ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика
Том 87, 1993

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques
Tome 87, 1993

ON THE CHURCH-ROSSER PROPERTY AND REDUCIBILITY OF NATURAL DERIVATIONS

CHAVDAR ILIEV

Чавдар Илиев. СВОЙСТВО ЧЕРЧ-РОССЕРА И РЕДУКЦИИ ЕСТЕСТВЕННЫХ ВЫВОДОВ

В статье рассмотрено свойство Черч-Россера для некоторых видов редукции. Предложен пример, что редукция дефинирована в [1], не обладает данным свойством, несмотря на то, что оно использовано для доказательства единственности нормальной формы и для отождествление выводов; показан вывод, который редуцируется до различных нормальных. Это свойство действительно для этой редукции в том случае, если отменим требование о приложении комутативных редукции только к максимальным сегментом. Так как свойство действительно для β- и η-редукции, можно предположить, что при добавлении комутативных редукции оно бы сохранилось — здесь показано, что это ведет к утрате свойства.

Chavdar Iliev. ON THE CHURCH-ROSSER PROPERTY AND REDUCIBILITY OF NATURAL DERIVATIONS

The paper contains a treatment of the Church-Rosser property with regard to several kinds of reductions. We give an example that the reducibility relation defined in [1] does not possess the property, although it is used to verify the uniqueness of the normal form and for stating the identity between proofs; we show a derivation that reduces to different normal ones. The property for this relation appears if we deny the restriction over commutative reductions to be applied only for maximal segments. The Church-Rosser property is a well-known fact for reducibility of derivations, and it might be expected that enlarging reducibility with commutative reductions will save the property. Here we illustrate that in that case the property is lost.

0. INTRODUCTION

The reducibility relation of natural derivation can be stated by analogy with λ-calculus: β- and η-reductions are defined for derivations. The question about the Church-Rosser property for these reductions seems to be clear, since they are analogous to reductions in λ-calculus and the Church-Rosser property is well-known fact for λ-terms [3]. According to the Curry-Howard isomorphism we have the same fact for implicative derivations. It is no difficulty to verify the property for derivations in the full language.

Another way of stating the reducibility of derivations is using the inversion principle, as it is made in [1]. Then the question about the Church-Rosser property needs some particularization. According to that principle the corresponding introduction and elimination inference rules are inverses of each other and nothing new is obtained by an elimination immediately following an introduction, so that such a sequence of inferences occuring in a derivation can be dispensed with — in other words, a proof of the conclusion of elimination is already "contained" in the proof of the premisses when the major premiss is inferred by introduction.

The notions *maximum formula* and *maximum segment* are based on this principle and make it explicit for the different cases that can arise; the inversion principle implies that they are unnecessary detours in a derivation which can be removed. A derivation is defined as normal (cf. [1]), when it contains no maximum formula and no maximum segment. A β-reduction removes the maximum formula. To remove maximum segments, commutative reductions are stated. They decrease the length of maximal segments. This allows to define a proper measure and by induction on it to prove that every derivation reduces to a normal form — the well-known Weak normal form theorem. By the Church-Rosser property it follows that the normal form of every derivation is unique — thence derivations with identical normal form can be equivalenced. In studies treating reducibility of derivations ([1, 2]) a detailed proof of the Church-Rosser property is missing, although it is used to verify the uniqueness of the normal form. Here we shall examplify that the normal form of derivations in the full language of intuitionistic logic is not unique when commutative reductions are carried out only for a maximum segment (this requirement for commutative reductions is essential for the proof of the Normal form theorem).

It is not necessary to require the maximum segment to carry out commutative reductions — then we call them free (commutative) reductions. Such a reduction does not always lead to any simplification of the derivation, but sometimes only changes the places of inference rules. Nevertheless, the refusal of the mentioned above requirement is essential for the Church-Rosser property, i.e. the reducibility relation defined by the β-reductions and the free reductions has the Church-Rosser property.

With the help of the Church-Rosser property we can state a natural equivalence relation between the proofs (derivations). As we mentioned above, the Church-Rosser property is valid for $\beta\eta$-reducibility. If we enrich this relation with the commutative reductions, using the Church-Rosser property, we could define a "better" equivalence relation between the proofs. We shall examplify that the adding of the free reductions to the β- and η-reductions leads to the lost of the Church-Rosser property.

1. DEFINITIONS

We shall represent derivations as terms and let denote them by e, f, g, h etc. The formulas that derivations are constructed by are in a language containing $\&$, $\vee$, $\supset$, $\perp$, $\forall$ and $\exists$. Derivations are constructed inductively starting from the atomic ones of the kind $[A]$, where $[A]$ is the trivial derivation of the formula A from a sequence of formulas (assumptions, hypotheses) Γ, with Γ containing A. For each logical constant σ (except $\perp$) we have two inference rules — introduction and elimination which we denote by σ^+ and σ^-, respectively. In certain steps some of the assumptions and parameters may be discharged (or closed). To avoid collisions in substitution operations, we shall put labels on the discharged formulas. We shall use natural numbers for labels, and we shall write them as an upper index of the labelled formula A, i.e. A^k. It is suitable to divide the sequence of assumptions into two ones — Δ with not labelled and Γ with labelled formulas, and to allow discharging formulas only from Γ. We write $d : \Delta\Gamma \longrightarrow A$ for "d is a derivation with conclusion formula A and all uneliminated (or open) assumptions belonging to $\Delta\Gamma$". The relation $d : \Delta\Gamma \longrightarrow A$ is defined inductively as it follows:

