that the solutions expressed as a multipole expansion yield quantities of interest, such as the hydrodynamic force, in a straight-forward fashion.

Now let us consider the flow due to a rotating spherical particle through an unbounded quiescent viscous fluid. We suppose that the spherical particle rotates with an angular velocity $\vec{\omega}$ and that the radius of the sphere is equal to a. If we take a cartesian co-ordinate system with an origin that coincides with the sphere centre then the boundary conditions of the problem considered are as follows:

$$\vec{v} \to 0 \quad \text{as} \quad r \to \infty, \tag{2.9}$$

$$\vec{v} = \vec{\omega} \times a\vec{e}_r \quad \text{at} \quad r = a, \tag{2.10}$$

where $\vec{e}_r$ is a unit vector in direction $\vec{r}$. The equation (2.10) suggests that the produced flow may be represented merely in terms of a rotlet (couplet) with strength $C_1\vec{\omega}$, located at the centre of the sphere, i. e.

$$\vec{v} = C_1\vec{\omega} \times \frac{\vec{r}}{r^3}. \tag{2.11}$$

With boundary condition (2.10) the velocity field (2.11) gives

$$\vec{\omega} \times a\vec{e}_r = C_1\omega \times \frac{a\vec{e}_r}{a^3},$$

whence $C_1 = a^3$. Therefore

$$\vec{v} = \frac{a^3}{r^3}(\vec{\omega} \times \vec{r}). \tag{2.12}$$

Using (2.12) we can calculate the torque acting on the particle:

$$\vec{M} = \iint_S \vec{r} \times (T.\vec{n})d\sigma = -8\pi\mu a^3\vec{\omega}.$$

3. A TRANSLATING SPHERICAL DROP IN A VISCOUS QUIESCENT FLUID

Consider a spherical drop moving slowly with velocity $\vec{U}$ in a viscous quiescent fluid. We suppose that the fluids both outside and inside the drop surface are immiscible, and that the surface tension, σ, at the interface is sufficiently strong to keep the drop approximately spherical against any deforming effect of viscous forces. It is also assumed that the Reynolds number of the motion within the drop is small compared with the unity, like that of the motion outside the drop. The two fluid motions are discribed by the equations (1.1) and (1.2) with different values of the viscosity — μ and $\hat{\mu}$ (here the caret indicates a quantity relating to the internal fluid and its motions).

We choose the origin of the co-ordinate system to be at the instantaneous position of the centre of the drop with radius a. The velocity $\vec{v}$ and the difference $p - p_\infty$ must vanish at infinity, and $\vec{\hat{v}}$ and $\hat{p} - \hat{p}_o$ are finite everywhere within the fluid particle.

The boundary conditions at the interface of the droplet, $r = a$, are as follows:

(i) Vanishing of the normal component of the velocity vectors:

$$(\vec{v} - \vec{U}).\vec{n} = 0, \quad (\vec{\hat{v}} - \vec{U}).\vec{n} = 0;$$

(ii) Continuity of the tangential velocity vectors:

$$\vec{v}.(I - \vec{n}\vec{n}) = \vec{\hat{v}}.(I - \vec{n}\vec{n});$$

(iii) Continuity of the tangential components of the stress vectors:

$$(T.\vec{n}).(I - \vec{n}\vec{n}) = (\hat{T}.\vec{n}).(I - \vec{n}\vec{n}).$$

With the equation

$$T.\vec{n} = -p\vec{n} + 2\mu E.\vec{n}$$

the equation (iii) has the form

$$(E.\vec{n}).(I - \vec{n}\vec{n}) = \lambda(\hat{E}.\vec{n}).(I - \vec{n}\vec{n}), \tag{3.1}$$

where $\lambda = \frac{\hat{\mu}}{\mu}$, the ratio of the drop and the solvent viscosities, respectively. A cursory inspection of the menu of the available singularities results in the following selections: a Stokeslet and a degenerate quadrupole outside the drop, and a Stokeson and a uniform field $\vec{U}$ inside the drop. Then the velocity fields outside and inside the drop can be written, respectively, as

$$\vec{V} = \frac{3a}{4}\vec{U}.(C_0 + C_2 a^2 \nabla^2) B(\vec{r}), \tag{3.2}$$

$$\hat{\vec{v}} = D_0 \vec{U} + D_2 a^{-2} \vec{U}.H(\vec{r}), \tag{3.3}$$

where the four unknown constants C_0, C_2, D_0 and D_2 are determined from the boundary conditions at the drop interface.

