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CHARACTERIZATION OF THE EFFECTIVE COMPUTABILITY IN $\mathfrak{f}$-ENUMERATIONS

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Румен Димитров. ХАРАКТЕРИЗАЦИЯ ЭФФЕКТИВНОЙ ВЫЧИСЛИМОСТИ ЧЕРЕЗ $\mathfrak{f}$-НУМЕРАЦИИ

В начале статьи данны дефиниции понятий $\mathfrak{f}$-базис, $\mathfrak{f}$-нумерация и $\mathfrak{f}$-допустимость. Основным результатом является эквивалентность $\mathfrak{f}$-допустимых функций и просто вычислимых функций в структуре $\mathfrak{N}$. В конце доказываются три варианта основной теоремы. Первые две дают обобщения части „$\Leftarrow$“ и „$\Rightarrow$“ главной теоремы. Третий характеризирует потенциальные $\mathfrak{f}$-допустимые функции.

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The main definitions of $\mathfrak{f}$-basis, $\mathfrak{f}$-enumeration and $\mathfrak{f}$-admissibility are given. As a main result the equivalence between $\mathfrak{f}$-admissibility and prime computability in $\mathfrak{N}$ is proved. Finally, three variants of the main theorem are proved. The first two ones are generalizations of the directions "$\Leftarrow$" and "$\Rightarrow$" of the main theorem. In the third variant potentially $\mathfrak{f}$-admissible functions are concerned.

1. INTRODUCTION

The notion of prime and search computability on abstract structures was introduced by Moschovakis [5] in 1969. An equivalent but more natural definition of prime computability was given by Skordev [8]. An important question is to characterize the prime computable functions on structures with domains the set of all natural numbers N. A well-known result is that all functions which can be computed using the functions S (successor), P (predecessor) and the predicate Z (zero recognition) are the μ-recursive functions. Here we study computability in the structure $\mathfrak{N} = (N; P; Z)$.

Our approach is external and is based on the characterizations of abstract computability by means of enumerations. This approach was initiated by Lacombe [4] and studied in [2, 3, 6, 9, 10]. In this paper we study a special class of enumerations of the structure $\mathfrak{N}$ — the $\mathfrak{f}$-enumerations. We prove the equivalence of $\mathfrak{f}$-admissibility and prime computability on $\mathfrak{N}$. As the set of $\mathfrak{f}$-enumerations is a proper subset of all enumerations, where P and Z are effective, in this case the result in one direction is stronger than that proved by Soskov in [10] or in [9].

2. NOTATIONS

Let $\mathfrak{N} = (N; P; Z)$ be the structure with a domain N, a single operation $P(x) = x \dot{-} 1$, and a single predicate $Z(x)$ which gives true for $x = 0$ and false otherwise.

Let p_i be the i-th prime number. We write $(x)_i$ for the primitive recursive function

$$\gamma(x, i) = \begin{cases} \max\{t : p_i^t / x\} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

and $\gamma_0(x)$ for $\gamma(x, 0)$.

We shall fix a coding $\langle\ \rangle$ of the finite sequences of natural numbers such that

$$\langle x_1, x_2, \ldots, x_n \rangle = \mu s\, [s > 0 \,\&\, (s)_0 = n \,\&\, (s)_1 = x_1 \,\&\, \ldots \,\&\, (s)_n = x_n],$$

i.e. $\langle x_1, x_2, \ldots x_n \rangle = 2^n . 3^{x_1} \ldots p_n^{x_n}$.

We shall write $\downarrow f(x_1, \ldots, x_n)$ if $f(x_1, \ldots, x_n)$ is defined, and $\uparrow f(x_1, \ldots, x_n)$ if it is not.

3. BASIC DEFINITIONS

Definition 1. A set $A \subseteq N$ is called $\mathfrak{f}$-basis if there exists a total function $\Psi : N \to N$ such that $A = \{\langle \Psi(0), \ldots, \Psi(i-1) \rangle : i \in N\}$.

Definition 2. The ordered pair $\langle A, \alpha \rangle$ is called $\mathfrak{f}$-enumeration if A is an $\mathfrak{f}$-basis and $\alpha = \gamma_0 \lceil A$ (i.e. $\alpha(\langle x_1, x_2, \ldots, x_n \rangle) = n$).

Note. If $\langle A, \alpha \rangle$ is an $\mathfrak{f}$-enumeration, then α is an 1,1 mapping from A to N.

