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ON AN APPLICATION OF THE CRUM–KREIN TRANSFORMATIONS*

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Евгени Христов, Йордан Мишев. ОБ ОДНОМ ПРИЛОЖЕНИИ ПРЕОБРАЗОВАНИЙ КРАММА-КРЕЙНА

На основе преобразования Крамма-Крейна построен простой алгорифм, который сводит задачу о разложениях по произведениям решений двух радиальных уравнений Шредингера, где у одного из них спектр чисто непрерывный, а у другого есть и одно собственое число к простейшему, когда у обоих уравнений спектр чисто непрерывный.

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Using the Crum-Krein transformations, a simple algorithm is constructed, which reduces the problem of expansion in products of solutions of two radial Schrödinger equations when one of them has purely continuous spectrum and the other one has in addition one descreet eigenvalue to the simplest case when the both equations have purely continuous spectra.

0. INTRODUCTION

In this paper it is shown how one can modify the method of transformation of expansions in products of solutions of two Schrödinger equations on semi-axis proposed in [4] to the case of different spectra. For brevity we consider in detail

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the case when one of the expansion formulas corresponds to the case when the both Schrödinger equations have purely continuous spectrum and the second expansion formula corresponds to the case when one of the Schrödinger equation has purely continuous spectrum and the second one has one discrete eigenvalue. The general case differs from this one only by some technical calculations similar to that in [1, 4] and we discuss it briefly in the end of this paper. The notations follow those in [2, 4].

A. In this section we recall some results which are basic in our constructions. The next theorem is well known.

Theorem 0.1 [1]. *Consider the equation*

$$y'' + (\lambda - v(x))y = 0, \quad a \le x \le b, \quad ' = \frac{d}{dx} \tag{0.1}$$

and let $z(x)$ be its solution for $\lambda = \lambda_0$, $z(x) \neq 0$, $a < x < b$. Then the function

$$y_1(x, \lambda) = \frac{W(z(x), y(x, \lambda))}{z(x)} \tag{0.2}$$

satisfies the equation

$$y_1'' + (\lambda - v_1(x))y_1 = 0, \quad v_1(x) = v(x) - 2\frac{d^2}{dx^2} \ln z(x). \tag{0.3}$$

The transformation inverse to (0.2) has the form

$$y(x, \lambda) = \frac{W(z_1(x), y_1(x, \lambda))}{(\lambda_0 - \lambda)z_1(x)}, \quad z_1(x) = \frac{1}{z(x)},$$

where $z_1(x)$ is the solution of equation (0.3) for $\lambda = \lambda_0$, $W(f, g) = fg' - f'g$.

The transformations of products of solutions of two equations of the type (0.1) are based on the following

Theorem 0.2. *Let us consider the two equations*

$$\frac{d^2}{dx^2}y^{(n)} + (k^2 - v^{(n)}(x))y^{(n)} = 0, \quad a \le x \le b, \quad n = 1, 2, \tag{0.4}$$

and let construct the equations

$$\frac{d^2}{dx^2}y_1^{(n)} + (k^2 - v_1^{(n)}(x))y_1^{(n)} = 0, \qquad v_1^{(n)}(x) = v^{(n)}(x) - 2\frac{d^2}{dx^2} \ln z^{(n)}(x),$$

where $z^{(n)}(x)$ satisfy the conditions of Theorem 0.1. Then we have the following relations:

$$W(Z(x), Y(x, \lambda)) = (\lambda_0 - \lambda)\frac{d}{dx}(Z(x)Y_1(x, \lambda)),$$

$$W(Z_1(x), Y_1(x, \lambda)) = (\lambda_0 - \lambda)^{-1}\frac{d}{dx}(Z_1(x)Y(x, \lambda)),$$

where

$$Y(x,\lambda) = y^{(1)}(x,\lambda)y^{(2)}(x,\lambda), \quad Z(x) = z^{(1)}(x)z^{(2)}(x),$$
$$Y_1(x,\lambda) = y_1^{(1)}(x,\lambda)y_1^{(2)}(x,\lambda), \quad Z_1(x) = z_1^{(1)}(x)z_1^{(2)}(x).$$

The proof is a direct consequence of Theorem 0.1 and the well-known identity

$$W(Y(x,\lambda), Z(x,\mu)) = \frac{1}{\lambda-\mu}\frac{d}{dx}\prod_{n=1,2} W(y^{(n)}(x,\lambda), z^{(n)}(x,\mu)), \tag{0.5}$$

which follows directly from (0.4) with $k^2 = \lambda, \mu$.

B. Now let us consider the boundary value problems

$$\frac{d^2}{dx^2}y^{(n)} + (k^2 - v^{(n)}(x))y^{(n)} = 0,\ 0 < x < \infty,\ n = 1,2, \tag{0.6}$$

$$y^{(n)}(0,k) = 0, \tag{0.7}$$

where the real-valued potentials $v^{(n)} \in X_1$, i.e.

$$\|v^{(n)}\|_{X_1} = \int_0^\infty (1+x)|v^{(n)}(x)|\,dx < \infty.$$

Let denote by $\varphi^{(n)}(x,k)$ and $f^{(n)}(x,k)$, respectively, the regular and the Jost's solutions of (0.6) defined by the conditions

$$\varphi^{(n)}(0,k) = 0, \quad \varphi^{(n)\prime}(0,k) = 1,$$
$$\lim_{x\to\infty} f^{(n)}(x,k)\exp(-ikx) = 1, \quad \operatorname{Im} k \geq 0,$$

and let denote by

$$f^{(n)}(k) = f^{(n)}(0,k) = W(f^{(n)}, \varphi^{(n)})$$

the Jost's functions of the boundary value problem (0.6)–(0.7). We put

$$\sigma(v^{(n)}) = \{k_{n,j} = i\kappa_{n,j} :\ \tau_{n,j} > 0,\ f^{(n)}(k_{n,j}) = 0,\ j = 1, 2, ..., N_n\}$$

for the set of zeros of $f^{(n)}(k)$ in the upper-half plane $\operatorname{Im} k > 0$ and put

$$\sigma = \sigma_1 \cup \sigma_2, \quad \sigma'' = \sigma_1 \cap \sigma_2, \quad \sigma' = \sigma \setminus \sigma''.$$

Recall that the square $\lambda_{n,j} = k_{n,j}^2$ determines the discreet spectra of the problem (0.6)–(0.7). We suppose for simplicity that $f^{(n)}(0) \neq 0$, $n = 1,2$. Denote by $\#\sigma(v^{(n)})$ the number of elements of $\sigma(v^{(n)})$ and let

$$\Omega(N,M) = \{(v^{(1)}, v^{(2)}) : v^{(n)} \in X_1,\ \#\sigma(v^{(1)}) = N,\ \#\sigma(v^{(2)}) = M\}.$$

Denote by

$$\Phi(x,k) = \varphi^{(1)}(x,k)\varphi^{(2)}(x,k), \quad F(x,k) = f^{(1)}(x,k)f^{(2)}(x,k)$$

the products of regular solutions and the Jost's solutions of (0.6), respectively, and let

$$F(k) = f^{(1)}(k) f^{(2)}(k)$$

be the product of Jost's functions of the boundary problem (0.6)–(0.7).