Taking into account formulas (1.3), (1.5), from (3.2) we obtain

$$\begin{aligned} v_i &= C_0 \frac{3a}{4} B_{ij} U_j + C_2 \frac{3}{4} a^3 \nabla^2 B_{ij} U_j \\ &= C_0 \frac{3a}{4} \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) U_j + C_2 \frac{3}{4} a^3 \left(\frac{2\delta_{ij}}{r^3} - \frac{6 x_i x_j}{r^3} \right) U_j. \end{aligned} \tag{3.4}$$

Since at $r = a$ we have $x_i = a n_i$, the equation (3.4) gives

$$v_i = \frac{3}{2} U_i \left(\frac{C_0}{2} + C_2 \right) + \frac{3}{2} n_i n_j U_j \left(\frac{C_0}{2} - 3C_2 \right) \tag{3.5}$$

and thus

$$\begin{aligned} v_i n_i|_{r=a} &= \frac{3}{2} n_i U_i \left(\frac{C_0}{2} + C_2 \right) + \frac{3}{2} n_i n_i n_j U_j \left(\frac{C_0}{2} - 3C_2 \right) \\ &= n_i U_i \left(\frac{3}{2} C_0 - 3C_2 \right). \end{aligned} \tag{3.6}$$

Therefore, from the first kinematic condition (i) we find

$$3C_0 - 6C_2 = 2. \tag{3.7}$$

Reverting to (3.4) we obtain

$$\hat{v}_i = D_0 U_i + D_2 \frac{1}{a^2} (2r^2 \delta_{ij} - x_i x_j) U_j;$$

$$\hat{v}_j|_{r=a} = (D_0 + 2D_2) U_i - n_i n_j U_j D_2,$$

and thus

$$\hat{v}_i n_i|_{r=a} = (D_0 + D_2) n_i U_i. \tag{3.8}$$

It follows from the second kinematic condition (i) that

$$D_0 + D_2 = 1. \tag{3.9}$$

The condition of continuity of the velocity at the surface of the drop (ii) requires to accomplish the following computations:

$$(I - \vec{n}\vec{n}).\vec{v}|_i = (\vec{v} - \vec{n}\vec{n}.\vec{v})_i = \left(\frac{3}{4} C_0 + \frac{3}{2} C_2 \right) (U_i - n_i n_j U_j)$$

and

$$(I - \vec{n}\vec{n}), \vec{v}|_i = (\vec{v} - \vec{n}\vec{n}.\vec{v})_i = (D_0 + 2D_2)(U_i - n_i n_j U_j).$$

Substituting these expressions into (ii) yields

$$3C_0 + 6C_2 = 4D_0 + 8D_2. \tag{3.10}$$

In order to apply the boundary condition (iii) we have need of the calculation of the rate of stress tensors E and $\hat{E}$. From (3.2) and (3.3) it follows that

$$\begin{aligned} e_{ij}|_{r=a} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{3}{8} a C_0 (B_{ik,j} + B_{jk,i}) U_k \\ &+ \frac{3}{8} a^3 C_2 (\nabla^2 B_{ik,j} + \nabla^2 B_{jk,i}) U_k, \end{aligned} \tag{3.11}$$

$$\hat{e}_{ij}|_{r=a} = \frac{1}{2}\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i}\right)|_{r=a} = \frac{D_2}{2a}(3n_i U_j - 3n_j U_i - 2\delta_{ij} n_k U_k). \tag{3.12}$$

Using (1.5) it is easy to obtain the formula

$$(\nabla^2 B_{ik,j} + \nabla^2 B_{jk,i})|_{r=a} = -\frac{12}{a^4}(\delta_{jk} n_i + \delta_{ki} n_j + \delta_{ij} n_k) + \frac{60}{a^4} n_i n_j n_k,$$

and thus

$$\begin{aligned} e_{ij}|_{r=a} &= \frac{3}{4}\frac{C_0}{a} n_k U_k (\delta_{ij} - 3n_i n_j) + \frac{9}{2}\frac{C_2}{a} n_k U_k (5n_i n_j + \delta_{ij}) \\ &- \frac{9}{2a} C_2 (n_i U_j + n_j U_i) \end{aligned}$$

and

$$e_{ij} n_j n_i|_{r=a} = \frac{n_k U_k}{a}\left(9C_2 - \frac{3}{2} C_0\right).$$

