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BOUNDS OF DEVIATION FROM EXPONENTIALITY FOR A PROBABILITY DENSITY CLASS*

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Апостол Обретенов. ОЦЕНКА ЭКСПОНЕНЦИАЛЬНОГО ОТКЛОНЕНИЯ ДЛЯ ОДНОГО КЛАССА ВЕРОЯТНОСТНЫХ ПЛОТНОСТЕЙ

Для одного класса вероятностных плотностей получена оценка экспоненциального отклонения.

Apostol Obretenov. BOUNDS OF DEVIATION FROM EXPONENTIALITY FOR A PROBABILITY DENSITY CLASS

In the present paper the estimate of deviation between a probability density function and an exponential one is given.

Let X be a positive random variable with a probability density $f(x)$ with mean $EX = \mu < \infty$. We assume $f(0) \neq 0$ and $f(\infty) = 0$. Let the derivative $f'(x)$ exist for every $x \geq 0$ and

$$f(x) \leq -f(x)\mu^{-1}. \tag{1}$$

Denote $f(0) = a$. We shall prove the following

Theorem. *For every probability density function* $f(x)$, *satisfying* (1), *the inequalities*

$$0 \leq f(x) - \mu^{-1}e^{-x/\mu} \leq a - \mu^{-1} \tag{2}$$

hold.

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Proof. Let

$$A(x) = -\mu^{-1} f(x) - f'(x). \tag{3}$$

Since $f(x)$ satisfies (1), we have $A(x) \geq 0$, so that the function

$$H(x) = \int_x^\infty A(y)\, dy \tag{4}$$

is a non-negative and non-increasing function. Putting

$$\Delta(x) = f(x) - \mu^{-1} e^{-x/\mu},$$

we easily get

$$\Delta(x) + \mu^{-1}\Delta'(x) = -A(x). \tag{5}$$

Solving the differential equation (1), according to $\Delta(x)$ we find

$$\Delta(x) = e^{-x/\mu}\left[\Delta(x) - \int_0^x A(y) e^{y/\mu}\, dy\right]. \tag{6}$$

Now, taking the derivative of (6), we get

$$\begin{aligned} \Delta'(x) &= -e^{-x/\mu}\left[\Delta(0) - \int_0^x A(y)e^{y/\mu}\, dy\right] + e^{-x/\mu}A(x)e^{x/\mu} \\ &= -\mu^{-1}e^{-x/\mu}\Delta(0) + \mu^{-1}e^{-x/\mu}\int_0^x A(y)e^{y/\mu}\, dy - A(x) \\ &\leq -\mu^{-1}e^{-x/\mu}\Delta(0) + \mu^{-1}e^{-x/\mu}H(0) - \mu^{-1}\int_x^\infty A(y)e^{y/\mu}\, dye^{-x/\mu}. \end{aligned} \tag{7}$$

We can see from (4) and the definition of $\Delta(x)$ that

$$H(0) = \Delta(0) = a - \mu^{-1}. \tag{8}$$

Then from (7) we find consequently

$$\Delta' \leq -\mu^{-1}e^{-x/\mu}\int_x^\infty A(y)e^{y/\mu}\, dy,$$

i.e. $\Delta'(x) \leq 0$. Therefore, $\Delta(x)$ is a non-increasing function. Since $\Delta(\infty) = 0$ and $\Delta(0) = a - \mu^{-1}$, we conclude that $0 \leq \Delta(x) \leq a - \mu^{-1}$, which is (2). Obviously, the bounds in (2) are sharp.

Let $f_n(x)$ be a sequence of densities, satisfying (1), and $f_n(0) = a_n$. Let μ_n be the corresponding means of $f_n(x)$. Then from (2) we see that $f_n(x) \to \mu^{-1}e^{-x/\mu}$ when $\mu_n \to \mu$ and $a_n - \mu_n^{-1} \to 0$.

Remark. If we change the direction of the inequality (1) and keep the other conditions, then the inequalities, similar to (2), hold for $-\Delta(x)$. The prove is the same as above.

Further, let us integrate (1) over the interval (y, ∞). We get

$$f(y) \geq \mu^{-1}[1 - F(y)], \tag{9}$$

where $F(x) = \int\limits_0^x f(y)\,dy$, and integrating (9) over (x, ∞) we find

$$1 - F(x) \geq \int\limits_x^\infty \mu^{-1}(1 - F(y))\,dy. \tag{10}$$

The inequality (10) shows that the distribution function

$$\mu^{-1} \int\limits_0^x (1 - F(y))\,dy$$

belongs to the so-called NBU-class (see [1]). Definition of this distribution class (and other classes) and certain bounds of deviation from exponentiality for it are given in [1].

REFERENCES

1. Obretenov, A., S. Rachev. Bounds of Deviation from Exponentiality of Distribution Functions Classes (to appear).

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