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ON THE CONNECTION BETWEEN THE ABSTRACT RECURSION THEORY AND THE METHOD OF SUCCESSIVE APPROXIMATIONS*

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Йордан Зашев. О СВЯЗИ АБСТРАКТНОЙ ТЕОРИИ РЕКУРСИИ С МЕТОДОМ ПОСЛЕДОВАТЕЛЬНОГО ПРИБЛИЖЕНИЯ

Вводится понятие ортогонализуемого полукольца, охватывающее как частный случай некоторые хорошо известные понятия классической чистой математики и пригодное, с другой стороны, для целей алгебраической теории рекурсии. Для таких полуколец доказывается так называемая теорема кодирования, являющейся фундаментальной для последней теории. Рассматриваются некоторые следствия главным образом для кольца, ограниченных линейных операторов в бесконечномерном пространстве Гильберта.

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We introduce the concept of orthogonalizable semiring which contains as a special case some familiar objects of classical pure mathematics and, on the other hand, is fit for the purposes of the algebraic recursion theory. A fundamental result of the last theory, called code evaluation theorem, is proved for such semirings. Some corollaries are considered, especially for the ring of bounded linear operators over an infinite dimensional Hilbert space.

There is a great deal of similarity between the principal problem of the algebraic recursion theory (that is the problem of fixed-point completion in the sense of [2]), on the one hand, and some problems about the existence of solutions of various kinds of systems of equations in the classical mathematics. Of course, in

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the algebraic recursion theory we have to do with least solutions of systems of inequalities, but this difference seems not to be so principal as another one, namely that in the algebraic recursion theory the emphasis is put on the expressibility of least solutions of those systems of inequalities by means of some basic operations, among which the most important one is the so-called iteration, while the existence problem is comparatively easy. On the other hand, the iteration is the least solution of one simple inequality of one unknown, which is clearly analogous to linear equations of one unknown. In the algebraic recursion theory an arbitrary "nonlinear" system of inequalities is reduced to one such inequality by a typical for the recursion theory process involving coding, so that the first system is interpreted in some sense "internally" as an object of the domain of solutions of the system. In this reducing a similarity may be observed with the iteration method in classical mathematics, but peculiarities also occur, which makes the comparison of the two methods not quite obvious. Such a peculiarity is the mentioned "internal" coding of the iteration process by which it is presented through one inequality.

In this paper we are going to inquire a little bit deeper into this analogy. Our special purpose is to consider corollaries of general results of algebraic recursion theory applied to classical mathematical objects. To this end we introduce the concept of orthogonalizable semiring, which is a kind of topological operative spaces in the sense of [1]. This concept is more suitable for our purposes, being closer to or comprising as a special case familiar objects of classical pure mathematics, for instance, the operator rings over infinite dimensional Hilbert spaces. On the other hand, the main result of the present paper — the code evaluation theorem in orthogonalizable semirings — does not follow from the corresponding result for operative spaces, which is ultimately due to the simpler algebraic structure of the former ones.

1. OTHOGONALIZABLE SEMIRINGS

By a semiring we shall mean an additive (commutative) monoid R in which an associative multiplication with an unit I, satisfying the usual distributive laws, is given, such that $\varphi 0 = 0 = 0\varphi$ for all $\varphi \in R$, where 0 is the zero of the addition in R. For all semirings, considered in the sequel, we shall suppose also that a topology and a partial order "$\leq$" are given in them, such that the addition is continuous as a function of two arguments and the following three conditions are fulfilled:

(i) if $\varphi \in R$ belongs to every neighborhood of $\psi \in R$, then $\psi \leq \varphi$;

(ii) every sequence $\{\varphi_\alpha\}_{\alpha \in D}$ of elements of the given semiring R (where D is a directed poset of indices), which has a limit, has also the greatest limit, i.e. a limit φ of this sequence such that for every other limit ψ of the same sequence we have $\psi \leq \varphi$;

(iii) if $\{\varphi_\alpha\}_{\alpha \in D}$ and $\{\psi_\alpha\}_{\alpha \in D}$ are two sequences of elements of R such that $\varphi_\alpha \leq \psi_\alpha$ for all $\alpha \in D$ and $\{\psi_\alpha\}_{\alpha \in D}$ has a limit, then $\{\varphi_\alpha\}_{\alpha \in D}$ also has a limit, and for the greatest limits ψ and φ of the last two sequences, respectively, we have $\varphi \leq \psi$.