Hence

$$\begin{aligned} &(I - \vec{n}\vec{n}).(E.\vec{n})|_i = E.\vec{n} - \vec{n}\vec{n}.(E.\vec{n})|_i \\ &= -\frac{9}{2a} C_2 U_i + \frac{9C_2}{2a} n_i n_k U_k = \frac{9C_2}{2a}(n_i n_k U_k - U_i) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} (I - \vec{n}\vec{n}).(\hat{E}.\vec{n})|_i &= \frac{D_2}{2a}(n_i n_k U_k + 3U_i) - \frac{2D_2}{a} n_i n_k U_k \\ &= \frac{3}{2}\frac{D_2}{a}(n_i n_k U_k - U_i). \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (ii) we find the fourth equation for the constants C_0, C_2, D_0 and D_2:

$$\lambda D_2 + 3C_2 = 0. \tag{3.15}$$

Solving the system (3.7), (3.9), (3.10) and (3.15) for the coefficients C_0, C_2, D_0 and D_2 we find

$$C_0 = \frac{2+3\lambda}{3(1+\lambda)}, \quad C_2 = \frac{\lambda}{6(1+\lambda)},$$

$$D_0 = \frac{3+2\lambda}{2(1+\lambda)}, \quad D_2 = -\frac{1}{2(1+\lambda)}.$$

Consequently the equations (3.2) and (3.3) have the form

$$\vec{v} = 2\pi\mu a \frac{2+3\lambda}{1+\lambda}\left[1 + \frac{\lambda a^2 \nabla^2}{2(2+3\lambda)}\right] \frac{B(\vec{r})}{8\pi\mu}.\vec{U}, \tag{3.16}$$

$$\hat{\vec{v}} = \frac{3+2\lambda}{2(1+\lambda)}\vec{U} - \frac{1}{2(1+\lambda)}\frac{1}{a^2}\vec{U}.H(\vec{r}). \tag{3.17}$$

The notable feature of (3.16) is that in the limit as $\lambda \to \infty$, C_0 and C_2 assume the values for the rigid particle given previously in (2.8) while in the limit as $\lambda \to 0$ the degenerate quadrupole vanishes and the Stokeslet alone provides the exact solution for a translating bubble.

In order to calculate the force on the fluid particle we have to integrate the surface stress vector over the drop surface:

$$\vec{F} = \iint_S (T.\vec{n})d\sigma = 4\pi\mu a\vec{U}\left(\frac{3\lambda+2}{2(\lambda+1)}\right). \tag{3.18}$$

In the limiting case, $\lambda \to \infty$, this expression for the drag becomes $\vec{F} = 6\pi\mu a\vec{U}$, which is simply the Stokes' law for the drag on a rigid spherical particle. In the limit $\lambda \to 0$ the expression (3.18) becomes

$$\vec{F} = 4\pi\mu a\vec{U}, \tag{3.19}$$

which is the drag on a spherical bubble at Re $\ll 1$.

4. SMALL DEFORMATIONS OF A FLUID PARTICLE IN A GENERAL LINEAR FLOW

In section 3 we solved the problem of a spherical drop of radius a in uniform flow at zero Reynolds number. It is of interest to compute the small deformations of a drop in a general linear flow at zero Reynolds number.

We assume that at large distances from the fluid particle the fluid undergoes a general linear flow

$$\vec{v}^{\infty} = G(E^{\infty} + \Omega^{\infty}).\vec{r}', \quad r' \to \infty, \tag{4.1}$$

where $\vec{r}'$ is a position vector, $e_{ij}^{\infty} = \text{const}$ and $\Omega_{ij}^{\infty} = \text{const}$ $(i, j = 1, 2, 3)$ are the dimensionless rates of strain and vorticity tensors, respectively, and G represents the magnitude of $\nabla\vec{v}$. We know from section 3 that a drop will deform in almost any viscous flow, other than in uniform (steady) translation through a stationary fluid. With characteristic velocity $V_c = Ga$ the dimensionless form of the equation (4.1) is

$$\vec{v}^{\infty} = (E^{\infty} + \Omega^{\infty}).\vec{r}, \quad r \to \infty. \tag{4.2}$$