Introduce the system of functions

$$\{\Phi\} : \Phi(x,k),\ k \in (0,\infty) \cup \sigma;\ \dot{\Phi}_j(x) = \dot{\Phi}(x,k_j),\ k_j \in \sigma'',$$

and

$$\{F\} : F(x,k),\ k \in (0,\infty) \cup \sigma;\ \dot{F}_j(x) = \dot{F}(x,k_j),\ k_j \in \sigma''.$$

Next, putting

$$F(k) = f_1(k) f_2(k),$$

we construct the system $\{\tilde{F}\}$ in the following way: if $k \in (0,\infty)$, let us put

$$\tilde{F}(x,k) = -\frac{4}{\pi} k \operatorname{Im}\{F(x,k) F^{-1}(k)\};$$

if $k_{n,j} \in \sigma'$, then let put

$$\tilde{F}_{n,j}(x) = a_{n,j} F(x,k_{n,j}); \quad a_{n,j} = 4 k_{n,j} \dot{F}^{-1}(k_{n,j}),$$

and to each $k_j \in \sigma''$ attach the pair of functions

$$\tilde{F}_{j,1}(x) = b_j(\dot{F}(x,k_j) + d_j F(x,k_j)), \quad \tilde{F}_{j,2}(x) = b_j F(x,k_j),$$

where $b_j = 8 k_j \ddot{F}^{-1}(k_j)$, $d_j = k_j^{-1} - \dddot{F}(k_j)(3\ddot{F}(k_j))^{-1}$.

The next two theorems are particular cases of the general expansion formulas listed in Theorem 4.1 bellow.

Theorem 0.3 [2]. *Let* $(v^{(1)}, v^{(2)}) \in \Omega(0,0)$, *then for any absolutely continuous function* $f(x) \in L^1(0,\infty)$ *we have the expansions*

$$f(x) = -\int_0^\infty \tilde{F}(x,k)(f, \Phi'(k))\, dk, \tag{0.8}$$

$$f(x) = -\int_0^\infty \Phi'(x,k)(f, \tilde{F}(k))\, dk. \tag{0.9}$$

Moreover, if

$$f(x) \in L_0^1 = \{f \in L^1 : \int_0^\infty f(x)\, dx = 0\},$$

then

$$f(x) = \int_0^\infty \tilde{F}'(x,k)(f, \Phi(k))\, dk. \tag{0.10}$$

Now let us consider together with (0.6)–(0.7) the problem

$$\frac{d^2}{dx^2}y_2^{(n)} + (k^2 - v_2^{(n)}(x))y_2^{(n)} = 0, \quad 0 < x < \infty, \quad n = 1, 2, \tag{0.11}$$

$$y_2^{(n)}(0, k) = 0. \tag{0.12}$$

Theorem 0.4 [2]. *Let $(v_2^{(1)}, v_2^{(2)}) \in \Omega(1,0)$, then for any absolutely continuous function $h(x) \in L^1(0,\infty)$ we have the expansions*

$$h(x) = -\int_0^\infty \tilde{F}_2(x,k)(h, \Phi_2'(k))\,dk - \tilde{F}_{1,1}^{(2)}(x, k_{1,1})(h, \Phi_2'(k_{1,1})), \tag{0.13}$$

$$h(x) = -\int_0^\infty \Phi_2'(x,k)(f, \tilde{F}_2(k))\,dk - \Phi_2'(x, k_{1,1})(h, \tilde{F}_{1,1}^{(2)}). \tag{0.14}$$

Moreover, if $h(x) \in L_0^1$, then

$$h(x) = \int_0^\infty \tilde{F}_2'(x,k)(f, \Phi_2(k))\,dk + \tilde{F}_{1,1}^{(2)'}(x)(h, \Phi_2(k_{1,1})), \tag{0.15}$$

where the systems of functions $\{\tilde{F}_2(x,k)\}$ and $\{\Phi_2(x,k)\}$ are defined as above.

We need also the following assertion, which is a direct consequence of the relation (0.5).

Theorem 0.5 [2]. *Let $(v_2^{(1)}, v_2^{(2)}) \in \Omega(1,0)$. Then for the systems $\{\tilde{F}_2(x,k)\}$ and $\{\Phi_2(x,k)\}$ the following biorthogonality relations hold:*

$$\begin{gathered}(\Phi_2(k_{1,1}), \tilde{F}_{1,1}^{(2)'}) = -(\Phi_2'(k_{1,1}), \tilde{F}_{1,1}^{(2)}) = 1, \quad k_{1,1} \in \sigma(v^{(1)}), \\ (\tilde{F}_2(k), \Phi_2'(k_{1,1})) = 0, \quad k \in (0,\infty), \\ (\tilde{F}_{1,1}^{(2)}, \Phi_2'(k)) = -(\Phi_2(k), \tilde{F}_{1,1}^{(2)'}) = 0.\end{gathered} \tag{0.16}$$

C. Let $\alpha_1 > 0$ and $k_1 = i\tau_1, \tau_1 > 0$. In [3] the next theorem is proved.

Theorem 0.6. *Let $(v^{(1)}, v^{(2)}) \in \Omega(0,0)$ and let $\varphi^{(1)}(x, k_1)$ be the regular solution of (0.12) for $n = 1$. Then for the potentials*

$$v_2^{(1)}(x) = v^{(1)}(x) - 2\frac{d^2}{dx^2}\ln I^{(1)}(\alpha_1, k_1; x), \quad v_2^{(2)}(x) = v^{(2)}(x), \tag{0.17}$$

where

$$I^{(1)}(\alpha_1, k_1; x) = 1 + \alpha_1 \int_0^x [\varphi^{(1)}(s, k_1)]^2\,ds, \tag{0.18}$$

we have $(v_2^{(1)}, v_2^{(2)}) \in \Omega(1,0)$. *The corresponding eigenfunction of* (0.11)–(0.12) *for* $n = 1$ *is*

$$\varphi_2^{(1)}(x, k_1) = \varphi^{(1)}(x, k_1)(I^{(1)}(\alpha_1, k_1; x))^{-1}, \quad \|\varphi_2^{(1)}\|_{L^2}^{-2} = \alpha_1.$$