Definition 3. Let $\alpha : A \to B$ be a surjective mapping, where $A \subseteq N$. A function $f : B^n \multimap B$ is called effective in $\langle A, \alpha \rangle$ if there exists a partial recursive function $\varphi : N^n \multimap N$ such that

$$(\forall a_1 \in A) \ldots (\forall a_n \in A) \left(f\left(\alpha(a_1), \ldots, \alpha(a_n)\right) \simeq \alpha\left(\varphi(a_1, \ldots, a_n)\right)\right).$$

Remark. It is clear that given a code $\langle \Psi(0), \ldots, \Psi(i-1) \rangle$ of an i-tuple, we can effectively recognize whether $i = 0$ (i.e. whether the sequence is empty), and if $i \neq 0$, then we can find the code of the sequence $\Psi(0), \ldots, \Psi(i-2)$. It means that in every $\mathfrak{f}$-enumeration P and Z are effective. Notice that in a fixed $\mathfrak{f}$-enumeration $\langle A, \alpha \rangle$ the function S (successor) is effective iff the function Ψ is recursive.

Definition 4. A partial function $f : N^n \multimap N$ is called $\mathfrak{f}$-admissible if it is effective in all $\mathfrak{f}$-enumerations.

Remark. The definition is correct, because for every $\mathfrak{f}$-enumeration $\langle A, \alpha \rangle$ the mapping α is surjective.

4. THE MAIN RESULT

Soskov has proved in [11] that a function f is prime computable (see [5]) in the structure $\mathfrak{N}$ iff it is partial recursive and

$$\forall x_1 \ldots \forall x_n \forall y \, (f(x_1, \ldots, x_n) = y \longrightarrow y \leqq \max(x_1, \ldots, x_n)),$$

i.e.

$$\forall x_1 \ldots \forall x_n \, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n)).$$

Here, for the proof of Skordev's conjecture (the main theorem), we are not going to use the prime computability.

Theorem 1. *A function $f : N^n \multimap N$ is $\mathfrak{f}$-admissible iff it is partial recursive and there exists a natural number c such that the following condition is true:*

$$\forall x_1 \ldots \forall x_n \, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)).$$

Proof. **A.** Let f be a partial recursive function and c be a natural number such that

$$\forall x_1 \ldots \forall x_n \, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)).$$

Given an $\mathfrak{f}$-enumeration $\langle A, \alpha \rangle$, we shall construct a partial recursive function φ, so that

$$f(\alpha(a_1), \ldots, \alpha(a_n)) \simeq \alpha(\varphi(a_1, \ldots, a_n))$$

for all $a_1, \ldots, a_n$ of A. The construction is standard and we shall not go into details. Since α is an 1,1 mapping from A to N, there exists $a \in A$ such that $\alpha(a) = c$. Fix a and let Max : $N^n \to N$ be the primitive recursive function such that for all a_1, $\ldots$, a_n of A

$$\mathrm{Max}(a_1, \ldots, a_n) = \begin{cases} a_i & \text{if } \gamma_0(a_i) = \max(\gamma_0(a), \gamma_0(a_1), \ldots, \gamma_0(a_n)), \\ a & \text{if } \gamma_0(a) = \max(\gamma_0(a), \gamma_0(a_1), \ldots, \gamma_0(a_n)). \end{cases}$$

Let PRED : $N \to N$ be the primitive recursive function such that PRED(x) gives the code $\langle a_0, \ldots, a_{n-1} \rangle$ of the sequence $a_0, \ldots, a_{n-1}$ if x is the code of the sequence $a_0, \ldots, a_{n-1}, a_n$, and PRED$(x) = x$ if x is the code of the empty sequence.

Define S1 : $N^2 \to N$ by the following equations:

$$\mathrm{S1}(x, 0) = x, \qquad \mathrm{S1}(x, t+1) = \mathrm{PRED}(\mathrm{S1}(x, t)).$$

It is clear that S1 is primitive recursive and for $x \in A$

$$\gamma_0(\mathrm{S1}(x, t)) = \gamma_0(x) \dot{-} t.$$

Let S2 : $N^n \longrightarrow\!\!\!\!\!\circ\; N$ be defined in the following way:

$$\mathrm{S2}(a_1,\ldots,a_n) \simeq \mu t\left[(\gamma_0(\mathrm{S1}(\mathrm{Max}(a_1,\ldots,a_n),t)) \dot{-} f(\gamma_0(a_1),\ldots,\gamma_0(a_n))) = 0\right].$$

Finally, if $\varphi : N^n \longrightarrow\!\!\!\!\!\circ\; N$ is defined by the equation

$$\varphi(a_1,\ldots,a_n) \simeq \mathrm{S1}(\mathrm{Max}_n(a_1,\ldots,a_n),\mathrm{S2}(a_1,\ldots,a_n)),$$

then it is easy to prove that φ is the function we are looking for.