We assume that the density of the fluid inside the drop is equal to that of the ambient fluid but the viscosities of the two fluids $\hat{\mu}$ and μ are different (here also the physical parameters pertaining to the interior of the drop will be distinguished from the corresponding exterior parameters by a caret). Further we assume that the surface tension σ is constant on the drop surface and the capillary number $\mathrm{Ca} = \dfrac{\mu V_c}{\sigma}$ is small, i. e. $\mathrm{Ca} \ll 1$.

The magnitude of the drop deformation depends on the capillary number and the ratio of the internal and external viscosity $\lambda \doteq \dfrac{\hat{\mu}}{\mu}$. For $\mathrm{Ca} \ll 1$ very small deviations from a spherical shape are possible. Taking into account this fact we express the drop shape in the form

$$F(\vec{r}, t) \equiv r - [1 + \mathrm{Ca} f(\vec{r})] \equiv 0. \tag{4.3}$$

So we assume that the deviation from sphericity is contained in the function $f(\vec{r})$ and that the magnitude of this deviation is proportional to Ca. It is also assumed that the Reynolds number of the motion within the drop is small compared with unit like that of the motion outside the drop.

In general, as one is solving for the motion inside and outside of the drop one has to determine the shape of the drop. It should be emphasized that the shape of a neutrally buoyant immiscible liquid drop immersed in a continuous liquid undergoing shear is not governed solely by the bulk and interfacial properties of the two phases, but also depends upon the rate of the shear. Despite the linearity of the Stokes equations governing the flow both inside and outside of the drop, in general the determination of the drop surface equation constitutes a non-linear problem, owing to the fact that the unknown shape has to be calculated simultaneously along with the solution of the equations of motion. In consequence of this non-linearity, the droplet shape has not yet been found in its full generality, but rather only for small departures from the spherical form. For small values of capillary number Ca the boundary conditions on the drop surface could be linearized about the boundary conditions for an exactly spherical drop and we shall see that for the above problem an approximate analytic solution could be obtained. The two fluid motions are described by the equations (1.1) and (1.2) with different values of the viscosity — μ and $\hat{\mu}$. We choose the origin of the co-ordinate system to be at the instantaneous position of the centre of the drop.

The boundary conditions in a dimensionless form at the interface of the drop are as follows:

(i) vanishing of the normal components of the velocity vectors, i. e.

$$\vec{v}.\vec{n} = 0, \tag{4.4}$$

$$\vec{\hat{v}}.\vec{n} = 0; \tag{4.5}$$

(ii) continuity of the tangential velocity vectors, i. e.

$$\vec{v}.(I-\vec{n}\vec{n})=\hat{\vec{v}}.(I-\vec{n}.\vec{n}); \tag{4.6}$$

(iii) continuity of the tangential components of the stress vectors

$$(E.\vec{n}).(I-\vec{n}\vec{n})=\lambda(\hat{E}.\vec{n}).(I-\vec{n}\vec{n}); \tag{4.7}$$

(iv) relation between the outside and the inside values of the normal components of the stress vectors

$$(T.\vec{n}-\lambda\hat{T}.\vec{n}).\vec{n}=\frac{\nabla_s.\vec{n}}{\mathrm{Ca}}. \tag{4.8}$$

Here ∇_s is acting over the drop surface and $\vec{n}$ is the outer normal. In addition, there is a boundary condition at infinity, namely (4.2), and the requirement that the solution be finite everywhere.

Applying the method of domain pertubations (described in section 1) we first postulate that the drop is spherical with radius "a", and the fields inside and outside it are solved, using only the equations (4.2) and (4.4)–(4.7).

Later the function $f(\vec{r})$ describing the deviation of the droplet from sphericity is determined using the boundary condition (4.8). Therefore, the solution presented herein should be considered as a first approximation of a much more complex problem [6].