1.1. Definition

1) If $A \in \Delta$, then $[A] : \Delta\Gamma \longrightarrow A$; if $B^k \in \Gamma$, then $[B^k] : \Delta\Gamma \longrightarrow B$;

2) If $d_0 : \Delta\Gamma \longrightarrow A$, $d_1 : \Delta\Gamma \longrightarrow B$, then $\&^+ d_0 d_1 : \Delta\Gamma \longrightarrow A \& B$;

3) If $d : \Delta\Gamma \longrightarrow A_0 \& A_1$, then $\&_0^- d : \Delta\Gamma \longrightarrow A_0$, $\&_1^- d : \Delta\Gamma \longrightarrow A_1$;

4) If $d : \Delta\Gamma \longrightarrow A$, then $\vee_0^+ d : \Delta\Gamma \longrightarrow A \vee B$, $\vee_1^+ d : \Delta\Gamma \longrightarrow B \vee A$;

5) If $d : \Delta\Gamma \longrightarrow A_0 \vee A_1$, $d_0 : \Delta A_0^{k_0}\Gamma \longrightarrow D$, $d_1 : \Delta A_1^{k_1}\Gamma \longrightarrow D$, then $\vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 : \Delta\Gamma \longrightarrow D$, and the occurrences of $[A_0^{k_0}]$ and $[A_1^{k_1}]$ *in* d_0 and d_1, respectively, are closed;

6) If $d : \Delta B^k\Gamma \longrightarrow C$, then $\supset^+ (B^k)d : \Delta\Gamma \longrightarrow B \supset C$, and the occurrences of $[B^k]$ *in* d are closed;

7) If $d : \Delta\Gamma \longrightarrow B \supset C$, $e : \Delta\Gamma \longrightarrow B$, then $\supset^- de : \Delta\Gamma \longrightarrow C$;

8) If $d : \Delta\Gamma \longrightarrow A(u)$, $u \notin \mathrm{Par}(\Delta\Gamma$, then $\forall^+(u)d : \Delta\Gamma \longrightarrow \forall z A_u(z)$, and the occurrences of u *in* d are closed;

9) If $d : \Delta\Gamma \longrightarrow \forall z B_u(z)$, t is a term, then $\forall^- td : \Delta\Gamma \longrightarrow B_u(t)$;

10) If $d : \Delta\Gamma \longrightarrow B_u(t)$, t is a term, then $\exists^+ td : \Delta\Gamma \longrightarrow \exists z B_u(t)$;

11) If $d : \Delta\Gamma \longrightarrow \exists z B(z)$, $e : \Delta B^k(u)\Gamma \longrightarrow C$, $u \notin \mathrm{Par}(\Delta\Gamma C)$, then $\exists^- d(B^k, u)e : \Delta\Gamma \longrightarrow C$, and the occurrences of $[B^k]$ and u *in* e are closed;

12) If $d : \Delta\Gamma \longrightarrow \perp$, then $\perp^- d : \Delta\Gamma \longrightarrow B$.

By $[F]$ we shall denote derivations of the kind $[A]$ or $[B^k]$. The occurrence of $[F]$ in a derivation is said to be *open* (or F is open) if it is not closed in the subderivation in which $[F]$ occurs. *Open parameters* are defined in the same way.

Note. According to the definition a labelled formula can be open, but if a formula is closed, it is surely labelled. Also, a formula or a parameter may have open and closed occurrences in a certain derivation, depending on the subderivations it is met in.

Derivations that only differ with respect to closed parameters or labels of closed formulas should be counted as identical.

A *conclusion of inference rule* is the conclusion of the derivation obtained by immediate applying of the rule. *Major premiss* of an elimination rule σ^- is the conclusion of the derivation written immediately after σ^-. *Length* of a derivation is defined as the number of the occurrences of strings of the kind σ^- or σ^+ in the derivation.

1.2. Substitution operations

By $[d \mid F : e]$ we denote the result of replacing the open occurrences of $[F]$ in d *with the derivation* e. *The result of replacing the open occurrences of a parameter* u in d with a term t is denoted by $[d \mid u : t]$ or shortly $d_u(t)$. We say that a collision appears at the substitution operation if a closed formula or a parameter of the derivation we substitute in occurs as open in the derivation or in the term we substitute with; or when substituting in a derivation or a subderivation of the kind $\exists^- d(B^k, u)e$, $B^k_u(t)$ for some term t occurs as open in the derivation we substitute with (by $B^k_u(t)$ we mean $(B_u(t))^k$). Collisions may be avoided by renaming closed parameters and proper change of labels of closed formulas. Since we have equivalenced derivations that differ with respect to closed parameters and labels of closed formulas, we shall assume that collisions do not appear at substitutions.

By $\Theta_u(t)$ we denote the replacing of the parameter u with the term t in all formulas of Θ (Θ is a sequence of formulas).

Lemma. a) *Let* $h : \Delta\Gamma \longrightarrow D$. *Then* $[h \mid v : t] : \Delta_v(t)\Gamma_v(t) \longrightarrow D_v(t)$.
b) *Let* $h : \Delta F \Gamma \longrightarrow C$ *and* $g : \Delta\Gamma \longrightarrow F$. *Then* $[h \mid F : g] : \Delta\Gamma \longrightarrow C$.