B. In this direction, we shall prove that if f is $\mathfrak{f}$-admissible, then f is partial recursive and

$$(*) \qquad \exists c \forall a_1 \ldots \forall a_n \,(\downarrow f(a_1,\ldots,a_n) \longrightarrow f(a_1,\ldots,a_n) \leqq \max(a_1,\ldots,a_n,c)).$$

From the result in [9] we can obtain that if f is effective in all enumerations of the structure $\mathfrak{N}$, then f is definable in $\mathfrak{N}$. Here we require that f be effective only in $\mathfrak{f}$-enumerations of $\mathfrak{N}$ and we prove something equivalent to definability.

First we shall prove that f is partial recursive. Let $\langle A,\alpha\rangle$ be an $\mathfrak{f}$-enumeration, where $A = \{\langle\Psi(0),\ldots,\Psi(i-1)\rangle \;:\; i \in N\}$ and Ψ is recursive. It is clear from the definitions that α and α^{-1} are partial recursive. We know that there exists a partial recursive function φ such that

$$f(\alpha(a_1),\ldots,\alpha(a_n)) \simeq \alpha(\varphi(a_1,\ldots,a_n)) \quad \text{for all } a_1,\ldots,a_n \text{ of } A.$$

Since α is bijective, we obtain

$$f(x_1,\ldots,x_n) \simeq \alpha\left(\varphi(\alpha^{-1}(x_1),\ldots,\alpha^{-1}(x_n))\right).$$

Hence f is a partial recursive function.

Let us now suppose that $(*)$ is not true, i.e.

$$(\bar{*}) \qquad \forall c \exists a_1 \ldots \exists a_n \,(\downarrow f(a_1,\ldots,a_n) \,\&\, (f(a_1,\ldots,a_n) > \max(a_1,\ldots,a_n,c))).$$

First we shall prove the following

Lemma. *The set* $\{(x_1,\ldots,x_n) : f(x_1,\ldots,x_n) > \max(x_1,\ldots,x_n)\}$ *is infinite.*

Proof. Suppose $\left(x_1^{(1)},\ldots,x_n^{(1)}\right)$, $\ldots$, $\left(x_1^{(k)},\ldots,x_n^{(k)}\right)$ are all elements of this set. Let $c_1 = \max\left(f\left(x_1^{(1)},\ldots,x_n^{(1)}\right), \ldots, f\left(x_1^{(k)},\ldots,x_n^{(k)}\right)\right)$, and $c_1 = 0$ if the set is empty. By $(\bar{*})$ we can find numbers y_1, $\ldots$, y_n such that $(f(y_1,\ldots,y_n) > \max(y_1,\ldots,y_n,\; c_1))$. Then, obviously, $f(y_1,\ldots,y_n) > \max(y_1,\ldots,y_n)$, but $f(y_1,\ldots,y_n) \neq \left(x_1^{(i)},\ldots,x_n^{(i)}\right)$ for all $i = 1,\ldots,k$. We have supposed that the set is finite and obtained a contradiction.

Let φ_0, φ_1, $\ldots$ be a list of all partial recursive functions of n variables.

Definition 5. Let $\langle A,\alpha\rangle$ be an $\mathfrak{f}$-enumeration. An n-tuple $(a_1,\ldots,a_n)$, where a_1, $\ldots$, a_n belong to A, is called witness for the condition

$$(i) \qquad \neg(f(\alpha(a_1),\ldots,\alpha(a_n)) \simeq \alpha(\varphi_i(a_1,\ldots,a_n)))$$

if $\downarrow f(\alpha(a_1),\ldots,\alpha(a_n))$ and one of the following is true:

1) $\uparrow \varphi_i(a_1,\ldots,a_n)$;

2) $\downarrow \varphi_i(a_1,\ldots,a_n)$, but $\varphi_i(a_1,\ldots,a_n) \notin A$, i.e. $\uparrow \alpha(\varphi_i(a_1,\ldots,a_n))$;

3) $\downarrow \alpha(\varphi_i(a_1, \ldots, a_n))$, but $f(\alpha(a_1), \ldots, \alpha(a_n)) \neq \alpha(\varphi_i(a_1, \ldots, a_n))$.

An f-basis A that consists of the numbers $a_0 < a_1 < a_2 < \cdots$ will be defined in steps. In each step l (for $l = -1, 0, 1, 2, \ldots$) we shall define a finite approximation $A_l = \{a_0, a_1, \ldots, a_{k_l}\}$. In other words, at the step l the values $\Psi(i)$ for $i = 0, \ldots, k_l - 1$ will be defined. In the step $(l+1)$ we shall build a set $A_{l+1} \supset A_l$ such that A_{l+1} will contain a witness for the condition $(l+1)$. Together with the set A we shall also define a set A^- such that $A \cap A^- = \varnothing$. In each step $(l+1)$ the condition $A^-_{l+1} \supseteq A^-_l$ will be met.

We shall prove that the set A is an f-basis. Next, if (A, α) is an f-enumeration, then we can find a witness for each of the conditions (i), where $i \in N$.