If we consider the disturbance flow $\vec{v}'$ due to the drop, then the total flow is $\vec{v}=\vec{v}^{\infty}+\vec{v}'$ and $\vec{v}'\to\infty$ as $r\to\infty$. Inspecting the functional form of the various singularities presented in section 1 we decided to represent the disturbance velocity field outside of the drop in terms of a Stokeslet dipole and a degenerate octupole:

$$\vec{v}'=2\pi\mu a^3(E^{\infty}.\nabla):(C_1+C_2a^2\nabla^2)\frac{B(\vec{r})}{8\pi\mu}, \tag{4.9}$$

where C_1 and C_2 are unknown constants.

Further we shall prove that in order to model correctly the flow inside the drop we have to use the velocity field

$$\hat{\vec{v}}=d_1E^{\infty}.\vec{r}+d_2\vec{\omega}^{\infty}\times\vec{r}+d_3[5r^2(E^{\infty}.\vec{r})-2\vec{r}(\vec{r}.E^{\infty}.\vec{r})], \tag{4.10}$$

where d_1, d_2 and d_3 are specified by the boundary conditions.

Since the Stokeson $H=2r^2I-\vec{r}\vec{r}$ is quadratic in r the gradient of H is a third-order tensor that is linear in r. It turned out that the symmetric and antisymmetric derivatives (which are known as roton and stresson) can be used as "building materials" in the construction of the interior flow field for a spherical drop in the linear field (4.2). That is so because roton and stresson simply correspond to a rigid body rotation and a constant rate of strain field. In (4.9) we also use a cubic field which is the less obvious portion of the solution. It should be emphasized that in order to avoid any singularity at the origin of the co-ordinate system we must use growing harmonics inside the drop.

The cubic field can be obtained by the appropriate linear combination of $E^{\infty}:\vec{r}\vec{r}\vec{r}$ and $r^2E^{\infty}.\vec{r}$. If we seek the velocity field $\hat{\vec{v}}$ inside the drop in the form

$$\hat{\vec{v}}=d_1E^{\infty}.\vec{r}+d_2\vec{\omega}\times\vec{r}+\hat{\vec{v}}',$$

then

$$\vec{v}\,' = AE^\infty : \vec{r}\vec{r}\vec{r} + Br^2 E^\infty . \vec{r},$$

where A and B are unknown constants, but it is easy to show that $A = -\frac{2}{5}B$. From the equation of continuity (4.2) it follows that

$$\nabla . \vec{v}\,' = 5Ae^\infty_{kl}x_k x_l + 2Be^\infty_{kl}x_k x_l$$
$$= (5A + 2B)e^\infty_{kl}x_k x_l = 0,$$

and thus

$$5A + 2B = 0.$$

Therefore we can write down

$$\vec{v}\,' = d_3(5r^2E^\infty . \vec{r} - 2E^\infty : \vec{r}\vec{r}\vec{r}).$$

Further we shall calculate the constants C_1, C_2, d_1, d_2 and d_3 from the boundary conditions (4.4)–(4.7).

Taking into account formulas (1.3) and (1.5) from (4.9) we obtain

$$v_i = e^\infty_{ki}x_k + \frac{1}{2}\varepsilon_{kji}\omega^\infty_k x_j + \frac{1}{4}C_1 e^\infty_{kj}\left(-\frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{jk}x_i}{r^3} + \frac{\delta_{ik}x_j}{r^3} - \frac{3}{r^5}x_i x_j x_k\right) + \frac{30}{4}C_2\frac{x_i x_j x_k}{r^7}e^\infty_{kj} - \frac{6}{4}\frac{C_2}{r^5}e^\infty_{kj}(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j). \tag{4.11}$$

Since at $r = 1$ we have $x_i = rn_i = n_i$ and (4.11) gives

$$v_i = e^\infty_{ki}n_k(1 - 3C_2) + e^\infty_{kj}n_i n_j n_k\left(-\frac{3}{4}C_1 + \frac{15}{2}C_2\right) + \frac{1}{2}\varepsilon_{kji}\omega^\infty_k. \tag{4.12}$$

Therefore

$$\vec{v}.\vec{n} = v_i n_i = e^\infty_{ki}n_i n_k\left(1 + \frac{9}{2}C_2 - \frac{3}{4}C_1\right) + \frac{1}{2}\varepsilon_{kji}\omega^\infty_k n_i n_j. \tag{4.13}$$