Note that the condition (i) implies that for every sequence $\{\varphi_\alpha\}_{\alpha \in D}$ of elements of R, which is a stationary one, i.e. for some $\alpha_0 \in D$ and $\varphi \in R$ we have $\varphi_\alpha = \varphi$

for all $\alpha \geq \alpha_0$, the element φ is the greatest limit of $\{\varphi_\alpha\}_{\alpha \in D}$.

Now let R be a semiring of this kind and let J be an arbitrary non-empty set of indices. By a *confinal* in J we shall mean a set D of finite subsets of J such that every finite subset of J is included in some element of D; every confinal is directed with respect to the partial order "$\subseteq$" in it. A series of the form $\sum_{j \in J} \varphi_j$, where $\varphi_j \in R$ for all $j \in J$, will be called *convergent* with respect to a confinal D in J iff the sequence $\{\sum_{j \in \alpha} \varphi_j\}_{\alpha \in D}$ of D-partial sums of that series has a limit in R as a directed set indexed sequence. The greatest limit of the last sequence will be called a *sum* of the series in question, corresponding to D. The series in question will be called *conventionally convergent* iff there is a confinal D in J such that it is convergent with respect to D. In the sequel the expressions of the form $\sum_{j \in J} \varphi_j$ will be used to denote also a sum of the series denoted in the same way, and the confinal D, to which this sum corresponds, will be clear from the context or it will be arbitrary otherwise.

If all but a finite number of the members of a series $\sum_{j \in J} \varphi_j$ in R are zeros, then the only sum of the last series is the algebraic sum of the non-zero members of it. This follows from condition (i) since in this case the sequence of D-partial sums of that series is stationary for any confinal D.

A mapping $f : R \to R$ will be called *additive* iff for every series $\sum_{j \in J} \varphi_j$ of elements of R and every confinal D in J such that the last series converges with respect to D in R, the series $\sum_{j \in J} f(\varphi_j)$ also converges with respect to D, and for the corresponding sums we have the equality

$$\sum_{j \in J} f(\varphi_j) = f\Big(\sum_{j \in J} \varphi_j\Big).$$

An element $\varphi \in R$ will be called *left* (respectively, *right*) additive iff the mapping $f : R \to R$, defined by $f(\xi) = \varphi\xi$ (respectively, $f(\xi) = \xi\varphi$), is additive.

A semiring R, satisfying all the suppositions above, will be called an *orthogonalizable* one (or, shortly, an *orthoring*) iff all elements of R are left additive and there is an *orthogonal quadruple* in R, i.e. a quadruple (T_+, T_-, F_+, F_-) of elements of R such that T_+ and F_+ are right additive as elements of R and the following equalities hold:

$$T_-T_+ = I = F_-F_+ \quad \text{and} \qquad T_-F_+ = 0 = F_-T_+.$$

An orthogonal quadruple (T_+, T_-, F_+, F_-) will be called complete iff

$$T_+T_- + F_+F_- = I.$$

More generally, a family $(\varphi^+(t), \varphi^-(t))_{t \in K}$ of pairs of elements of R will be called an *orthogonal system* iff the following three conditions hold:

a) for all $t \in K$ the element $\varphi^+(t)$ is right additive;

b) for all $t \in K$ we have $\varphi^-(t)\varphi^+(t) = I$; and

c) for all $t, s \in K$, for which $t \neq s$, we have $\varphi^-(t)\varphi^+(s) = 0$.