Conversely, let $(v_2^{(1)}, v_2^{(2)}) \in \Omega(1,0)$ *and* $\varphi_2^{(1)}(x, k_1)$ *be the eigenfunction of the problem* (0.11)–(0.12) *for* $n = 1$ *such that* $\|\varphi_2^{(1)}\|_{L^2}^{-2} = \alpha_1$. *If*

$$v^{(1)}(x) = v_2^{(1)}(x) - 2\frac{d^2}{dx^2}\ln I_2^{(1)}(\alpha_1, k_1; x), \tag{0.19}$$

$$v^{(2)}(x) = v_2^{(2)}(x), \tag{0.20}$$

where $I_2^{(1)}(\alpha_1, k_1; x) = \alpha_1 \int_x^\infty [\varphi_2^{(1)}(s, k_1)]^2\, ds$, *then* $(v^{(1)}, v^{(2)}) \in \Omega(0,0)$ *and the regular solution of* (0.6) *for* $n = 1$ *is given by the formula*

$$\varphi^{(1)}(x, k_1) = \varphi_2^{(1)}(x, k_1)(I_2^{(1)}(\alpha_1, k_1; x))^{-1}.$$

In Sections 1 and 2 we construct operators connecting the functions $\tilde{F}(x,k)$, $\Phi(x,k)$ with the systems $\{\tilde{F}_2\}, \{\Phi_2\}$, where the potentials $v^{(n)}(x)$ and $v_2^{(n)}(x)$ are related as in Theorem 0.6. The main steps of the transformation from the expansions, listed in Theorem 0.3, to the expansions of Theorem 0.4 are exposed in Section 3.

1. OPERATORS $S^{(0)}$ AND $\tilde{S}^{(0)}$

Let us denote by $y_2^{(n)}(x) = \varphi_2^{(n)}(x, k_1)$ the regular solution of (0.11), where $k_1 = k_{1,1}$ is the eigenvalue of the problem (0.11)–(0.12) for $n = 1$. Recall that the transformation [1, 3]

$$y_1^{(n)}(x, k) = \frac{W(y_2^{(n)}(x), y_2^{(n)}(x, k))}{(k_1^2 - k^2)y_2^{(n)}(x)}, \quad k \neq k_1, \tag{1.1}$$

gives the equation

$$\frac{d^2}{dx^2}y_1^{(n)} + (k^2 - v_1^{(n)}(x))y_1^{(n)} = 0, \ \ 0 < x < \infty, \tag{1.2}$$

where

$$v_1^{(n)}(x) = v_2^{(n)}(x) - 2\frac{d^2}{dx^2}\ln y_2^{(n)}(x), \tag{1.3}$$

and for any $a > 0$ we have $\int_0^a \left|v_1^{(n)}(x) - \frac{2}{x^2}\right| dx + \int_a^\infty x\left|v_1^{(n)}(x)\right| dx < \infty.$

Next, following Theorem 0.6, let us denote $z_1^{(1)}(x) = I_2^{(1)}(\alpha_1, k_1; x)(\varphi_2^{(1)}(x, k_1))^{-1}$,

$z_1^{(2)}(x) = (\varphi_2^{(2)}(x, k_1))^{-1}$ and transform the solutions of Eq. (1.2) as follows:

$$y^{(n)}(x,k) = \frac{W(z_1^{(n)}(x), y_1^{(n)}(x,k))}{z_1^{(n)}(x)}, \quad k \neq k_1. \tag{1.4}$$

In this way we obtain Eq. (0.6), where the potentials $v^{(n)}(x)$ are connected with the potentials $v_2^{(n)}(x)$ via formulas (0.19), (0.20). Equality (0.19) is obtained because of Theorem 0.3 (see [4]), the relation (0.20) expresses the fact that for $n = 2$ the transformation (1.4) is inverse to the transformation (1.1).

Let us denote $Z(x) = z_1^{(1)}(x)z_1^{(2)}(x)$, $Y_2(x) = y_2^{(1)}(x)y_2^{(2)}(x) = \Phi_2(x, k_1)$. The following asymptotics hold (see, e. g. [3]):

$$\begin{aligned} &Z_1(x) \sim x^{-2},\ x \to 0; \quad Z_1(x) \sim \text{const}.e^{-2\tau_1 x},\ x \to \infty, \\ &Y_2(x) \sim x^2,\ x \to 0; \quad Y_2(x) \sim \text{const}.,\ x \to \infty, \end{aligned} \tag{1.5}$$

which could be differentiated with respect to x. In particular, we have

$$Y_2'(x) \sim 2x,\ x \to 0;\ x|Y_2'(x)| \in L_1(A, \infty),\ A > 0. \tag{1.6}$$

Lemma 1.1. *Let the potentials $v^{(n)}(x)$ and $v_2^{(n)}(x)$ in (0.6) and (0.11), respectively, be connected by formulas (0.19), (0.20). Then the following relations hold:*

$$W(Z_1(x), \Phi_1(x,k)) = (k_1^2 - k^2)^{-1}\frac{d}{dx}(Z_1(x)\Phi(x,k)), \tag{1.7}$$

$$W(Y_2(x), \Phi_2(x,k)) = (k_1^2 - k^2)\frac{d}{dx}(Y_2(x)\Phi_1(x,k)), \tag{1.8}$$

$$W(Z_1(x), F_1(x,k)) = \frac{k - k_1}{k + k_1}\frac{d}{dx}(Z_1(x)F(x,k)), \tag{1.9}$$

$$W(Y_2(x), F_2(x,k)) = \frac{d}{dx}(Y_2(x)F_1(x,k)), \tag{1.10}$$

$$F(k) = \frac{k + k_1}{k - k_1}F_2(k). \tag{1.11}$$

Here $\Phi_1(x,k) = \varphi_1^{(1)}(x,k)\varphi_1^{(2)}(x,k)$, $F_1(x,k) = f_1^{(1)}(x,k)f_1^{(2)}(x,k)$, where $\varphi_1^{(n)}(x,k)$ is the regular solution and $f_1^{(n)}(x,k)$ is the Jost's solution of Eqs. (1.2) *with potentials defined by* (1.3). *The functions $\Phi(x,k)$, $F(x,k)$ and $\Phi_2(x,k)$, $F_2(x,k)$ are defined in a similar way using Eqs.* (0.6) *and* (0.11). *Finally, $F(k)$ and $F_2(k)$ are the products of the Jost's functions of the boundary problems* (0.6), (0.7) *and* (0.11), (0.12), *respectively.*