Proof. An induction on the length of h is applied.

2. REDUCIBILITY OF DERIVATIONS

2.1. β-reductions and commutative reductions

We define reductions as formal expressions of the form $g \rightarrowtail g'$ as follows:

(β1) $\&^-_i \&^+ d_0 d_1 \rightarrowtail d_i$, $i < 2$;
(β2) $\vee^- \vee^+_i d(A^{k_0}_0)d_0(A^{k_1}_1)d_1 \rightarrowtail [d_i \mid A^{k_i}_i : d]$, $i < 2$;
(β3) $\supset^- \supset^+ (B^k)de \rightarrowtail [d \mid B^k : e]$;
(β4) $\forall^- t \forall^+(u)d \rightarrowtail [d \mid u : t]$;
(β5) $\exists^- \exists^+ td(B^k, u)e \rightarrowtail [[e \mid u : t] \mid B^k_u(t) : d]$;
(c1) $\sigma^- \exists^- d(B^k, u)eY \rightarrowtail \exists^- d(B^k, u)\sigma^- eY$;
(c2) $\sigma^- \vee^- d(A^{k_0}_0)d_0(A^{k_1}_1)d_1 Y \rightarrowtail \vee^- d(A^{k_0}_0)\sigma^- d_0 Y(A^{k_1}_1)\sigma^- d_1 Y$.

The expressions (β1)–(β5) we call β-reductions and (c1), (c2) — free commutative reductions. Y is an expression depending on σ^-. For (c1) we assume that u and $B^k_u(t)$ (for any term t) do not occur as open in Y; and for (c2) we assume that $A_0^{k_0}$ and $A_1^{k_1}$ are not open in Y (these requirements are not restrictions over ($c1$) and ($c2$), since they concern closed parameters and labels of closed formulas). In the expression of the kind $g \rightarrowtail g'$ the right hand side g' is called a reduction of the left hand side g. We write $d \longmapsto_1 d'$ for "d' is obtained from d by replacing a subderivation of d by a reduction of it". The *reducibility relation* is defined as the reflexive and transitive closure of $\longmapsto_1$, and it is denoted by $\longmapsto$. A derivation is said to be *irreducible* if it reduces only to itself.

2.2. Maximum formula and maximum segment

Maximum formula is a formula which is a conclusion of introduction rule and magor premiss of the corresponding elimination rule. A derivation of the form $\sigma^- dY$ is a *maximum segment* if there exists a sequence of derivations d_i $(i \leqq n)$ such that d_0 is obtained by immediate applying of σ^+ (the corresponding of σ^-), for every $i > 0$ d_i is obtained from d_{i-1} either by $\vee^-$ or by $\exists^-$, for every $i > 0$ the conclusions of d_i and d_{i-1} are identical, and $d_n = d$. The number n is called *length of the maximum segment.* A derivation is defined as *normal* if it contains no maximum formula and no maximum segment. By β-reductions the maximum formulas are removed (although new ones may appear) and the commutative reductions decrease the length of maximal segments in the cases they occur. It is not necessary to require a maximum segment to carry out a free commutative reduction. When we require the left hand sides of ($c1$) and ($c2$) to be maximal segments, the reducibility relation that arises is denoted by $\underset{P}{\longmapsto}$ (as defined by Prawitz in [1]). Obviously, a derivation is normal if and only if it is irreducible (in the sense of $\underset{P}{\longmapsto}$). According to the Weak normal form theorem every derivation reduces to a normal one. If $\underset{P}{\longmapsto}$ had the Church-Rosser property, every derivation should reduce to unique normal form. Here we give an example of derivation which reduces to different normal forms, i.e. $\underset{P}{\longmapsto}$ does not possess the Church-Rosser property and the conjecture ([1]) that two derivations may be equivalenced only if the normal derivations to which they are reduced are identical is not valid.

2.2.1. Example. Let A, B and C be formulas, $u \notin \mathrm{Par}(AC)$, $D = B \,\&\, (A \,\&\, C)$, and $F = \exists z D_u(z)$. We construct the following derivations:

$d_0 = \&^+[A^k]\&_1^-\&_1^-[D^n] : FA^kD^k \longrightarrow A \,\&\, C,$

$d_1 = \&_1^-[D^n] : FB^mD^n \longrightarrow A \,\&\, C,$

$d = \vee_1^+\&_0^-[D^n] : FD^n \longrightarrow A \vee B.$

Then $e = \vee^- d(A^k)d_0(B^m)d_1 : FD^n \longrightarrow A \,\&\, C$. Let

$$h = \&_1^-\exists^-[F](D^n, u)e : F \longrightarrow C.$$

We shall show that h reduces to two different ones.