Infinite orthogonal systems exist into every non-trivial orthoring. Indeed, such is, for instance, the system $(\mathbf{n}^+, \mathbf{n}^-)_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers and for every $n \in \mathbb{N}$

$$\mathbf{n}^+ = F_+^n T_+ \quad \text{and} \quad \mathbf{n}^- = T_- F_-^n.$$

Example 1. Let M be an infinite set, and let $T_+ : M \to M$ and $F_+ : M \to M$ be two injective functions such that $T_+(M) \cap F_+(M) = \varnothing$. Denote by R the set of all partial multivalued functions from M to M; the elements of R may be identified with (arbitrary) binary relations in M. The set R is a semiring with respect to the union of two relations as the addition operation and the composition of multivalued functions as multiplication; the zero 0 and the unit I are the nowhere defined function and the identity mapping in M, respectively. This semiring is an orthogonalizable one with respect to the inclusion relation "$\subseteq$" as partial order, the Scott topology with respect to this partial order (which may be described as follows: those sets $Y \subseteq R$ are open, for every element φ of which there is a finite $\iota \subseteq \varphi$ such that all $\psi \in R$, containing ι, belong to Y), and the orthogonal quadruple (T_+, T_-, F_+, F_-), where T_- and F_- are the partial mappings defined for $x \in R$ by

$$T_-(x) = \begin{cases} y & \text{if } x = T_+(y), \\ \text{undefined} & \text{if there is no such } y, \end{cases}$$

and

$$F_-(x) = \begin{cases} y & \text{if } x = F_+(y), \\ \text{undefined} & \text{if there is no such } y. \end{cases}$$

Example 2. Let V be the set of all infinite sequences $(x_0, x_1, \ldots)$ of real numbers x_i with the product topology. V is a real vector space and let L be the ring of linear operators $\varphi : V \to V$ with the topology induced by the product one in V^V. Denote by R the ring which differs from L only in the order of writing the multiplication: $\varphi\psi$ in R means $\psi\varphi$ in L. Take the identity "$=$" as partial order "$\leq$" in R. Then R is an orthogonalizable ring (i.e. an orthoring which is a ring with respect to the algebraic operations in it); an orthogonal quadruple in R is, for instance, the following one:

$$\begin{aligned} T_+((x_0, x_1, \ldots)) &= (x_0, x_2, x_4, \ldots); \\ F_+((x_0, x_1, \ldots)) &= (x_1, x_3, x_5, \ldots); \\ T_-((x_0, x_1, \ldots)) &= (x_0, 0, x_1, 0, \ldots); \\ F_-((x_0, x_1, \ldots)) &= (0, x_0, 0, x_1, \ldots). \end{aligned}$$

Example 3. Let H be an infinite dimensional (real or complex) Hilbert space and let R be the ring of bounded linear operators in H. Take the trivial partial order, i.e. the identity "$=$", for "$\leq$" in R. Then R is an orthogonalizable ring with respect to any of the known operator topologies. For us, however, the weak operator topology will be the most important one and, unless otherwise indicated, we shall always have in view this topology in the context of this example. Orthogonal quadruples in R may be found, for instance, as it follows: Let T_+ and F_+ be two isometrical (preserving the scalar product) operators in R, the images $\mathrm{Im}T_+$ and $\mathrm{Im}F_+$ of which are orthogonal to each other subspaces of H, and let $T_- = T_+^*$ and $F_- = F_+^*$ be the corresponding adjoint operators. Then (T_+, T_-, F_+, F_-) is an orthogonal quadruple, which is a complete one iff the images of T_+ and

F_+ are orthogonal completions to each other. It will be convenient in the sequel to call the orthogonal quadruples, arising in this way, *isometrical* ones and the positive components of such quadruples (i.e. T_+ and F_+) — *semiunitary* operators. Accordingly, by an *isometrical* orthogonal system we mean an orthogonal system $(K^+(j), K^-(j))_{j\in J}$ such that the operators $K^+(j)$ are isometrical and $K^-(j)$ is the adjoint operator of $K^+(j)$ for all $j \in J$. Here is a simple lemma about such systems.