The proof is a direct consequence of Theorem 0.2, having in mind the following representations:

$$\varphi^{(n)}(x,k) = \frac{W(z_1^{(n)}(x), \varphi_1^{(n)}(x,k))}{z_1^{(n)}(x)}, \quad \varphi_1^{(n)}(x,k) = \frac{W(y_2^{(n)}(x), \varphi_2^{(n)}(x,k))}{(k_1^2 - k^2)y_2^{(n)}(x)},$$

$$f^{(n)}(x,k) = \frac{iW(z_1^{(n)}(x), f_1^{(n)}(x,k))}{(k_1-k)z_1^{(n)}(x)},$$

$$f_1^{(1)}(x,k) = i(k_1+k)\frac{iW(z_1^{(n)}(x), f_2^{(1)}(x,k))}{y_2^{(1)}(x)},$$

$$f_1^{(2)}(x,k) = -(k_1-k)\frac{iW(y_2^{(2)}(x), f_2^{(2)}(x,k))}{y_2^{(2)}(x)}.$$

The relation (1.11) follows from the equalities (see Theorem 0.3 [4])

$$f_1^{(1)}(k) = \frac{k-k_1}{k+k_1}f^{(1)}(k), \quad f_2^{(2)}(k) = f^{(2)}(k).$$

Now, following [1], let introduce the operators

$$\mathbf{A_1}f = f(x) + 2Z_1^{-1}(x)\int_x^\infty Z_1'(s)f(s)\,ds,$$

$$\tilde{\mathbf{A}}_\mathbf{1}f = f(x) - 2(Z_1^{-1}(x))'\int_x^\infty Z_1(s)f(s)\,ds.$$

Using the function $Y_2(x)$, let us also construct the operators

$$\mathbf{S}_\mathbf{2}^{(0)}f = f(x) - 2Y_2^{-1}(x)\int_0^x Y_2'(s)f(s)\,ds, \tag{1.12}$$

$$\tilde{\mathbf{S}}_\mathbf{2}^{(0)}f = f(x) + 2(Y_2^{-1}(x))'\int_0^x Y_2(s)f(s)\,ds.$$

Recall [4] that $\mathbf{A_1}, \tilde{\mathbf{A}}_\mathbf{1} \in \mathcal{L}(L_1, L_1), \mathcal{L}(L_\infty, L_\infty)$, where $L_1 = L_1(0,\infty)$, $L_\infty = L_\infty(0,\infty)$; here, as usual, $\mathcal{L}(X,Y)$ denotes the space of the linear bounded operators defined in X with image in Y.

From the estimates (1.5), (1.6) it follows that

$$\mathbf{S}_\mathbf{2}^{(0)}, \tilde{\mathbf{S}}_\mathbf{2}^{(0)} \in \mathcal{L}(L_\infty, L_\infty), \quad \tilde{\mathbf{S}}_\mathbf{2}^{(0)} \in \mathcal{L}(L_1, L_1).$$

Lemma 1.2. *The condition*

$$f \in L_1(Y_2') = \{f(x) : f \in L_1, (f, Y_2') = 0\}$$

is necessary and sufficient for the relations

$$\mathbf{S_2}f = \mathbf{S}_\mathbf{2}^{(0)}f = \mathbf{S}_\mathbf{2}^{(\infty)}f = f(x) + 2Y_2^{-1}(x)\int_x^\infty Y_2'(s)f(s)\,ds \in \mathcal{L}(L_1(Y_2'), L_1).$$

The proof is easily obtained by (1.5), (1.6) (see, also, Lemma 1.1 [4]).

Note that if $f' \in L_1, L_\infty$, we have

$$DA_1 f = \tilde{A}_1 Df, \quad DS_2^{(0)} f = \tilde{S}_2^{(0)} Df, \quad D = \frac{d}{dx}. \tag{1.13}$$

Following the proof of Theorem 1.1 [1], we obtain

Theorem 1.1. *Let us denote*

$$\mathbf{S}^{(0)} = \mathbf{A_1}\mathbf{S}_2^{(0)}, \quad \tilde{\mathbf{S}}^{(0)} = \tilde{\mathbf{A}}_1\tilde{\mathbf{S}}_2^{(0)}.$$

Then the following representations hold:

$$\Phi(x,k) = \mathbf{S}^{(0)}\Phi_2(x,k), \quad \Phi'(x,k) = \tilde{\mathbf{S}}^{(0)}\Phi_2'(x,k), \quad k \in (0,\infty), \tag{1.14}$$

$$\tilde{F}(x,k) = \mathbf{S}^{(0)}\tilde{F}_2(x,k), \quad \tilde{F}'(x,k) = \tilde{\mathbf{S}}^{(0)}\tilde{F}_2'(x,k), \quad k \in (0,\infty). \tag{1.15}$$

Proof. From (1.13) it follows that it is sufficient to prove only the first formulas in (1.14) and (1.15). Integrating equalities (1.7) and (1.9) from x to ∞, we obtain

$$\Phi(x,k) = (k_1^2 - k^2)\mathbf{A_1}\Phi_1(x,k), \quad F(x,k) = \frac{k+k_1}{k-k_1}\mathbf{A_1}F_1(x,k).$$

Integrating equalities (1.8) and (1.10) from 0 to x, we obtain

$$\Phi_1(x,k) = (k_1^2 - k^2)^{-1}\mathbf{S}_2^{(0)}\Phi_2(x,k),$$

$$F_1(x,k) = \frac{F_2(k)}{(k_1^2 - k^2)Y_2(x)} + \mathbf{S}_2^{(0)}F_2(x,k).$$

In this way we get (1.14) and the equality

$$F(x,k) = \frac{k+k_1}{k-k_1}\mathbf{S}^{(0)}F_2(x,k) + \frac{F_2(k)}{k_1^2 - k^2}\mathbf{A_1}Y_2^{-1}(x),$$

which together with (1.11) and the obvious equality $\operatorname{Im}\{(k_1^2 - k^2)^{-1}\mathbf{A_1}Y_2^{-1}(x)\} = 0$ give (1.15). The theorem is proved.

2. OPERATORS $\mathbf{T}^{(\infty)}$ AND $\tilde{\mathbf{T}}^{(\infty)}$

Let introduce the operators

$$\mathbf{B_1}f = f(x) - 2Z_1(x)\int_0^x (Z_1^{-1}(s))' f(s)\,ds,$$

$$\tilde{\mathbf{B}}_1 f = f(x) + 2Z_1'(x)\int_0^x Z_1^{-1}(s) f(s)\,ds.$$

In [1] we have shown that $\mathbf{B_1}, \tilde{\mathbf{B}}_1 \in \mathcal{L}(L_1, L_1), \mathcal{L}(L_\infty, L_\infty)$ and if $f \in L_1, L_\infty$, then $\tilde{\mathbf{B}}_1 Df = D\mathbf{B_1}f$.