$A \,\&\, C$ is a conclusion of $\&^+[A^k]\&_1^-\&_1^-[D^n]$, which is obtained by immediate applying of $\&^+$, and next the rules $\vee^-$, $\exists^-$ and $\&_1^-$ are applied consequently, i.e. we have a maximum segment, so we can carry out the following reductions:

$$h = \&_1^-\exists^-[F](D^n, u)e \underset{P}{\longmapsto} \exists^-[F](D^n, u)\&_1^- e \qquad \text{(by } (c1))$$

$$= \exists^-[F](D^n, u)\&_1^- \vee^- d(A^k)d_0(B^m)d_1 \qquad \text{(by } (c2) \text{ in } \&_1^- \vee^- d(A^k)d_0(B^m)d_1)$$

$$\underset{P}{\longmapsto} \exists^-[F](D^n, u) \vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1.$$

$$\vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 = \vee^- \vee_1^+ \&_0^- [D^n](A^k)\&_1^- d_0(B^m)\&_1^- \&_1^- [D^n] \quad \text{(by } (\beta 2))$$
$$\underset{P}{\mapsto} [\&_1^- \&_1^- [D^n] \mid B^m : \&_0^- [D^n]] = \&_1^- \&_1^- [D^n],$$

i.e.

$$\vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 \underset{P}{\mapsto} \&_1^- \&_1^- [D^n].$$

Then

$$\exists^- [F](D^n, u) \vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 \underset{P}{\mapsto} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n].$$

We have

$$h = \&_1^- \exists^- [F](D^u, u)e \underset{P}{\mapsto} \exists^- [F](D^u, u)\&_1^- \&_1^- [D^n] = h_0,$$

and h_0 is normal.

On the other hand,

$$e = \vee^- \vee_1^+ \&_0^- [D^n](A^k)d_0(B^m)\&_1^- [D^n] \quad \text{(by } (\beta 2))$$
$$\underset{P}{\mapsto} [\&_1^- [D^n] \mid B^m : \&_0^- [D^n]] = \&_1^- [D^n].$$

Then

$$h = \&_1^- \exists^- [F](D^n, u)e \underset{P}{\mapsto} \&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] = h_1.$$

In h_1 we can not carry out the reduction

$$\&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] \underset{P}{\mapsto} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n],$$

since $\&_1^- \exists^- [F](D^n, u)\&_1^- [D^n]$ is not a maximum segment, hence h_1 is normal. We have

$$h \underset{P}{\mapsto} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n] = h_0 \quad \text{and} \quad h \underset{P}{\mapsto} \&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] = h_1,$$

where h_0 and h_1 are normal and not identical.

There may be objection in connection with the derivation $\vee^- d(A^k)d_0(B^m)d_1$, since no assumption is closed in d_1 by $\vee^-$, but if we take $\&_1^- \&^+[B^m]d_1$ instead of d_1, by a similar way we can reduce h to h_0 and h_1.

As it is seen from the example, the reason for the different normal forms of one and the same derivation is the restriction over commutative reductions to be applied only for maximal segments.

We can redefine the notion "maximum segment" by changing "d_0 is obtained by immediate applying of σ^+ (the corresponding of σ^-)" in the above definition with "d_0 is not of the kind $\exists^- g(B^k, u)e$ or $\vee^- f(C^i)f_0(D^j)f_1$". Then we have: a derivation is normal (contains no maximum formula and no maximum segment) if and only if it is irreducible (in the sense of $\mapsto$).

3. CHURCH-ROSSER PROPERTY

In this section we shall give a sketch of the proof of the Church-Rosser property for reducibility relation constructed by β-reductions and the free commutative reductions. Also, we shall examplify that the extention of this reducibility relation with η-reductions leads to the lost of the property.

To verify the property, we shall use the Tait's idea for λ-terms.

3.1. Fast reduction

The relation "fast reduction" is defined inductively using the already defined reductions. We shall denote it by "$\Vdash\!\!\rightarrow$".

Definition.

(R1) $d \Vdash\!\!\rightarrow d$;

(R2) if $d_i \Vdash\!\!\rightarrow e_i$ for every $i < 2$, then $\&^+ d_0 d_1 \Vdash\!\!\rightarrow \&^+ e_0 e_1$;

(R3) if $d \Vdash\!\!\rightarrow d'$, then $\&_i^- d \Vdash\!\!\rightarrow \&_i^- d'$ for every $i < 2$;

(R4) if $d \Vdash\!\!\rightarrow e$, then $\vee_i^+ d \Vdash\!\!\rightarrow \vee_i^+ e$ for every $i < 2$;

(R5) if $d \Vdash\!\!\rightarrow d'$, $d_i \Vdash\!\!\rightarrow e_i$ for every $i < 2$, then

$$\vee^- d(B^k)d_0(C^m)d_1 \Vdash\!\!\rightarrow \vee^- d'(B^k)e_0(C^m)e_1;$$

(R6) if $d \Vdash\!\!\rightarrow d'$, then $\supset^+ (B^k)d \Vdash\!\!\rightarrow \supset^+ (B^k)d'$;

(R7) if $d \Vdash\!\!\rightarrow d'$, $e \Vdash\!\!\rightarrow e'$, then $\supset^- de \Vdash\!\!\rightarrow \supset^- d'e'$;

(R8) if $d \Vdash\!\!\rightarrow d'$, then $\forall^+(u)d \Vdash\!\!\rightarrow \forall^+(u)d'$;

(R9) if $d \Vdash\!\!\rightarrow d'$, then $\forall^- td \Vdash\!\!\rightarrow \forall^- td'$;

(R10) if $d \Vdash\!\!\rightarrow d'$, then $\exists^+ td \Vdash\!\!\rightarrow \exists^+ d'$;

(R11) if $d \Vdash\!\!\rightarrow d'$, $e \Vdash\!\!\rightarrow e'$, then $\exists^- d(B^k, u)e \Vdash\!\!\rightarrow \exists^- d'(B^k, u)e'$;