Step -1. Let $k_{-1} = 0$, $a_0 = 1$, $A_{-1} = \{1\}$, $A^-_{-1} = \varnothing$.

Suppose that in step (l) we have built the finite set A^-_l and the finite set A_l which consists of the elements $a_0, a_1, \ldots, a_{k_l}$. Suppose $\Psi(i)$ has been defined for $i < k_l$.

Step $l+1$. We shall define the sets A_{l+1} and A^-_{l+1}, so that A_{l+1} contains a witness for the condition $(l+1)$.

Let $(k^1_{l+1}, k^2_{l+1}, \ldots, k^n_{l+1})$ be an n-tuple such that

$$f(k^1_{l+1}, \ldots, k^n_{l+1}) > \max(k^1_{l+1}, \ldots, k^n_{l+1}) > k_l.$$

The choice of such n-tuple is possible because the set

$$\{(x_1, \ldots, x_n) \,:\, (\forall i \leqq n)(x_i \leqq k_l)\}$$

is finite while the set

$$\{(x_1, \ldots, x_n) \,:\, f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)\}$$

is infinite by the previous Lemma. Let $k^1_{l+1} = \max(k^1_{l+1}, \ldots, k^n_{l+1})$. Note that $k_{l+1} > k_l$. Let $h = (k_{l+1} - k_l)$. We shall define $a_{k_l+1}, \ldots, a_{k_{l+1}}$ and $\Psi(k_l), \ldots, \Psi(k_{l+1} - 1)$ successively, so that the following is true for $i = 1, \ldots, h$:

1) $a_{k_l+i} = 2a_{k_l+i-1}p_{k_l+i}^{\Psi(k_l+i-1)}$;

2) $a_{k_l+i} \notin A^-_l$.

The first will ensure that $a_i = \langle \Psi(0), \ldots, \Psi(i-1) \rangle$ for all $i \in N$ and thus A will be an f-basis. The second will ensure that the requirements (1), ..., (l) are not injured for the sake of $(l+1)$.

Since A^-_l is finite, we can define $\Psi(k_l + i - 1)$ and a_{k_l+i} successively for $i = 1, \ldots, h$ in the following way:

$$\Psi(k_l + i - 1) = \mu t \left[2a_{k_l+i-1}p^t_{k_l+i} \notin A^-_l\right] \text{ and } a_{k_l+i} = 2a_{k_l+i-1}p_{k_l+i}^{\Psi(k_l+i-1)}.$$

Note that 1) and 2) are true now.

Let $A_{l+1} = A_l \cup \{a_{k_l+1}, \ldots, a_{k_l+1}\}$ and

$$A^-_{l+1} = \begin{cases} A^-_l \cup \{\varphi_{l+1}(a_{k^1_{l+1}}, \ldots, a_{k^n_{l+1}})\} & \text{if } \downarrow \varphi_{l+1}(a_{k^1_{l+1}}, \ldots, a_{k^n_{l+1}}) \notin A_{l+1}, \\ A^-_l & \text{otherwise.} \end{cases}$$

From these definitions we can see that A_{l+1} and A^-_{l+1} are finite, $A_l \subset A_{l+1}$ and $A^-_l \subseteq A^-_{l+1}$.

Let $A = \bigcup_{i=-1}^{\infty} A_i$ and $A^- = \bigcup_{i=-1}^{\infty} A_i^-$. Now we have to prove that A is an $\mathfrak{f}$-basis and $\langle A, \alpha \rangle$ ($\alpha = \gamma_0 \upharpoonright A$) is the $\mathfrak{f}$-enumeration that we are looking for. It is clear that $a_0, a_1, \ldots$ are all the elements of A. In the following lemma we prove that A is an $\mathfrak{f}$-basis.

Lemma 1. *$a_i = \langle \Psi(0), \ldots, \Psi(i-1) \rangle$ for $i \in N$.*

Proof. Using the definitions, we can prove Lemma 1 easily by induction.

Next we shall see that $A^- \cap A = \varnothing$. For this purpose we shall prove

Lemma 2. *$A_k^- \cap A_k = \varnothing$ for all $k \geqq -1$.*

Proof. An induction is applied.

For $k = -1$ we know that $A_{-1} = \{a_0\}$ and $A_{-1}^- = \varnothing$ and obviously the statement is true.

Let suppose that for some natural l we have $A_l^- \cap A_l = \varnothing$. By construction $A_{l+1} = A_l \cup \{a_{k_l+1}, \ldots, a_{k_{l+1}}\}$ and $a_{k_l+1}, \ldots, a_{k_{l+1}} \notin A_l^-$. Using the induction hypothesis we derive that $A_l^- \cap A_{l+1} = \varnothing$. If $A_l^- = A_{l+1}^-$, then there is nothing to prove. Else

$$A_{l+1}^- = A_l^- \cup \left\{\varphi_{l+1}(a_{k_{l+1}^1}, \ldots, a_{k_{l+1}^n})\right\} \quad \text{and} \quad \varphi_{l+1}(a_{k_{l+1}^1}, \ldots, a_{k_{l+1}^n}) \notin A_{l+1}.$$

In this case it is obvious that $A_{l+1}^- \cap A_{l+1} = \varnothing$.