Similarly from (4.10) one gets

$$\hat{v}_i = d_1 e^\infty_{ki}n_k + d_2\varepsilon_{kji}\omega^\infty_k n_j + 5d_3 e^\infty_{ki}n_k - 2d_3 e^\infty_{kj}n_i n_j n, \tag{4.14}$$

$$\hat{\vec{v}}.\vec{n} = \hat{v}_i n_i = e^\infty_{ki}n_k n_i(d_1 + 5d_2 - 2d_3) + d_2\varepsilon_{kji}\omega^\infty_k n_j n. \tag{4.15}$$

According to (4.4), (4.5), (4.12) and (4.15) we have

$$3C_1 - 18C_2 = 4, \tag{4.16}$$

$$d_1 + 3d_3 = 0. \tag{4.17}$$

From the boundary condition (4.6) we obtain

$$e^\infty_{ki}n_k(1 - 3C_2) - e^\infty_{kj}n_k n_j n_i(1 - 3C_2) + \frac{1}{2}\varepsilon_{kji}\omega^\infty_k n_j$$
$$= e^\infty_{ki}n_k(d_1 + 5d_3) - e^\infty_{kj}.n_k n_j n_i(d_1 + 5d_3) + d_2\varepsilon_{kji}\omega^\infty_k n_i.$$

Therefore $d_2 = \frac{1}{2}$ and

$$1 - 3C_2 = d_1 + 5d_3. \tag{4.18}$$

In order to apply the boundary condition (4.7) we have need of the following quantities:

$$e_{ip}n_pn_i|_{r=1} = e_{kj}^{\infty}n_kn_j\left(\frac{3}{2}C_1 - 18C_2 + 1\right), \tag{4.19}$$

$$\hat{e}_{ip}n_pn_i|_{r=1} = e_{kj}^{\infty}n_kn_j(d_1 + 9d_3). \tag{4.20}$$

After some algebra we find that the left and the right-hand side of (4.7) at $r = 1$ are equal to

$$e_{kj}n_i(\delta_{jk} - n_kn_j)|_{r=1} = \left(1 - \frac{3}{4}C_1 + 12C_2\right)(-e_{kj}^{\infty}n_kn_jn_i + e_{ki}^{\infty}n_k),$$

$$\hat{e}_{kj}n_i(\delta_{jk} - n_kn_j)|_{r=1} = (d_1 + 8d_3)(-e_{kj}^{\infty}n_kn_jn_i + e_{ki}^{\infty}n_k).$$

Substituting these expressions into (4.7) yields

$$1 - \frac{3}{4}C_1 + 12C_2 = \lambda(d_1 + 8d_3). \tag{4.21}$$

In this way we have four equations, namely (4.16)–(4.18) and (4.21) for four unknown constants C_1, C_2, d_1 and d_3.

Solving this system we get

$$C_1 = \frac{2}{3}\frac{5\lambda + 2}{\lambda + 1}, \quad C_2 = \frac{\lambda}{3(\lambda + 1)}, \quad d_1 = -\frac{3}{2(\lambda + 1)}, d_3 = \frac{1}{2(\lambda + 1)}. \tag{4.22}$$

Therefore

$$\vec{v} = E^{\infty}.\vec{r} + \frac{1}{2}\vec{\omega} \times \vec{r} + 2\pi\mu a^3(E^{\infty}.\vec{r})\left(\frac{2}{3}\frac{5\lambda + 2}{\lambda + 1} + \frac{\lambda a^2}{3(\lambda + 1)}\nabla^2\right)\frac{B(\vec{r})}{8\pi\mu}$$

$$\hat{\vec{v}} = -\frac{3}{2(\lambda + 1)}E^{\infty}.\vec{r} + \frac{1}{2}\vec{\omega} \times \vec{r} + \frac{1}{2(\lambda + 1)}[5r^2(E^{\infty}.\vec{r}) - 2E^{\infty} : \vec{r}\vec{r}\vec{r}].$$