(R12) if $d \Vdash\!\!\rightarrow d'$, then $\bot^- d \Vdash\!\!\rightarrow \bot^- d'$;

(R13) if $d_i \Vdash\!\!\rightarrow e_i$, then $\&_i^- \&^+ d_0 d_1 \Vdash\!\!\rightarrow e_i$ for every $i < 2$;

(R14) if $d \Vdash\!\!\rightarrow d'$, $d_i \Vdash\!\!\rightarrow e_i$, then

$$\vee^- \vee_i^+ d(A_0^{k_0})d_0(A_1^{k_1})d_1 \Vdash\!\!\rightarrow [e_i \mid A_i^{k_i} : d'] \text{ for every } i < 2;$$

(R15) if $d \Vdash\!\!\rightarrow d'$, $e \Vdash\!\!\rightarrow e'$, then $\supset^- \supset^+ (B^k)de \Vdash\!\!\rightarrow [d' \mid B^k : e']$;

(R16) if $d \Vdash\!\!\rightarrow d'$, then $\forall^- t \forall^+(u)d \Vdash\!\!\rightarrow [d' \mid u : t]$;

(R17) if $d \Vdash\!\!\rightarrow d'$, $e \Vdash\!\!\rightarrow e'$, then $\exists^- \exists^+ td(B^k, u)e \Vdash\!\!\rightarrow [[e' \mid u : t] \mid B_u^k(t) : d']$;

(C1) if $\sigma^- eY \Vdash\!\!\rightarrow e'$, $d \Vdash\!\!\rightarrow d'$, then $\sigma^- \exists^- d(B^k, u)eY \Vdash\!\!\rightarrow \exists^- d'(B^k, u)e'$;

(C2) if $\sigma^- d_i Y \Vdash\!\!\rightarrow d_i'$ for every $i < 2$, $d \Vdash\!\!\rightarrow d'$, then

$$\sigma^- \vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 Y \Vdash\!\!\rightarrow \vee^- d'(A_0^{k_0})d_0'(A_1^{k_1})d_1'.$$

As for $(c1)$ and $(c2)$ we have similar requirements for u, B^k, $A_0^{k_0}$ and $A_1^{k_1}$ in (C1) and (C2).

We shall call (R1)–(R2) simple (fast) reductions.

It is almost obvious that the transitive closure of $\Vdash\!\!\rightarrow$ coincides with $\vdash\!\!\rightarrow$. To prove that fact, it is enough to verify:

1) if $d \vdash\!\!\rightarrow_1 g$, then $d \Vdash\!\!\rightarrow g$, and

2) if $d \Vdash\!\!\rightarrow g$, then $d \vdash\!\!\rightarrow g$.

The first condition verifies by induction on d, and the second — by induction on the definition of $\Vdash\!\!\rightarrow$.

Definition. The relation $\vdash\!\!\rightarrow$ has the Church-Rosser property if for every d, d_0, d_1 that $d \vdash\!\!\rightarrow d_0$ and $d \vdash\!\!\rightarrow d_1$ there exists d^* such that $d_0 \vdash\!\!\rightarrow d^*$ and $d_1 \vdash\!\!\rightarrow d^*$.

Using the fact that if a relation has the Church-Rosser property, then its reflexive and transitive closure also has the property (cf. [3]); to verify the Church-Rosser property for $\longmapsto$, it is enough to verify it for $\Vdash\!\!\rightarrow$. We can not prove the property directly for $\longmapsto$, since $\longmapsto_1$ does not possess it (it is easy to give an example).

First we state two commutational lemmata which say that $\Vdash\!\!\rightarrow$ commutates with the substitution operations from Section 1.

3.1.1. Lemma. *If $h \Vdash\!\!\rightarrow h'$, then $[h \mid w : t] \Vdash\!\!\rightarrow [h' \mid w : t]$.*

3.1.2. Lemma. *If $h : \Delta F \Gamma \longrightarrow C$, $h_1 : \Delta\Gamma \longrightarrow F$ and $h \Vdash\!\!\rightarrow h'$, $h_1 \Vdash\!\!\rightarrow h'_1$, then $[h \mid F : h_1] \Vdash\!\!\rightarrow [h' \mid F : h'_1]$.*

The proof of the lemmata is carried out by induction on the length of h.

3.2. Theorem. *The relation $\Vdash\!\!\rightarrow$ has the Church-Rosser property: if $h \Vdash\!\!\rightarrow h^0$ and $h \Vdash\!\!\rightarrow h^1$, then there exists h^* such that $h^0 \Vdash\!\!\rightarrow h^*$ and $h^1 \Vdash\!\!\rightarrow h^*$.*

Proof. An induction on the length of h is applied. If $h = [F]$, then the only possibility for h^0 and h^1 is $h^0 = [F]$ and $h^1 = [F]$. Then $h^* = [F]$. We shall treat in details only one of the cases concerning commutative reductions — $h = \sigma^- \exists^- d(B^k, u)eY$. The case $h = \sigma^- \vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 Y$ is similar. A treatment of the cases concerning β-reductions may be found in [2].