The connection between the operators $\mathbf{B_1}, \tilde{\mathbf{B}}_1$ and $\mathbf{A_1}, \tilde{\mathbf{A}}_1$ is given by the following

Lemma 2.1 [1]. *For any $f \in L_1, L_\infty$ and $h \in L_\infty, L_1$ the following relations hold:*

$$(\mathbf{A_1}f, h) = (f, \tilde{\mathbf{B}}_1 h), \quad (\tilde{\mathbf{A}}_1 f, h) = (f, \mathbf{B_1}h).$$

Moreover, in the spaces L_1, L_∞ we have $\mathbf{B_1} = \mathbf{A_1^{-1}}, \tilde{\mathbf{B}}_1 = \tilde{\mathbf{A}}_1^{-1}$.

Note also the equalities

$$\int_0^\infty \tilde{\mathbf{A}}_1 f(x)\,dx = -\int_0^\infty f(x)\,dx, \quad \int_0^\infty \tilde{\mathbf{B}}_1 f(x)\,dx = -\int_0^\infty f(x)\,dx, \ f \in L_1.$$

From here it follows that if $f \in L_0^1$, then $\tilde{\mathbf{A}}_1 f, \tilde{\mathbf{B}}_1 f \in L_0^1$.

We introduce the operators

$$\mathbf{T}_2^{(\infty)} f = f(x) + 2Y_2(x)\int_x^\infty (Y_2^{-1}(s))' f(s)\,ds,$$

$$\tilde{\mathbf{T}}_2^{(\infty)} f = f(x) - 2Y_2'(x)\int_x^\infty Y_2^{-1}(s) f(s)\,ds.$$

From the estimates (1.5), (1.6) follows

Lemma 2.2. *The operator $\mathbf{T}_2^{(\infty)} \in \mathcal{L}(L_1, L_1)$, $\mathcal{L}(L_\infty, L_\infty)$ and $\tilde{\mathbf{T}}_2^{(\infty)} \in \mathcal{L}(L_1, L_1)$.*

Next let us mention that by changing the order of integration it is easy to prove

Lemma 2.3. *The following equalities hold:*

$$(\tilde{\mathbf{S}}_2^{(0)} h, f) = (h, \mathbf{T}_2^{(\infty)} f), \quad h \in L_1, L_\infty, \ f \in L_\infty, L_1, \tag{2.1}$$

$$(\mathbf{S}_2^{(0)} h, f) = (h, \tilde{\mathbf{T}}_2^{(\infty)} f), \quad h \in L_\infty, \ f \in L_1, \tag{2.2}$$

i. e. the adjoint operator to $\mathbf{T}_2^{(\infty)} \in \mathcal{L}(L_\infty, L_\infty), \mathcal{L}(L_1, L_1)$ is the operator $\tilde{\mathbf{S}}_2^{(0)} \in \mathcal{L}(L_1, L_1), \mathcal{L}(L_\infty', L_\infty)$ and the adjoint operator to $\tilde{\mathbf{T}}_2^{(\infty)} \in \mathcal{L}(L_1, L_1)$ is the operator $\mathbf{S}_2^{(0)} \in \mathcal{L}(L_\infty, L_\infty)$.

Remark 2.1. Lemma 2.2 allows the following more precise formulation:

$$\mathbf{T}_2^{(\infty)} \in \mathcal{L}(L_1, L_1(Y_2')), \mathcal{L}(L_\infty, L_\infty(Y_2')),$$
$$\tilde{\mathbf{T}}_2^{(\infty)} \in \mathcal{L}(L_1, L_1(Y_2)).$$

In fact, from (1.5), (1.6) it follows that $Y_2 \in L_\infty, Y_2' \in L_1, L_\infty$. Further, since $\mathbf{S}_2^{(0)} Y_2 = 0$, $\tilde{\mathbf{S}}_2^{(0)} Y_2' = 0$, replacing h in (2.1), (2.2) by Y_2' and Y_2, respectively, we obtain

$$(Y_2', \mathbf{T}_2^{(\infty)} f) = 0, \ f \in L_1, L_\infty, \quad (Y_2, \tilde{\mathbf{T}}_2^{(\infty)} f) = 0, \ f \in L_1, \tag{2.3}$$

which we had to prove.

Now let us introduce the protective operators on the subspaces $L_1(Y_2)$, $L_{1(\infty)}(Y_2')$ as follows:

$$\mathbf{P}(Y_2)h \in L_1(Y_2), \quad h \in L_1, \qquad \mathbf{P}(Y_2')h \in L_{1(\infty)}(Y_2'), \quad h \in L_{1(\infty)}.$$

From the biorthogonality relations (see Theorem 0.5) we have the representations

$$\mathbf{P}_{1(\infty)}(Y_2')h = h(x) + \tilde{F}_{1,1}^{(2)}(x)(h, Y_2'), \quad h \in L_{1(\infty)}, \tag{2.4}$$

$$\mathbf{P}(Y_2)h = h(x) + \tilde{F}_{1,1}^{(2)'}(x)(h, Y_2), \quad h \in L_1. \tag{2.5}$$

From the asymptotics (1.5), (1.6) we get

$$(Y_2, Y_2') = \frac{1}{2} \lim_{x\to\infty} Y_2^2(x) = C_0^{-1}, \tag{2.6}$$

hence the following representations hold:

$$\mathbf{P}(Y_2)h = h(x) - C_0 Y_2'(x)(h, Y_2), \quad h \in L_1, \tag{2.7}$$

$$\mathbf{P}_{1(\infty)}(Y_2')h = h(x) - C_0 Y_2(x)(h, Y_2'), \quad h \in L_{1(\infty)}. \tag{2.8}$$

Note that from (1.12) it follows

$$\int_0^\infty \tilde{\mathbf{S}}_2^{(0)} f(x)\, dx = -\int_0^\infty f(x)\, dx + 2C_\infty(f, Y_2),$$

where $C_\infty = \lim_{x\to\infty} Y_2^{-1}(x)$. Hence $\tilde{\mathbf{S}}_2^{(0)} f \in L_0^1$ if and only if $f \in L_0^1 \cap L^1(Y_2)$. Note also that from (2.5) follows $\int_0^\infty \mathbf{P}(\mathbf{Y_2})h(x)\, dx = \int_0^\infty h(x)\, dx$, $h \in L_1$. In this way we get

Corollary 2.1. *Let $h \in L_0^1$ and $\mathbf{P}(Y_2)h$ be defined by (2.5). Then $\tilde{\mathbf{S}}_2^{(0)}\mathbf{P}(Y_2)h \in L_0^1$.*