The solution just obtained can now be used to calculate from (4.8) the deformation of the drop for small values of the capillary number Ca. If $\vec{e}_r$ is the unit vector in the radial direction of a spherical co-ordinate system, then a first approximation to the unit normal vector $\vec{n}$ for small Ca is just $\vec{e}_r$. According to the definition of $\vec{n}$ in terms of F and the equation $\nabla r = \frac{\vec{r}}{r} = \vec{e}_r$ it follows that

$$\vec{n} \equiv \frac{\nabla E}{|\nabla F|} = \frac{\nabla r - \mathrm{Ca}\nabla f}{|\nabla r - \mathrm{Ca}\nabla f|} = \frac{\vec{e}_r - \mathrm{Ca}\nabla f}{\sqrt{1 + \mathrm{Ca}[(\nabla f)^2 - 2(\vec{e}_r.\nabla f)]}}. \tag{4.23}$$

Next it is important to observe that

$$\vec{n} = \vec{e}_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2), \tag{4.24}$$

$$\nabla.\vec{n} = \nabla.\vec{e}_r - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2), \tag{4.25}$$

$$\nabla.\vec{e}_r = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{\delta_{ii}}{r} - \frac{x_i x_i}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}. \tag{4.26}$$

From (4.24)–(4.26) it follows that the surface curvature $\nabla.\vec{n}$ can be expressed as

$$\begin{aligned}\nabla.\vec{n} &= \frac{2}{r} - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2) = 2(1-\mathrm{Ca}.f) - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2)\\ &= 2 - \mathrm{Ca}(2f + \nabla^2 f) + O(\mathrm{Ca}^2).\end{aligned} \tag{4.27}$$

We shall note that the surface curvature can be expressed as the sum of the inverse principle radii of curvature, that is

$$\nabla.\vec{n} = \frac{1}{R_1} + \frac{1}{R_2}.$$

With (4.27) the boundary condition (4.8) gives

$$\begin{aligned}&\mathrm{Ca}\{T.[e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)] - \lambda\hat{T}.[e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)]\}\\ &\quad\times [e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)] = 2 - \mathrm{Ca}(2f + \nabla^2 f).\end{aligned} \tag{4.28}$$

It is clear that to order $O(\mathrm{Ca}^2)$ the boundary condition (4.28) takes the form

$$(T.e_r - \lambda\hat{T}.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 - \mathrm{Ca}(2f + \nabla^2 f)],$$

or

$$(E.e_r - \lambda\hat{E}.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 - \mathrm{Ca}(2f + \nabla^2 f)]. \tag{4.29}$$

The shape function $f(\vec{r})$ is a true scalar and linearly related to the variables $\vec{v}$, $\hat{\vec{v}}$ and $(T.\vec{n} - \lambda\hat{T}.\vec{n})$. Therefore $f(\vec{r})$ must be expressible in invariant form as a linear function of E^∞ (or Ω^∞), i. e.

$$f(\vec{r}) = b\vec{r}.E^\infty.\vec{r}, \tag{4.30}$$

where b is an unknown constant.

From (4.27) and (4.30) it follows that the surface curvature $\nabla.\vec{n}$ is equal to

$$\nabla.\vec{n} = 2 + 4(\vec{r}.E^\infty.\vec{r})b\mathrm{Ca} + O(\mathrm{Ca}^2), \tag{4.31}$$

and thus (4.28) becomes

$$(E^\infty.e_r - \lambda\hat{E}^\infty.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 + 4(\vec{r}.E^\infty.\vec{r})b\mathrm{Ca}]. \tag{4.32}$$

In order to apply the boundary condition (4.32) we have to calculate the pressures p and $\hat{p}$ from the Stokes equations inside and outside of the drop. After some algebra one obtains

$$\frac{\partial v_i'}{\partial x_j} = 5d_3 e_{ij}^\infty x_p x_p + 10 d_3 e_{ki}^\infty x_k x_j - 2d_3 e_{kl}^\infty x_k x_l \delta_{ij} - 4d_3 e_{kj}^\infty x_k x_i, \tag{4.33}$$

$$(\nabla^2 \vec{v}')_i = 42 d_3 e_{ki}^\infty x_k, \tag{4.34}$$

$$(4.35) \qquad (\nabla\hat{p})_i = \hat{\mu}(\nabla^2\vec{v})_i = \hat{\mu}42d_3e^{\infty}_{ki}x_k.$$

Integrating the equation (4.35) with respect to x_i we find that

$$(4.36) \qquad \hat{p} = \hat{\mu}21d_3e^{\infty}_{ki}x_kx_i + \hat{p}_0 = 21\vec{\mu}d_3E^{\infty} : \vec{r}\vec{r} + \hat{\mu}\hat{p}_0,$$

where $\hat{p}_0 =$ const.