Let h be of the kind $\sigma^- \exists^- d(B^k, u)eY$. The following subcases arise:

1. $\sigma^- \exists^- d(B^k, u)eY \Vdash\!\!\rightarrow \exists^- d^0(B^k, u)e^0 - h^0$ (by (C1)) and
$\sigma^- \exists^- d(B^k, u)eY \Vdash\!\!\rightarrow \sigma^- \exists^- d^1(B^k, u)e^1 Y^1 = h^1$ (by simple reduction),
where by hypothesis we have $\sigma^- eY \Vdash\!\!\rightarrow e^0$, $eY \Vdash\!\!\rightarrow e^1 Y^1$, $d \Vdash\!\!\rightarrow d^j$ for $j = 0, 1$.

We have to show that there exists h^* such that $h^0 \Vdash\!\!\rightarrow h^*$ and $h^1 \Vdash\!\!\rightarrow h^*$. The induction hypothesis is valid for $\sigma^- eY$. We have $\sigma^- eY \Vdash\!\!\rightarrow \sigma^- e^1 Y^1$ and $\sigma^- eY \Vdash\!\!\rightarrow e^0$, hence there exists e^* such that $\sigma^- e^1 Y^1 \Vdash\!\!\rightarrow e^*$ and $e^0 \Vdash\!\!\rightarrow e^*$.

By the induction hypothesis for d we have $d^j \Vdash\!\!\rightarrow d^*$ for $j = 0, 1$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Then
$h^0 = \exists^- d^0(B^k, u)e^0 \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*$ (by (R17)),
$h^1 = \sigma^- \exists^- d^1(B^k, u)e^1 Y^1 \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*$ (by (C1)).

2. $\sigma^- \exists^- d(B^k, u)eY \Vdash\!\!\rightarrow \exists^- d^0(B^k, u)e^0 = h^0$ (by (C1)),
$\sigma^- \exists^- d(B^k, u)eY \Vdash\!\!\rightarrow \exists^- d^1(B^k, u)e^1 = h^1$ (by (C1)),
where $\sigma^- eY \Vdash\!\!\rightarrow e^l$ for $l = 0, 1$, and $d \Vdash\!\!\rightarrow d^j$ for $j = 0, 1$. By $\sigma^- eY \Vdash\!\!\rightarrow e^j$ for $j = 0, 1$, and by the induction hypothesis for $\sigma^- eY$ we have that there exists e^* such that $e^l \Vdash\!\!\rightarrow e^*$ for $l = 0, 1$. By the induction hypothesis for $d : d^j \Vdash\!\!\rightarrow d^*$ for $j = 0, 1$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Then by (R17) we have

$$h^0 = \exists^- d^0(B^k, u)e^0 \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*$$

and

$$h^1 = \exists^- d^1(B^k, u)e^1 \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*.$$

3. $h = \sigma^- \exists^- \exists^+ td(B^k, u)eY$.
$h = \sigma^- \exists^- \exists^+ td(B^k, u)eY \Vdash\!\!\rightarrow \exists^- \exists^+ td^0(B^k, u)e'' = h^0$ (by (C1)),
$\exists^- \exists^+ td(B^k, u)e \Vdash\!\!\rightarrow [[e' \mid u : t]B_u^k(t) : d^1]$ (by (R17)), and
$h = \sigma^- \exists^- \exists^+ td(B^k, u)eY \Vdash\!\!\rightarrow \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y^1 = h^1$ (by simple reduction), where we have $d \Vdash\!\!\rightarrow d^j$ for $j = 0, 1$, $e \Vdash\!\!\rightarrow e'$, $\sigma^- eY \Vdash\!\!\rightarrow e''$ and $Y \Vdash\!\!\rightarrow Y^1$.

We assume that $B_u^k(t)$ is not an open assumption in Y and u is not open in Y, hence in Y^1. (New open parameters and formulas do not appear by reductions.)

By the induction hypothesis for d: $d^j \Vdash\!\!\rightarrow d^*$ for $j = 0,1$. By $e \Vdash\!\!\rightarrow e'$ and $Y \Vdash\!\!\rightarrow Y^1$ it follows $\sigma^- eY \Vdash\!\!\rightarrow \sigma^- e'Y^1$ by simple reduction.

The induction hypothesis is valid for $\sigma^- eY$ and $\sigma^- eY \Vdash\!\!\rightarrow \sigma^- e'Y^1$, $\sigma^- eY \Vdash\!\!\rightarrow e''$, hence there exists e^* such that $\sigma^- e'Y^1 \Vdash\!\!\rightarrow e^*$ and $e'' \Vdash\!\!\rightarrow e^*$.

$B_u^k(t)$ is not open in Y^1; u is not open in Y^1, hence

$$\sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y^1 = [[\sigma^- e'Y^1 \mid u : t] \mid B_u^k(t) : d^1].$$

By Lemma 3.1.1 and $\sigma^- e'Y^1 \Vdash\!\!\rightarrow e^*$ we have $[\sigma^- e'Y^1 \mid u : t] \Vdash\!\!\rightarrow [e^* \mid u : t]$, and by Lemma 3.1.2 it follows

$$[[\sigma^- e'Y^1 \mid u : t] \mid B_u^k(t) : d^1] \Vdash\!\!\rightarrow [[e^* \mid u : t] \mid B_u^k(t) : d^*],$$

i.e.

$$h^1 = \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y \Vdash\!\!\rightarrow [[e^* \mid u : t] \mid B_u^k(t) : d^*].$$