In addition to Lemma 2.3, the connection between the operators $\mathbf{S}_2^{(0)}$, $\tilde{\mathbf{S}}_2^{(0)}$ and $\mathbf{T}_2^{(\infty)}$, $\tilde{\mathbf{T}}_2^{(\infty)}$ gives the following

Theorem 2.1. (i) *The general solution of the equation*

$$\mathbf{S}_2^{(0)} h(x) = f(x), \quad f \in L_\infty, \tag{2.9}$$

in the space L_∞ is

$$h(x) = \mathbf{T}_2^{(\infty)} f(x) + C_0 Y_2(x)(h, Y_2'). \tag{2.10}$$

In particular, the unique solution of the equation

$$\mathbf{S}_2 g(x) = f(x), \quad f \in L_{1(\infty)},$$

in the subspace $L_{1(\infty)}(Y_2')$ *is*

$$g(x) = \mathbf{T}_2^{(\infty)} f(x). \tag{2.11}$$

(ii) *The general solution of the equation*

$$\tilde{\mathbf{S}}_2^{(0)} h(x) = f(x), \quad f \in L_1,$$

in the space L_1 *is*

$$h(x) = \tilde{\mathbf{T}}_2^{(\infty)} f(x) + C_0 Y_2'(x)(h, Y_2).$$

In particular, the unique solution of the equation

$$\tilde{\mathbf{S}}^{(0)} g(x) = f(x), \quad f \in L_1,$$

in the subspace $L_{1(\infty)}(Y_2)$ *is*

$$g(x) = \tilde{\mathbf{T}}_2^{(\infty)} f(x).$$

Proof. From Lemma 2.1 [4] it follows that the general solution of (2.9) is given by the formula

$$h(x) = \mathbf{T}_2^{(\infty)} f(x) + C Y_2(x),$$

where the constant C is determined using (2.3), (2.6). This gives (2.10). Next note that from (2.8) it follows that (2.10) could be rewritten in the form $\mathbf{P}(Y_2')h = \mathbf{T}_2^{(\infty)} f$. Now, in order to obtain (2.11), it remains to mention that if $g \in L_{1(\infty)}(Y_2')$, then $\mathbf{P}_{1(\infty)}(Y_2')g = g$. The theorem is proved.

Remark 2.2. Having in mind the definitions of the projectors $\mathbf{P}_{1(\infty)}(Y_2')$ and $\mathbf{P}(Y_2)$, one can say that the solution of equation $\mathbf{S}_2\mathbf{P}_{1(\infty)}(Y_2')h = f$, $f \in L_{1(\infty)}$, in $L_{1(\infty)}(Y_2')$ is $\mathbf{P}_{1(\infty)}(Y_2')h = \mathbf{T}_2^{(\infty)} f$ and the solution of equation $\tilde{\mathbf{S}}_2\mathbf{P}(Y_2)h = f$, $f \in L_1$, in $L_1(Y_2)$ is $\mathbf{P}_1(Y_2)h = \tilde{\mathbf{T}}_2^{(\infty)} f$.

3. TRANSFORMATIONS OF THE EXPANSION FORMULAS

First we shall show how one can obtain Theorem 0.3 from Theorem 0.4.

Theorem 3.1. *Let by the potentials* $v_2^{(n)}$ *in equations* (0.11) *are constructed the potentials* $v^{(n)}(x)$ (0.19)–(0.20). *Then if we have the expansion formulas of Theorem* 0.4, *the expansion formulas of Theorem* 0.3 *are true if the functions* $\Phi(x,k)$ *and* $\tilde{F}(x,k)$ *are constructed as in Theorem* 1.1.

Proof. From the definition (2.4) of the operator $\mathbf{P}_1(Y_2')$ it follows that the expansion (0.13) can be written in the form

$$g(x) \equiv \mathbf{P}_1(Y_2')h(x) = -\int_0^\infty \tilde{F}_2(x,k)(h, \Phi_2'(k))\, dk. \tag{3.1}$$

Now let us apply to the both sides of this equality the operator $\mathbf{S} = \mathbf{A_1 S_2}$. If we remark that from (0.16) it follows $\tilde{F}_2(x,k) \in L_\infty(Y_2')$ and the transformation (1.15) can be written in the form $\tilde{F}(x,k) = \mathbf{S}\tilde{F}_2(x,k)$, we obtain

$$\mathbf{SP}(Y_2')h(x) = -\int_0^\infty \tilde{F}(x,k)(h, \Phi_2'(k))\,dk. \tag{3.2}$$

Here and further on we suppose that the convergence of the integral in the right-hand side of (3.1) is absolute with respect to k and uniform with respect to x (see, e. g. [2]). In particular, since $\mathbf{S} \in \mathcal{L}(L_\infty, L_\infty)$, we can apply the operator $\mathbf{S}$ under the sign of the integration. Now let construct by $f \in L_1$ the function $h = \mathbf{T}^{(\infty)} f = \mathbf{T}_2^{(\infty)} \mathbf{B_1} f$ and insert it in (3.2). Let notice that from (2.3) follows $h \in L_1(Y_2')$, hence (3.2) gives

$$\mathbf{ST}^{(\infty)} f(x) = -\int_0^\infty \tilde{F}(x,k)(\mathbf{T}^{(\infty)} f, \Phi_2'(k))\,dk.$$

Here we have also used the obvious fact that for any $h \in L_1(Y_2')$ we have $\mathbf{P}(Y_2')h = h$. Further, from the equality $\mathbf{T}^{(\infty)*} = \tilde{\mathbf{S}}^{(0)}$ and the representation (1.14) we get

$$(\mathbf{T}^{(\infty)} f, \Phi_2'(k)) = (f, \tilde{\mathbf{S}}^{(0)} \Phi_2'(k)) = (f, \Phi'(k)).$$

Finally, in order to obtain (0.8), it remains to mention that $\mathbf{ST}^{(\infty)} f = f$.