Lemma 1. *Let $(K^+(j), K^-(j))_{j\in J}$ be an isometrical orthogonal system in R and let D be a confinal in J. Then for any sequence of operators $\varphi_j \in R$, $j \in J$, the following two conditions are equivalent:*

(a) *the sum of $\sum\limits_{j\in J} K^+(j)\varphi_j$, corresponding to D, exists with respect to the strong operator topology in R; and*

(b) *for all vectors $x \in H$ the series $\sum\limits_{j\in J} \|\varphi_j x\|^2$ converges.*

Indeed, if (a) holds and S is the sum of the series in question, corresponding to D, then using the isometricality of the operators $K^+(j)$ we have

$$\sum_{j\in J} \|\varphi_j x\|^2 = \sum_{j\in J} \|K^+(j)\varphi_j x\|^2 = \|Sx\|^2 < \infty.$$

Conversely, let we have (b). Then for all finite $a \subseteq J$

$$\|\sum_{j\in a} K^+(j)\varphi_j x\|^2 = \sum_{j\in a} \|K^+(j)\varphi_j x\|^2 = \sum_{j\in a} \|\varphi_j x\|^2.$$

Since the series

$$\sum_{j\in J} \|\varphi_j x\|^2$$

converges, using the completeness of the space H we conclude that the series

$$Sx = \sum_{j\in J} K^+(j)\varphi_j x$$

converges in it as well; whence by the Banach–Steinhaus theorem we get (a).

Remark. We may avoid quoting the last theorem if we replace the condition (b) by the following one:

(b′) *there is a positive real number C such that for all vectors $x \in H$ we have* $\sum\limits_{j\in J} \|\varphi_j x\|^2 \leq C\|x\|^2$,

which may serve our purposes below as well.

2. THE CODE EVALUATION THEOREM

Let R be an orthoring. Suppose a semigroup G with an unit e and a homomorphism $\chi : G \to R'$ into the multiplicative semigroup R' of right additive elements of R are given. Let $G(\overline{X}) = G(X_0, \ldots, X_{n-1})$ be the semigroup of monomials of the variables $X_0, \dots, X_{n-1}$ with coefficients in G, i.e. the set of all formal expressions of the form

$$q = g_0 Y_0 g_1 \cdots g_{m-1} Y_{m-1} g_m, \tag{1}$$

where $g_0, \dots, g_m \in G$ and $Y_0, \dots, Y_{m-1} \in \{X_0, \dots, X_{n-1}\}$, with the obvious multiplication operation. The homomorphism χ extends uniquely to a homomorphism $\widetilde{\chi} : G(\overline{X}) \to R^{R^n}$ into the semigroup of all functions $f : R^n \to R$ (with respect to the usual multiplication operation $(fg)(\overline{\xi}) = f(\overline{\xi})g(\overline{\xi})$, $\overline{\xi} \in R^n$) such that for all $i < n$, $\widetilde{\chi}(X_i)$ is the i-th projection $R^n \to R$, i.e. $\widetilde{\chi}(X_i)(\xi_0, \dots \xi_{n-1}) = \xi_i$ for all $(\xi_0, \dots, \xi_{n-1}) \in R^n$. By a *coding* for $G(\overline{X})$ with respect to χ we shall mean an orthogonal system

$$(k^+(q), k^-(q))_{q \in G(\overline{X})} \tag{2}$$

in the orthoring R for which the following two conditions hold:

a) for all $q \in G(\overline{X})$ both of the series

$$\sum_{g \in G} k^+(qg)k^-(g) \tag{3}$$

and

$$\sum_{s \in G(\overline{X}) \setminus G} k^+(qs)k^-(s) \tag{4}$$

are conventionally convergent; and

b) there is an element $\tau \in R$ such that

$$\tau k^+(g) = \chi(g)$$

for all $g \in G$, and for all $q \in G(\overline{X}) \setminus G$

$$\tau k^+(q) = 0.$$

The last condition b) can be replaced by the conventional convergence of the series