Now we are ready to prove

Lemma 3. *$A \cap A^- = \varnothing$.*

Proof. Suppose that there exists a number a such that $a \in A \cap A^-$. We can find i and j such that $a \in A_i$ and $a \in A_j^-$. If $k = \max(i, j)$, then $A_k^- \cap A_k \neq \varnothing$, which contradicts Lemma 2.

We shall see next that for each $i \in N$ $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i). Let i be a fixed natural number. The n-tuple $(k_i^1, k_i^2, \ldots, k_i^n)$ has been chosen in such a way that $f(k_i^1, \ldots, k_i^n) > \max(k_i^1, \ldots, k_i^n) = k_i$. Note that $f(\alpha(a_{k_i^1}), \ldots, \alpha(a_{k_i^n}))$ is defined. We shall consider the following cases for $\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})$:

1. $\downarrow \varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}) \in A_i$.

Since $A_i = \{a_0, \ldots, a_{k_i}\}$, then

$$(\forall a \in A_i)(\alpha(a) = \gamma_0(a) \leqq k_i) \quad \text{and} \quad \alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})) \leqq k_i.$$

We know that $f(k_i^1, \ldots, k_i^n) > k_i$ and hence

$$\alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})) \neq f(\alpha(a_{k_i^1}), \ldots, \alpha(a_{k_i^n})),$$

i.e. $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i) by 3) of Definition 5.

2. $\uparrow \varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})$.

Since $\uparrow \alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}))$, we obtain that $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i) by 1) of Definition 5.

3. $\downarrow \varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})$, but $\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}) \notin A_i$.

In this case $\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}) \in A_i^- \subseteq A^-$. By Lemma 3 $A \cap A^- = \varnothing$ and hence $\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}) \notin A$. We obtain $\uparrow \alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}))$ and then $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i) by 2) of Definition 5.

We have derived that $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i) and the theorem is proved.

Remark 1. In the construction of the set A we have not used the partial recursiveness of f.

Remark 2. The construction of the set A^- could be avoided because the requirement 2) (i.e. $a_{k_{l+1}^n} \notin A_l^-$) in step $(l+1)$ of the construction of the set A may be changed by the condition:

$$\text{if for some } l_1 \leqq l \quad y = \varphi_{l_1}(a_{k_{l_1}^1}, \ldots, a_{k_{l_1}^n}), \text{ then } a_{k_{l+1}^n} \neq y.$$

5. THREE VARIANTS OF THE THEOREM

First we shall prove a stronger result than that proved in the direction "$\Leftarrow$" of the main result.

Definition 6. Let $\alpha : A \to B$ be a surjective mapping, where $A \subseteq N$. A predicate $P(x_1, x_2, \ldots, x_n)$ on B is called effective in the enumeration $\langle A, \alpha \rangle$ if there exists a partial recursive function $\varphi : N^n \multimap \{0, 1\}$ such that

$$(\forall a_1 \in A) \ldots (\forall a_n \in A)(\downarrow \varphi(a_1, \ldots, a_n) \,\&\, (P(\alpha(a_1), \ldots, \alpha(a_n)) \Leftrightarrow \varphi(a_1, \ldots, a_n) = 1)).$$

Theorem 2. *If f is a function such that*

$$\exists c \forall x_1 \ldots \forall x_n(\downarrow f(x_1, \ldots, x_n) \to f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)),$$

then f is effective in all enumerations of the structure $\mathfrak{N}$.

Proof. First we shall note that enumerations of the structure $\mathfrak{N}$ are those for which the functions and predicated of the structure are effective. Obviously, the $\mathfrak{f}$-enumerations are enumerations of $\mathfrak{N}$.

Let $\langle A, \alpha \rangle$ be an enumeration of $\mathfrak{N}$. Let $Z : N \multimap \{0, 1\}$ and PRED : $N \multimap N$ be partial recursive functions such that for all $a \in A$:

1) $\alpha(\mathrm{PRED}(a)) \simeq \alpha(a) \dot{-} 1$;

2) $Z(a) = \begin{cases} 1 & \text{if } \alpha(a) = 0, \\ 0 & \text{if } \alpha(a) \neq 0. \end{cases}$

Obviously, $\downarrow \mathrm{PRED}(a) \in A$ and $\downarrow Z(a)$ for all $a \in A$. Thus $\downarrow Z(\mathrm{PRED}^t(a))$ for all $a \in A$ and $t \in N$.