Now we shall show how one can obtain the expansion (0.10) from (0.15). Having in mind the definition (2.5) of the operator $\mathbf{P}(Y_2)$, let rewrite (0.15) in the form

$$\mathbf{P}(Y_2)h(x) = \int_0^\infty \tilde{F}_2'(x,k)(h, \Phi_2(k))\,dk, \quad h \in L_0^1.$$

Applying to the both sides of this equation the operator $\tilde{\mathbf{S}}^{(0)} = \tilde{\mathbf{A}}_1 \tilde{\mathbf{S}}_2^{(0)}$ and taking into account (1.15), we get

$$\tilde{\mathbf{S}}^{(0)} \mathbf{P}(Y_2)h(x) = \int_0^\infty \tilde{F}'(x,k)(h, \Phi_2(k))\,dk, \quad h \in L_0^1. \tag{3.3}$$

Further, let us construct through $f \in L_0^1$ the function $h = \tilde{\mathbf{T}}^{(\infty)} f \in L_0^1$. Note that the last inclusion follows from the equality

$$\int_0^\infty \tilde{\mathbf{T}}^{(\infty)} f(x)\,dx = -\int_0^\infty f(x)\,dx, \quad f \in L_1.$$

Inserting h in (3.3), we obtain

$$\tilde{\mathbf{S}}^{(0)} \mathbf{P}(Y_2) \tilde{\mathbf{T}}^{(\infty)} f(x) = \int_0^\infty \tilde{F}'(x,k)(\tilde{\mathbf{T}}^{(\infty)} f, \Phi_2(k))\,dk. \tag{3.4}$$

In the left-hand side of (3.4) we have $f(x)$ since $\mathbf{P}(Y_2)\tilde{\mathbf{T}}^{(\infty)}f = \tilde{\mathbf{T}}^{(\infty)}f$ and $\tilde{\mathbf{S}}\tilde{\mathbf{T}}^{(\infty)}f = f$. In order to obtain (0.10), it remains to notice that from (1.14) we get

$$(\tilde{\mathbf{T}}^{(\infty)}f, \Phi_2(k)) = (f, \mathbf{S}^{(0)}\Phi_2(k)) = (f, \Phi(k)).$$

Here we have also used the fact that according to Corollary 2.1 the left-hand side of (3.4) is in L_0^1 for any $f \in L_0^1$.

In a similar way, taking into account the representation (2.7) of $\mathbf{P}(Y_2)$ and the equation $\tilde{\mathbf{S}}^{(0)}Y_2' = 0$, one could obtain (0.9) from (0.14). The theorem is proved.

The transition from Theorem 0.3 to Theorem 0.4 is given by the following

Theorem 3.2. *Let by the equations* (0.11) *the system of functions* $\{\Phi_2\}$ *and* $\{\tilde{F}_2\}$ *be constructed. Consider together with* (0.11) *the equations* (0.6), *where the potentials* $v^{(n)}$ *are connected with* $v_2^{(n)}$ *by the formulas* (0.17)–(0.18), *and the functions* $\Phi(x,k)$ *and* $\tilde{F}(x,k)$ *are expressed by means of* $\{\Phi_2\}$ *and* $\{\tilde{F}_2\}$ *as in Theorem* 1.1. *Then the expansion formulas of Theorem* 0.4 *are a consequence of the expansion formulas of Theorem* 0.3.

Proof. First we shall show how one can obtain (0.13) from (0.8). Let us set in (0.8) $f = \mathbf{SP}(Y_2')h$, $h \in L_1$, where the operator $\mathbf{P}(Y_2')$ is defined by (2.4). We have the expansion

$$\mathbf{SP}(Y_2')h(x) = -\int_0^\infty \tilde{F}(x,k)(\mathbf{SP}(Y_2')h, \Phi'(k))\,dk,$$

which in view of Theorem 1.1 can be written in the form

$$\mathbf{SP}(Y_2')h(x) = \int_0^\infty \mathbf{S}\tilde{F}_2(x,k)(\mathbf{SP}(Y_2')h, \tilde{\mathbf{S}}^{(0)}\Phi_2'(k))\,dk. \tag{3.5}$$

From $\tilde{\mathbf{S}}^{(0)*} = \mathbf{T}^{(\infty)}$ and Lemma 2.4 it follows that

$$\mathbf{T}^{(\infty)}\mathbf{SP}(Y_2')h = \mathbf{P}(Y_2')h, \quad h \in L_1, L_\infty, \tag{3.6}$$

and

$$(\mathbf{SP}(Y_2')h, \tilde{\mathbf{S}}^{(0)}\Phi_2'(k)) = (\mathbf{P}(Y_2')h, \Phi_2'(k)).$$

These equations together with the representation (2.4) and biorthogonality relations of Theorem 0.5 give

$$(\mathbf{SP}(Y_2')h, \tilde{\mathbf{S}}^{(0)}\Phi_2'(k)) = (h, \Phi_2'(k)). \tag{3.7}$$

Let us apply the operator $\mathbf{T}^{(\infty)}$ to the both sides of (3.5). Having in mind (3.6) and (3.7), we get

$$\mathbf{P}(Y_2')h(x) = \int_0^\infty \tilde{F}_2(x,k)(h, \Phi_2'(k))\,dk.$$

Note that $\tilde{F}_2(x,k) \in L_\infty(Y_2')$ and $\int_0^\infty \tilde{F}_2(x,k)(h,\Phi_2'(k))\,dk \in L_1(Y_2')$ (see the proof in [2]). This, together with (2.4), gives (0.13). In a similar way one could obtain

(0.15) from (0.10) starting from the definition (2.6) of the operator $\mathbf{P}(Y_2)$. We shall omit here the details of the proof.

Next we shall show how one can obtain (0.14) from (0.9). Let us set in (0.9)

$$f(x) = \tilde{\mathbf{S}}^{(0)}\mathbf{P}(Y_2)h(x), \quad h \in L_1,$$

where $\mathbf{P}(Y_2)$ is defined by (2.7). As above, taking into account the biorthogonality relations from Theorem 0.5, we get

$$(\tilde{\mathbf{S}}^{(0)}\mathbf{P}(Y_2)h, \tilde{F}(k)) = (h, \tilde{F}_2(k)).$$

From here it follows

$$\tilde{\mathbf{S}}^{(0)}\mathbf{P}(Y_2)h(x) = -\tilde{\mathbf{S}}^{(0)}\left[\int_0^\infty \Phi_2'(x,k)(h, \tilde{F}_2(k))\,dk\right],$$

where we suppose that

$$\tilde{f}(x) = \int_0^\infty \Phi_2'(x,k)(h, \tilde{F}_2(k))\,dk \in L_1.$$

The relation $\tilde{f} \in L_1$ could be proved as in [6]. Applying Theorem 2.1, we have

$$\mathbf{P}(Y_2)h(x) = -\int_0^\infty \Phi_2'(x,k)(h, F_2(k))\,dk + C_0 Y_2'(x)(\tilde{f}, Y_2).$$

This, together with (2.7), leads to the expansion

$$h(x) = -\int_0^\infty \Phi_2'(x,k)(h, F_2(k))\,dk + C(h)Y_2'(x), \tag{3.8}$$

where we denote $C(h) = C_0(h, Y_2) + C_0(\tilde{f}, Y_2)$. Taking the scalar product of (3.8) with $\tilde{F}_2(x,k)$ and using again the biorthogonality relations from Theorem 0.5, we obtain that

$$C(h) = -(h, \tilde{F}_{1,1}^{(2)}).$$

The theorem is proved.