$$\sum_{g \in G} \chi(g)k^-(g).$$

Indeed, if this series is conventionally convergent, then we may take any sum of it for τ and the equalities in b) will follow from the definition of orthogonal system. Conversely, if τ satisfies b), then the last series coincides with

$$\sum_{g \in G} \tau k^+(g)k^-(g),$$

which is conventionally convergent by a) since τ is left additive. The element τ will be called a starting element of the coding in question.

To every coding (2) we naturally assign two idempotent elements $\varkappa'$ and $\varkappa''$ of R defined as the sums of the series

$$\sum_{g \in G} k^+(g)k^-(g) \quad \text{and} \quad \sum_{s \in G(\overline{X}) \setminus G} k^+(s)k^-(s),$$

respectively, the existence of which is supposed in a). The element $\varkappa = \varkappa' + \varkappa''$ is also idempotent and the codings (2), for which $\varkappa = I$, will be called complete. It follows from the above that for any starting element τ of a coding (2) the equality

$$\tau\varkappa = \tau\varkappa' = \sum_{g\in G} \chi(g)k^-(g) \tag{5}$$

holds (where the last sum corresponds to the confinal supposed in condition a) for the case $q = e$); and for complete codings the starting element is unique.

In the sequel we shall usually have the homomorphism χ fixed and we shall write $\widetilde{q}$ for $\widetilde{\chi}(q)$.

Let $G[\![X_0, \ldots, X_{n-1}]\!] = G[\![\overline{X}]\!]$ be the set of all formal series of the form

$$P(\overline{X}) = \sum_{p\in M} \lambda_p p(\overline{X}),$$

where $M \subseteq G(\overline{X})$, and for all $p \in M$ the element $\lambda_p \in R$ belongs to the center of the orthoring R, i.e. commutes with all elements of R. Here in the set M a confinal is supposed to be given, so that to the last series we assign a *value*

$$\widetilde{P}(\overline{\xi}) = \sum_{p\in M} \lambda_p \widetilde{p}(\overline{\xi})$$

for every n-tuple $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in R^n$ for which the last sum, corresponding to that confinal, exists in R. Consider a formal system of inequalities of the form

$$P_i(\overline{X}) \leq X_i, \quad i < n, \tag{6}$$

where

$$P_i(\overline{X}) = \sum_{p\in M_i} \lambda_{ip} p(\overline{X})$$

is a formal series in $G[\![\overline{X}]\!]$ for all $i \leq n$. By a solution of the system (6) we mean an n-tuple $\overline{\xi} \in R^n$ such that the values $\widetilde{P}_i(\overline{\xi})$ exist and satisfy the inequalities $\widetilde{P}_i(\overline{\xi}) \leq \xi_i$ in R for all $i < n$.

An element $\rho \in R$ will be called a *governing* element of the system (6) with respect to the coding (2) iff the equalities

$$\rho k^+(g) = 0 \tag{7}$$

and

$$\rho k^+(tX_i g) = \sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g) \tag{8}$$

are satisfied for all $g \in G$, $t \in G(\overline{X})$ and $i < n$. Here the right hand side of the last equality has to be understood as the sum corresponding to the confinal in M_i, which is supposed in the definition of formal series, and thus the existence of the last sum is supposed in the definition of governing element. A necessary and sufficient condition for the existence of governing element for (6) with respect to (2) is the existence of the sum in (8) and the conventional convergence of the series

$$\sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g)\Big) k^-(tX_i g). \tag{9}$$

Indeed, if the sum in (8) exists and (9) is conventionally convergent, then any sum ρ of it satisfies (7) and (8). Conversely, if ρ is a governing element, then (9) is conventionally convergent, because it coincides with the series obtained from (4) by putting $q = e$ and multiplying memberwise from left by ρ. We see as well that the governing element ρ satisfies the equalities

$$(10) \qquad \rho\varkappa = \rho\varkappa'' = \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g)\Big) k^-(tX_i g),$$

where the external sum corresponds to the confinal supposed in the condition a) for $q = e$, and for complete codings ρ is unique.