Now we shall see that there exists a partial recursive function γ such that $\gamma\lceil A = \alpha$. Let us define $\gamma : N \multimap N$ in the following way:

$$\gamma(x) \simeq \mu t \left[Z(\mathrm{PRED}^t(x)) = 1\right].$$

We shall prove that $\gamma(a) = \alpha(a)$ for all $a \in A$, i.e.

$$\alpha(a) = \mu t \left[Z(\mathrm{PRED}^t(a)) = 1\right]. \tag{1}$$

Since $\downarrow \alpha(a) \in N$ for $a \in A$, then (1) could be proved by induction on $\alpha(a)$. The proof of this fact is left to the reader.

We are looking for a partial recursive function $\varphi : N^n \multimap N$ such that

$$(\forall a_1 \in A)\dots(\forall a_n \in A)(f(\alpha(a_1),\dots,\alpha(a_n)) \simeq \alpha(\varphi(a_1,\dots,a_n))).$$

The construction of φ is the same as the construction of the function φ in the proof of the main result.

In the proof of the main result we observed that if

$$\forall c\exists x_1\dots\exists x_n(\downarrow f(x_1,\dots,x_n)\,\&\, f(x_1,\dots,x_n) > \max(x_1,\dots,x_n,c)),$$

then there exists an f-enumeration $\langle A, \alpha\rangle$ such that f is not effective in $\langle A, \alpha\rangle$. We built the set A in steps. In each step we built a finite approximation of A. We shall analyze that construction and obtain a stronger result than the one proved in the direction "$\Rightarrow$" of Theorem 1. We have noted that the construction of A could be modified in such a way that the use of A^- be avoided. We shall use that f is partial recursive and modify the construction of A in the following way.

On the step $(l+1)$ we define effectively a code of an n-tuple $(k^1_{l+1}, k^2_{l+1},\dots,k^n_{l+1})$ such that

$$f(k^1_{l+1},\dots,k^n_{l+1}) > \max(k^1_{l+1},\dots,k^n_{l+1}) > k_l.$$

Later we define the recursive function $S : N \to N$ such that $S(l+1)$ gives the code of the n-tuple $(k^1_{l+1}, k^2_{l+1},\dots,k^n_{l+1})$, which was defined on step $(l+1)$.

Let F be a primitive recursive function such that

$$(x_1,x_2,\dots,x_n,y) \in G_f \Leftrightarrow \exists z(F(x_1,x_2,\dots,x_n,y,z) = 0)$$

and let $g(x_1,x_2,\dots,x_n,t) = F(x_1,x_2,\dots,x_n,L(t),R(t))$.

We know that f has the normal form

$$f(x_1,x_2,\dots,x_n) \simeq L(\mu t[g(x_1,x_2,\dots,x_n,t) = 0]).$$

Let J be a standard coding of ordered pairs in N. We denote

$$J^2 = J, \quad J^{n+1}(x_1,x_2,\dots,x_n,x_{n+1}) = J(J^n(x_1,x_2,\dots,x_n),x_{n+1}) \text{ for } n > 2.$$

Let Pr^n_m ($m \leqq n$) be a primitive recursive function such that

$$\mathrm{Pr}^n_m(J^n(x_1,x_2,\dots,x_n)) = x_m$$

(for $n = 2$ we write L for Pr^2_1 and R for Pr^2_2).

Define the functions $S : N \to N$ and $M : N \to N$ as it follows:

1) $S(-1) = 0$ and $M(-1) = 0$;

2) $S(l+1) = L(\mu s[g(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s), \mathrm{Pr}_{n+1}^{n+1}(s)) = 0$
$\quad \& \ L(\mathrm{Pr}_{n+1}^{n+1}(s)) > \max(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s))$
$\quad \& \ \max(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s)) > M(l)])$ and
$M(l+1) = \max(\mathrm{Pr}_1^n(S(l+1)), \mathrm{Pr}_2^n(S(l+1)), \ldots, \mathrm{Pr}_n^n(S(l+1)))$.

Remark. We shall define S and M for $n = -1$ in order to unify the definitions of S and M for $n = 0$ and $n > 0$, but we shall think that they are defined only for $n \geqq 0$.

By induction we shall prove that S and M are totally defined.

1. For $l = -1$ we have $S(-1) = M(-1) = 0$.
2. Let $\downarrow S(l)$ and $\downarrow M(l)$ for some natural l.
3. We know that the set $\{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)\}$ is infinite and there exists $(x_1, \ldots, x_n)$ such that

$$\downarrow f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n) > M(l).$$

Using the normal form of f, we derive that there exists s such that

$$g(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s), \mathrm{Pr}_{n+1}^{n+1}(s)) = 0$$
$$\& \ L(\mathrm{Pr}_{n+1}^{n+1}(s)) > \max(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s))$$
$$\& \ \max(\mathrm{Pr}_1^{n+1}(s), \mathrm{Pr}_2^{n+1}(s), \ldots, \mathrm{Pr}_n^{n+1}(s)) > M(l).$$

From here we can easily see that $\downarrow S(l+1)$ and then $\downarrow M(l+1)$. Thus, using the definition of the functions S and M, we derive that they are recursive. We can see that $S(l+1)$ is the code of the n-tuple $(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n)$, which was defined on step $(l+1)$, and that $M(l+1) = \max(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n) = k_{l+1}$.