4. TRANSFORMATION OF THE EXPANSION FORMULAS IN THE GENERAL CASE

Let consider two boundary value problems (0.6)–(0.7), where the potentials $(v^{(1)}, v^{(2)}) \in \Omega(N, M)$. The main result of [2] is the following

Theorem 4.1. *Let $(v^{(1)}, v^{(2)}) \in \Omega(N, M)$, then for any absolutely continuous function $f(x) \in L_1$ the following expansion formulas hold:*

$$f(x) = -\int_0^\infty \tilde{F}(x,k)(f,\Phi'(k))\,dk$$

$$-\sum\nolimits_{\sigma'} \tilde{F}_{n,j}(x)(f,\Phi'(k_{n,j})) - \sum\nolimits_{\sigma''}\{\tilde{F}_{j,1}(x)(f,\Phi'(k_j)) + \tilde{F}_{j,2}(x)(f,\dot{\Phi}'(k_j))\},$$

$$f(x) = -\int_0^\infty \Phi'(x,k)(f,\tilde{F}(k))\,dk$$

$$-\sum\nolimits_{\sigma'} \Phi'_{n,j}(x)(f,\tilde{F}_{n,j}) - \sum\nolimits_{\sigma''}\{\Phi'(x,k_j)(f,\tilde{F}_{j,1}) + \dot{\Phi}'(x,k_j)(f,\tilde{F}_{j,2})\},$$

and if $f(x) \in L^1_0$, *then*

$$f(x) = \int_0^\infty \tilde{F}'(x,k)(f,\Phi(k))\,dk$$

$$+\sum\nolimits_{\sigma'} \tilde{F}'_{n,j}(x)(f,\Phi(k_{n,j})) + \sum\nolimits_{\sigma''}\{\tilde{F}'_{j,1}(x)(f,\Phi(k_j)) + \tilde{F}'_{j,2}(x)(f,\dot{\Phi}(k_j))\}.$$

Let take a number $k_1 = i\tau_1, \tau_1 > 0, |k_1| > \max_\sigma |k_{n,j}|$. Making the transformations similar to those in Theorem 0.6, we obtain new potentials $(v_2^{(1)}, v_2^{(2)}) \in \Omega(N+1, M)$. The analog of Theorem 1.1 here is the next

Lemma 4.1. *The following representations hold:*

(4.1) $$\Phi(x,k) = \mathbf{S}^{(0)}\Phi_2(x,k), \quad \Phi'(x,k) = \tilde{\mathbf{S}}^{(0)}\Phi'_2(x,k), \ k \in (0,\infty)\cup\sigma,$$

(4.2) $$\dot{\Phi}(x,k_j) = \mathbf{S}^{(0)}\dot{\Phi}_2(x,k_j), \quad \dot{\Phi}'(x,k_j) = \tilde{\mathbf{S}}^{(0)}\dot{\Phi}'_2(x,k_j), \ k_j \in \sigma'',$$

(4.3) $$\tilde{F}(x,k) = \mathbf{S}^{(0)}\tilde{F}_2(x,k), \quad \tilde{F}'(x,k) = \tilde{\mathbf{S}}^{(0)}\tilde{F}'_2(x,k), \ k \in (0,\infty),$$

(4.4) $$\tilde{F}_{n,j}(x) = \mathbf{S}^{(0)}\tilde{F}^{(2)}_{n,j}(x), \quad \tilde{F}'_{n,j}(x) = \tilde{\mathbf{S}}^{(0)}\tilde{F}^{(2)'}_{n,j}(x), \ k_{n,j} \in \sigma',$$

(4.5) $$\tilde{F}_{j,m}(x) = \mathbf{S}^{(0)}\tilde{F}^{(2)}_{j,m}(x), \quad \tilde{F}'_{j,m}(x) = \tilde{\mathbf{S}}^{(0)}\tilde{F}^{(2)'}_{j,m}(x), \ k_j \in \sigma'', m = 1,2.$$

Proof. One could obtain (4.1)–(4.3) following the proof of Theorem 1.1. In order to obtain (4.4), (4.5), one could take into account in addition the relations

$$a^{(2)}_{n,j} = \left(\frac{k_{n,j}+k_1}{k_{n,j}-k_1}\right) a_{n,j}, \quad k_{n,j} \in \sigma',$$

$$b^{(2)}_{n,j} = \left(\frac{k_j+k_1}{k_j-k_1}\right) b_j, \quad d^{(2)}_j = d_j + \frac{2k_1}{k_1^2-k_j^2}, \quad k_j \in \sigma'',$$

where the constants $a_{n,j}$, b_j and d_j, which determine the functions $\tilde{F}_{n,j}(x)$, $\tilde{F}_{j,1}(x)$ and $\tilde{F}_{j,2}(x)$, are defined as in (0.8).

Using this lemma and following the constructions of Section 3, it is not difficult to transform the expansion formulas given in Theorem 4.1 to the expansions corresponding to the case $(v_2^{(1)}, v_2^{(2)}) \in \Omega(N+1, M)$, where the potentials $v_2^{(1)}$, $v_2^{(2)}$ are expressed in terms of $v^{(1)}$, $v^{(2)}$ as in (0.17).

Remark 4.1. In [4] we have constructed the transformations $\Omega(M, N) \leftrightarrow \Omega(M \pm 1, N \pm 1)$. The restriction $\sigma' = \varnothing$ could be avoided, because from the easily verifiable relation

$$a_{n,j}^{(2)} = \left(\frac{k_{n,j} + k_0}{k_{n,j} - k_0}\right)^2 a_{n,j}, \quad k_{n,j} \in \sigma',$$

follow the representations

$$\tilde{F}_{n,j}(x) = \mathbf{A}\tilde{F}_{n,j}^{(2)}(x), \quad \tilde{F}'_{n,j}(x) = \tilde{\mathbf{A}}\tilde{F}_{n,j}^{(2)}(x),$$

where the operators $\mathbf{A}$, $\tilde{\mathbf{A}}$ are defined as in [4].

In this way, combining the transformations obtained here and in [4], we get a simple procedure which reduces the problem of expansions in products of solutions of two equations with $(v^{(1)}, v^{(2)}) \in \Omega(M, N)$ to the simplest one $(v^{(1)}, v^{(2)}) \in \Omega(0, 0)$. In more details the similar constructions for the Schrödinger equations with potentials $v_l^{(n)}(x) = l(l+1)x^{-2} + v^{(n)}(x)$, $v^{(n)} \in X_1$; $n = 1, 2$; $l = 1, 2, \ldots$, is considered in [5].

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