Let ρ be a governing element for the system (6) with respect to the coding (2) with a starting element τ. Then the inequality

$$(11) \qquad \tau\varkappa + \xi\rho\varkappa \le \xi,$$

which is a linear one with respect to ξ, will be called *iterational* inequality for (6). For any two elements $\varphi, \psi \in R$, by *iteration* of φ starting from ψ we shall mean an element $\vartheta \in R$ such that

$$\psi + \vartheta\varphi \le \vartheta$$

and for all $\alpha, \xi \in R$ we have

$$\alpha\psi + \xi\varphi \le \xi \implies \alpha\vartheta \le \xi.$$

Theorem 1. *Let ρ be a governing element for the system (6) with respect to a coding (2) for $G(\overline{X})$ with a starting element τ. If ω is an iteration of $\rho\varkappa$ starting from $\tau\varkappa$, then the n-tuple*

$$\omega k^+(\overline{X}) = (\omega k^+(X_0), \ldots, \omega k^+(X_{n-1}))$$

is the least solution of (6) in the set

$$E = \{\overline{\xi} \in R^n \mid \sum_{q\in G(\overline{X})} \widetilde{q}(\overline{\xi})k^-(q) \text{ is conventionally convergent}\},$$

and for all $q \in G(\overline{X})$ it satisfies the equality

$$\widetilde{q}(\omega k^+(\overline{X})) = \omega k^+(q).$$

Conversely, if there is a solution of (6) in E, then the iterational inequality (11) has a solution with respect to ξ in R.

Proof. Since by the suppositions of the theorem ω is the least solution of (11), it should satisfy the equality $\tau\varkappa + \omega\rho\varkappa = \omega$, whence by a multiplication from right we get

$$(12) \qquad \chi(g) = \tau k^+(g) = \tau\varkappa k^+(g) = \omega k^+(g)$$

for all $g \in G$, and

$$(13) \qquad \sum_{p\in M_i} \lambda_{ip}\omega k^+(qp)\chi(g) = \omega\rho k^+(qX_i g) = \omega k^+(qX_i g)$$

for all $g \in G$, $q \in G(\overline{X})$ and $i < n$. Hence we get also that for all such g and q

$$(14) \qquad \omega k^+(q)\chi(g) = \omega k^+(qg).$$

Indeed, if $q \in G$, then by (12) we have

$$\omega k^+(q)\chi(g) = \chi(q)\chi(g) = \chi(qg) = \omega k^+(qg),$$

and if $q = q_1 X_i g_1$, then using (13) we obtain

$$\omega k^+(q)\chi(g) = \sum_{p\in M_i} \lambda_{ip}\omega k^+(q_1p)\chi(g_1)\chi(g)$$

$$= \sum_{p\in M_i} \lambda_{ip}\omega k^+(q_1p)\chi(g_1g) = \omega k^+(q_1Xg_1g) = \omega k^+(qg).$$

We shall show that for all $q, s \in G(\overline{X})$

$$\omega k^+(q)\omega k^+(s) \leq \omega k^+(qs). \tag{15}$$

By condition a) in the definition of coding it follows that there are elements $\zeta \in R$ such that for all $s \in G(\overline{X})$ the equality

$$\zeta k^+(s) = \omega k^+(qs)$$

holds; such is, for instance, the element

$$\zeta = \omega \sum_{g\in G} k^+(qg)k^-(g) + \omega \sum_{s\in G(\overline{X})\setminus G} k^+(qs)k^-(s).$$