Using $e'' \Vdash\!\!\rightarrow e^*$ and $d^j \Vdash\!\!\rightarrow d^*$, $j = 0,1$, we have

$$\exists^-\exists^+ t d^0(B^k, u)e'' \Vdash\!\!\rightarrow [[e^* \mid u : t] \mid B_u^k(t) : d^*] \quad \text{(by (R7))}.$$

Let $h^* = [[e^* \mid u : t] \mid B_u^k(t) : d^*]$. Then

$$h^0 = \exists^-\exists^+ t d^0(B^k, u)e'' \Vdash\!\!\rightarrow [[e^* \mid u : t] \mid B_u^k(t) : d^*]$$

and

$$h^1 = \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y \Vdash\!\!\rightarrow [[e^* \mid u : t] \mid B_u^k(t) : d^*].$$

4. $h = \sigma^-\exists^-\exists^- d(B^k, u)e(C^m, v)gY$.

$$h = \sigma^-\exists^-\exists^- d(B^k, u)e(C^m, v)gY \Vdash\!\!\rightarrow \sigma^-\exists^- d^0(B^k, u)e'Y^1 = h^0$$

(by simple reduction, where $\exists^- e(C^m, v)g \Vdash\!\!\rightarrow e'$ and then

$$\exists^-\exists^- d(B^k, u)e(C^m, v)g \Vdash\!\!\rightarrow \exists^- d^0(B^k, u)e' \quad \text{by (C1))}$$

and

$$h = \sigma^-\exists^-\exists^- d(B^k, u)e(C^m, v)gY \Vdash\!\!\rightarrow \exists^-\exists^- d^1(B^k, u)e^1(C^m, v)g' = h^1 \quad \text{(by (C1))},$$

where $d \Vdash\!\!\rightarrow d^j$, $j = 0,1$, $\exists^- e(C^m, v)g \Vdash\!\!\rightarrow e'$, $\sigma^- gY \Vdash\!\!\rightarrow g'$, $e \Vdash\!\!\rightarrow e^1$ and $Y \Vdash\!\!\rightarrow Y^1$.

By the induction hypothesis for d we have $d^j \Vdash\!\!\rightarrow d^*$, $j = 0,1$. It is necessary to exist e^* such that $\sigma^- e'Y^1 \Vdash\!\!\rightarrow e^*$ and $\exists^- e^1(C^m, v)g' \Vdash\!\!\rightarrow e^*$. Using $\exists^- e(C^m, v)g \Vdash\!\!\rightarrow e'$ and $Y \Vdash\!\!\rightarrow Y^1$, we have $\sigma^-\exists^- e(C^m, v)gY \Vdash\!\!\rightarrow \sigma^- e'Y^1$ by simple reduction. Also $\sigma^- gY \Vdash\!\!\rightarrow g'$ and $e \Vdash\!\!\rightarrow e^1$, hence

$$\sigma^-\exists^- e(C^m, v)gY \Vdash\!\!\rightarrow \exists^- e^1(C^m, v)g' \quad \text{(by (C1))}.$$

The induction hypothesis is valid for $\sigma^-\exists^- e(C^m, v)gY$, hence there exists e^* such that $\sigma^- e'Y^1 \Vdash\!\!\rightarrow e^*$ and $\exists^- e^1(C^m, v)g' \Vdash\!\!\rightarrow e^*$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Using $\sigma^- e'Y^1 \Vdash\!\!\rightarrow e^*$, $d^j \Vdash\!\!\rightarrow d^*$ for $j = 0,1$ and (C1), we have

$$h^0 = \sigma^-\exists^- d^0(B^k, u)e'Y^1 \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*.$$

By $\exists^- e^1(C^m, v)g' \Vdash\!\!\rightarrow e^*$, $d^j \Vdash\!\!\rightarrow d^*$ and (C1) we get

$$h^1 = \exists^-\exists^- d^1(B^k, u)e^1(C^m, v)g' \Vdash\!\!\rightarrow \exists^- d^*(B^k, u)e^*.$$

5. $h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) eY$.
$h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) eY \Vdash\!\!\to \exists^- \vee^- f^0(A_0^{k_0}) f_0^0(A_1^{k_1}) f_1^0(B^k, u) e' = h^0$ (by (C1)), $\exists^- f_i(B^k, u) e \Vdash\!\!\to f_i'$ for $i = 0, 1$, and then

$$\exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) e \Vdash\!\!\to \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' \quad \text{(by (C2))},$$

$$h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) eY \Vdash\!\!\to \sigma^- \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' Y^1 = h^1$$

(by simple reduction).

We have $\sigma^- eY \Vdash\!\!\to e'$, $f_i \Vdash\!\!\to f_i^0$, $\exists^- f_i(B^k, u) e \Vdash\!\!\to f_i'$, $i = 0, 1$, $f \Vdash\!\!\to f^j$ for $j = 0, 1$, and $Y \Vdash\!\!\to Y^1$. We need f_0^* and f_1^* such that

$$\sigma^- f_0' Y^1 \Vdash\!\!\to f_0^*, \quad \exists^- f_0^0(B^k, u) e' \Vdash\!\!\to f_0^*, \quad \sigma^- f_1' Y^1 \Vdash\!\!\to f_1^*, \quad \exists^- f_1^0(B^k, u) e' \Vdash\!\!\to f_1^*.$$

By $\exists^- f_0(B^k, u) e \Vdash\!\!\to f_0'$ and $Y \Vdash\!\!\to Y^1$ we have $\sigma^- \exists^- f_0(B^k, u) eY \Vdash\!\!\to \sigma^- f_0' Y^1$ with a simple reduction.