Let $\mathfrak{F}$ be a universal for the partial recursive functions and recursive predicate such that: $\varphi_l(x_1, \ldots, x_n) = y \Leftrightarrow \exists z \mathfrak{F}(l, x_1, \ldots, x_n, y, z)$. We shall define a predicate $\mathfrak{B}(x)$ such that $\mathfrak{B}(x) \Leftrightarrow x \in A$. For this purpose first we shall define the predicate $\mathfrak{C}_0$ in the following way:

$$\mathfrak{C}_0(s, x, i) \Leftrightarrow \forall l_1 \leqq \mu l[(M(l) < ((x)_0 \dot{-} i)) \& (((x)_0 \dot{-} i) \leqq M(l+1))],$$
$$(\forall y \forall z(\mathfrak{F}(l_1, \mathrm{PRED}^{(x)_0 \dot{-} \mathrm{Pr}_1^n(S(l_1))}(x), \ldots, \mathrm{PRED}^{(x)_0 \dot{-} \mathrm{Pr}_n^n(S(l_1))}(x), y, z)$$
$$\longrightarrow (2\,\mathrm{PRED}^{i+1}(x) p^s_{(x)_0 \dot{-} i} \neq y))).$$

Next define the predicate $\mathfrak{C}_1$ as it follows:

$$\mathfrak{C}_1(s, x, i) \Leftrightarrow \mathfrak{C}_0(s, x, i) \& \forall s_1 < s \neg \mathfrak{C}_0(s_1, x, i).$$

Note that if $x = \langle a_1, \ldots, a_j \rangle$ $(j > i)$, then $\mathfrak{C}_1(s, x, i)$ is true if and only if s is defined just the way $\Psi(j - i - 1)$ is defined in the construction of the set A in the main theorem.

By the expression $(M(l) < ((x)_0 \dot{-} i)) \& (((x)_0 \dot{-} i) \leqq M(l+1))$ we find a number l such that $(l+1)$ is the number of the step, where $\mathrm{PRED}^i(x)$ is defined. Then for all $l_1 \leqq l$ we calculate the code of the first n-tuple $(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n)$ (i.e. $S(l_1)$) such that

$$f(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n) > \max(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n).$$

Further we find the least s such that for all $l_1 \leqq l$ if $\varphi_{l_1}(x_{k^1_{l_1}}, \ldots, x_{k^n_{l_1}}) = y$, then $2\,\mathrm{PRED}^{i+1}(x)p^s_{(x)_0 \dot{-} i} \neq y$.

Since M and S are recursive functions and $\mathfrak{F}$ is a recursive predicate, there exists a recursive predicate $\mathfrak{P}_1$ such that

$$\mathfrak{C}_1(s,x,i) \Leftrightarrow \forall q_1 \mathfrak{P}_1(s,x,i,q_1) \,\&\, (\forall s_1 < s)\neg\forall q_2 \mathfrak{P}_1(s_1,x,i,q_2),$$

i.e.

$$(**) \qquad \mathfrak{C}_1(s,x,i) \Leftrightarrow \forall q_1 \mathfrak{P}_1(s,x,i,q_1) \,\&\, \exists q_3 \mathfrak{P}_2(s,x,i,q_3),$$

where $\mathfrak{P}_2$ is again a recursive predicate.

Let us define $\mathfrak{B}$ in the following way:

$$\mathfrak{B}(x) \Leftrightarrow (\mathrm{PRED}^{(x)_0}(x) = 1 \,\&\, (\forall i < (x)_0)(\mathfrak{C}_1((x)_{((x)_0 \dot{-} i)}, x, i)).$$

Let notice that $\mathfrak{B}(x)$ is true if and only if $x = 2^n.3^{\Psi(0)} \ldots p_n^{\Psi(n-1)}$, where $\Psi(0)$, $\Psi(1)$, $\ldots$, $\Psi(n-1)$ are exactly like those ones, defined in the construction of the set A in the proof of the main result. In other words, $\mathfrak{B}(x) \Leftrightarrow x \in A$.