Similarly, one obtains the pressure outside the drop, namely

$$(4.37) \qquad p = -\frac{3}{2}\mu C_1 E^{\infty} : \vec{r}\vec{r} + \mu p_0,$$

where $p_0 =$ const.

We observe that the pressure field inside and outside of the droplet can be calculated from Stokes equations up to a constant. The constant p_0 involved in the outside field is determined from the known pressure for the droplet. As we shall see, the constant $\hat{p}_0$ involved in the interior pressure field can be determined from the boundary condition (4.8) for the normal components of the stress vectors. Using the equations (4.19), (4.20), (4.32), (4.36) and (4.37) we obtain the following equation for the constant:

$$(4.38) \qquad \begin{aligned} &-p_0 + \hat{p}_0 + 21d_3\hat{\mu}E^{\infty} : \vec{r}\vec{r} + \frac{3}{2}\mu C_1 E^{\infty}\vec{r}\vec{r} \\ &+2\mu\left[1 + 12C_2 - \frac{3}{4}C_1 + \frac{9}{4}C_1 - 30C_2 - \lambda(d_1 + 8d_3 + d_3)\right]E^{\infty} : \vec{r}\vec{r} \\ &= \mu\left[\frac{2}{\mathrm{Ca}} + 4bE^{\infty} : \vec{r}\vec{r}\right]. \end{aligned}$$

It follows from the equation (4.38) that

$$\hat{p}_0 - p_0 = \frac{2}{\mathrm{Ca}}$$

and

$$(4.39) \qquad \frac{3}{4}C_1 + \frac{21}{2}\lambda d_3 + 1 + \frac{3}{2}C_1 - 18C_2 - \lambda(d_1 + 9d_3) = 2b.$$

Finally, substituting the constants C_1, C_2, d_1, d_2 and d_3 from (4.22) in the equation (4.39) we obtain

$$b = \frac{19\lambda + 16}{8(\lambda + 1)},$$

and thus the corresponding drop shape is

$$(4.40) \qquad r = 1 + \mathrm{Ca}\frac{19\lambda + 16}{8(\lambda + 1)}(\vec{r}.E^{\infty}.\vec{r}).$$

In order to illustrate the result that we have obtained in (4.40) we shall note that for a simple shear flow ($v_1 = x_2$, $v_2 = 0$, $v_3 = 0$)

$$E = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$r = 1 + \mathrm{Ca}x_1x_2\frac{19\lambda + 16}{8(\lambda + 1)},$$

whereas for an extensional flow ($v_1 = -\frac{1}{2}x_1$, $v_2 = -\frac{1}{2}x_2$, $v_3 = x_3$)

$$E = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$r = 1 + \mathrm{Ca}\frac{19\lambda + 16}{8(\lambda + 1)}(x_3^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2)$$

These two flows are sketched in Fig. 1. One can see that although the deformation is small in all cases for the limit $\mathrm{Ca} \ll 1$, the slight difference in shape of the two flows shows that extensional flow is more efficient at stretching deformable particles than the simple shear flow.

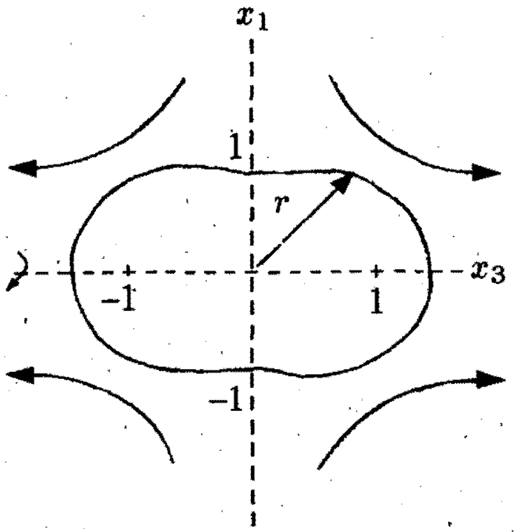

Fig. 1

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