By $\sigma^- eY \Vdash\!\!\to e'$, $f_0 \Vdash\!\!\to f_0^0$ and (C1) we get $\sigma^- \exists^- f_0(B^k, u) eY \Vdash\!\!\to \exists^- f_0^0(B^k, u) e'$. The induction hypothesis is valid for $\sigma^- \exists^- f_0(B^k, u) eY$, hence there exists f_0^* such that $\sigma^- f_0' Y^1 \Vdash\!\!\to f_0^*$ and $\exists^- f_0^0(B^k, u) e' \Vdash\!\!\to f_0^*$. Similar, we have f_1^* such that $\sigma^- f_1' Y^1 \Vdash\!\!\to f_1^*$ and $\exists^- f_1^0(B^k, u) e' \Vdash\!\!\to f_1^*$.

By the induction hypothesis for f we have $f^0 \Vdash\!\!\to f^*$ and $f^1 \Vdash\!\!\to f^*$. Let $h^* = \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*$. By $f^0 \Vdash\!\!\to f^*$, $\exists^- f_0^0(B^k, u) e' \Vdash\!\!\to f_0^*$, $\exists^- f_1^0(B^k, u) e' \Vdash\!\!\to f_1^*$ and (C2):

$$h^0 = \exists^- \vee^- f^0(A_0^{k_0}) f_0^0(A_1^{k_1}) f_1^0(B^k, u) e' \Vdash\!\!\to \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*.$$

By $\sigma^- f_0' Y^1 \Vdash\!\!\to f_0^*$, $\sigma^- f_1' Y^1 \Vdash\!\!\to f_1^*$ and (C2):

$$h^1 = \sigma^- \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' Y^1 \Vdash\!\!\to \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*.$$

The proof is completed.

We define the following relation: $d_0 \equiv d_1$ if there exists d^* such that $d_0 \vdash\!\!\to d^*$ and $d_1 \vdash\!\!\to d^*$. Using the Church-Rosser property it is easy to verify that $\equiv$ is an equivalence relation (cf. [3]).

3.3. η-reductions

We can enrich $\vdash\!\!\to$ with the following reductions, called η-reductions:

(η1) $\&^+ \&_0^- d \&_1^- d \rightarrowtail d$;
(η2) $\vee^- d(A_0^{k_0}) \vee_0^+ [A_0^{k_0}](A_1^{k_1}) \vee_1^+ [A_1^{k_1}] \rightarrowtail d$;
(η3) $\supset^+ (B^k) \supset^- d[B^k] \rightarrowtail d$ if B^k is not open in d;
(η4) $\forall^+(u) \forall^- ud \rightarrowtail d$;
(η5) $\exists^- d(B^k, u) \exists^+ u[B^k] \rightarrowtail d$.

It is easy to show that when $\vdash\!\!\to$ is enlarged with the η-reductions, it does not possess the Church-Rosser property. Let $D = A \,\&\, (B \,\&\, C)$, $u \notin \mathrm{Par}(BC)$ and $F = \exists z D_u(z)$. Obviously, $g = \exists^-[F](D^k, u) \&_1^-[D^k]$, for which we have $g : \exists z (A \,\&\, (B \,\&\, C))_u(z) \longrightarrow B \,\&\, C$, is irreducible.

Let $h = \&^+ \&_0^- \exists^-[F](D^k, u) \&_1^-[D^k] \&_1^- \exists^-[F](D^k, u) \&_1^-[D^k]$. We have

$$\begin{aligned} &\&^+ \&_0^- \exists^-[F](D^k, u) \&_1^-[D^k] \&_1^- \exists^-[F](D^k, u) \&_1^-[D^k] \\ &\quad \vdash\!\!\to \exists^-[F](D^k, u) \&_1^-[D^k] = g, \qquad \text{(by } (\eta 1)) \end{aligned}$$

and

$$\&^+\&_0^-\exists^-[F](D^k,u)\&_1^-[D^k]\&_1^-\exists^-[F](D^k,u)\&_1^-[D^k]$$
$$\longmapsto \&^+\exists^-[F](D^k,u)\&_0^-\&_1^-[D^k]\exists^-[F](D^k,u)\&_1^-\&_1^-[D^k] \quad \text{(by (c1))}.$$

Obviously, $\&^+\exists^-[F](D^k,u)\&_0^-\&_1^-[D^k]\exists^-[F](D^k,u)\&_1^-\&_1^-[D^k]$ is also irreducible and different from $\exists^-[F](D^k,u)\&_1^-[D^k]$, i.e. h reduces to two different irreducible ones. This verifies that the Church-Rosser property is lost when $\longmapsto$ is enriched with η-reductions. This example also illustrates that the reducibility relation which arises from η-reductions and commutative reductions does not possess the Church-Rosser property.

REFERENCES

1. Prawitz, D. Ideas and results in proof theory. In: Proceeding of Second Scandinavian Logic Symp., Amsterdam, 1971.
2. Girard, J. Proof theory and logical complexity. Vol. 1, Bibliopolis, edizioni di filosofia e science, 1987.
3. Barendregt, H. P. The Lambda-Calculus. North-Holland Publ. Company, 1981.

Received on 14.03.1994