Using $(**)$, we can find recursive predicates $\mathfrak{P}_3$ and $\mathfrak{P}_4$ such that

$$x \in A \Leftrightarrow \mathfrak{B}(x) \Leftrightarrow \forall q_4 \mathfrak{P}_3(q_4, x) \,\&\, \exists q_5 \mathfrak{P}_4(q_5, x).$$

The set $\{x : \exists q_5 p_4(q_5, x)\}$ is recursively enumerable. The set $\{x : \forall q_4 p_3(q_4, x)\}$ is co-recursively enumerable. Thus the set A could be represented as a difference of two recursively enumerable sets. We proved the following result:

Theorem 3. *If the function $f : N \rightharpoonup N$ is effective in every $\mathfrak{f}$-enumeration $\langle A, \alpha\rangle$ such that A can be represented as an intersection of a recursively enumerable and a co-recursively enumerable set, then f is partial recursive and there exists $c \in N$ such that*

$$\forall x_1 \ldots \forall x_n (\downarrow f(x_1,\ldots,x_n) \longrightarrow f(x_1,\ldots,x_n) \leqq \max(x_1,\ldots,x_n,c)).$$

For the proof of the main theorem we have defined the term witness for the condition (i) $\neg(f(\alpha(a_1),\ldots,\alpha(a_n)) \simeq \alpha(\varphi_i(a_1,\ldots,a_n)))$, where the left-hand side of this conditional equality was defined for every witness. Now we shall use this fact to prove another variant of the main theorem.

Definition 7. If $\alpha : A \to B$, where $A \subseteq N$ is a surjective mapping, then $f : B^n \rightharpoonup B$ is said to be potentially effective in the enumeration $\langle A, \alpha\rangle$ if there exists a partial recursive function $\varphi : N^n \rightharpoonup N$ such that

$$(\forall a_1 \in A)\ldots(\forall a_n \in A)(\downarrow f(\alpha(a_1),\ldots,\alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1,\ldots,a_n)) \\ \& \, f(\alpha(a_1),\ldots,\alpha(a_n)) = \alpha(\varphi(a_1,\ldots,a_n))).$$

Definition 8. $f : N^n \rightharpoonup N$ is called potentially $\mathfrak{f}$-admissible if f is potentially effective in all $\mathfrak{f}$-enumerations.

Theorem 4. *A function $f : N \rightharpoonup N$ is potentially $\mathfrak{f}$-admissible if and only if f is potentially partial recursive and there exists $c \in N$ such that*

$$\forall x_1 \ldots \forall x_n (\downarrow f(x_1,\ldots,x_n) \longrightarrow f(x_1,\ldots,x_n) \leqq \max(x_1,\ldots,x_n,c)).$$

Proof. **A.** Let $\langle A, \alpha\rangle$ be a functional enumeration, f be potentially partial recursive, and let exist $c \in N$ such that

$$\forall x_1 \ldots \forall x_n(\downarrow f(x_1,\ldots,x_n) \longrightarrow f(x_1,\ldots,x_n) \leqq \max(x_1,\ldots,x_n,c)).$$

We shall find a partial recursive function φ such that

$$(\forall a_1 \in A)\ldots(\forall a_n \in A)(\downarrow f(\alpha(a_1),\ldots,\alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1,\ldots,a_n)) \\ \& f(\alpha(a_1),\ldots,\alpha(a_n)) = \alpha(\varphi(a_1,\ldots,a_n))).$$

The construction of the function φ is the same as the construction of φ in the proof of the main result, but here instead of the function f we shall use its partial recursive continuation.

B. In the direction "$\Rightarrow$" for the proof that f is potentially partial recursive we can see that $\alpha(\varphi(\alpha^{-1}(x_1),\ldots,\alpha^{-1}(x_n)))$ is a partial recursive continuation of f. For the proof that $\exists c \in N$ such that

$$\forall x_1 \ldots \forall x_n(\downarrow f(x_1,\ldots,x_n) \longrightarrow f(x_1,\ldots,x_n) \leqq \max(x_1,\ldots,x_n,c))$$

we can construct a set A just the way we built it in the main theorem. We can see that $\downarrow f(\alpha(x_{k_i^1}),\ldots,\alpha(x_{k_i^n}))$, but either

$$\uparrow \alpha(\varphi_i(x_{k_i^1},\ldots,x_{k_i^n}))$$

or

$$(\downarrow \alpha(\varphi_i(x_{k_i^1},\ldots,x_{k_i^n}))\,\&\,\alpha(\varphi_i(x_{k_i^1},\ldots,x_{k_i^n})) \neq f(\alpha(x_{k_i^1}),\ldots,\alpha(x_{k_i^n})))$$

for all natural i. Thus there exists φ such that

$$(\forall a_1 \in A)\ldots(\forall a_n \in A)(\downarrow f(\alpha(a_1),\ldots,\alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1,\ldots,a_n)) \\ \& f(\alpha(a_1),\ldots,\alpha(a_n)) = \alpha(\varphi(a_1,\ldots,a_n))).$$

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