Taking an arbitrary such element ζ and using (5), (10), (13) and (14), we have

$$\omega k^+(q)\tau\varkappa + \zeta\rho\varkappa = \omega k^+(q)\tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip}\zeta k^+(tp)\chi(g)\Big) k^-(tX_ig)$$

$$= \omega k^+(q)\sum_{g\in G}\chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip}\omega k^+(qtp)\chi(g)\Big) k^-(tX_ig)$$

$$= \sum_{g\in G}\omega k^+(qg)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \omega k^+(qtX_ig)k^-(tX_ig) = \zeta',$$

where the last two sums correspond to the confinals D' and D'', respectively, with respect to which the series (3) and (4) are supposed to converge in condition a) for the case $q = e$. Since, however, the element ζ' still satisfies $\zeta' k^+(s) = \omega k^+(qs)$ for all monomials s, the above calculations hold for ζ' as well and prove the equality

$$\omega k^+(q)\tau\varkappa + \zeta'\rho\varkappa = \zeta',$$

whence by the definition of iteration it follows that

$$\omega k^+(q)\omega \leq \zeta',$$

and multiplying from right by $k^+(s)$, we obtain (15). Using (12), (14), (15) and a simple induction on the degree of the monomial $q \in G(\overline{X})$, we get the inequality

$$\widetilde{q}(\omega k^+(\overline{X})) \leq \omega k^+(q), \tag{16}$$

where $\omega k^+(\overline{X}) \in R^n$ is $(\omega k^+(X_0), \ldots, \omega k^+(X_{n-1}))$. Thence by the help of (13) and conditions (ii) and (iii) in Section 1 it follows that $\omega k^+(\overline{X})$ is a solution of (6):

$$\widetilde{P}_i(\omega k^+(\overline{X})) = \sum_{p\in M_i} \lambda_{ip}\widetilde{p}(\omega k^+(\overline{X})) \leq \sum_{p\in M_i} \lambda_{ip}\omega k^+(p) \leq \omega k^+(X_i).$$

Moreover, by (16) the series

$$\sum_{q\in G(\overline{X})} \widetilde{q}(\omega k^+(\overline{X}))k^-(q)$$

is memberwise less or equal to

$$\sum_{q\in G(\overline{X})} \omega k^+(q)k^-(q);$$

and the last series converges with respect to the confinal $\{a\cup b \mid a\in D' \,\&\, b\in D''\}$, as it follows in an obvious way from the convergence of the sums in the definitions of $\varkappa'$ and $\varkappa''$ with respect to D' and D'', respectively, using the condition of continuity of the addition $+$. Therefore by the same conditions (ii) and (iii) it follows that $\omega k^+(\overline{X}) \in E$. To show that this solution is the least one in E, suppose that the n-tuple $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in E$ is a solution of (6) in E, and let v be the element

$$\sum_{g\in G} \chi(g)k^-(g) + \sum_{q\in G(\overline{X})\setminus G} \widetilde{q}(\overline{\xi})k^-(q),$$

where the last two sums correspond to the confinals D' and D'', respectively. The existence of the second one of those sums follows from that for the sum in the definition of the idempotent $\varkappa''$ by a memberwise multiplication from left by an arbitrary sum ψ of the series

$$\sum_{q\in G(\overline{X})} \widetilde{q}(\overline{\xi})k^-(q);$$

and the existence of the last sum is the condition $\overline{\xi}\in E$. Then, using the supposition that the elements $\chi(g)$ are right additive, we have

$$\begin{aligned}
\tau\varkappa + v\rho\varkappa &= \tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} v k^+(tp)\chi(g)\Big) k^-(tX_i g)\\
&= \tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} \widetilde{t}(\overline{\xi})\widetilde{p}(\overline{\xi})\chi(g)\Big) k^-(tX_i g)\\
&= \sum_{g\in G} \chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \widetilde{t}(\overline{\xi})\widetilde{P}_i(\overline{\xi})\chi(g)k^-(tX_i g)\\
&\le \sum_{g\in G} \chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \widetilde{t}(\overline{\xi})\xi_i\chi(g)k^-(tX_i g) = v.
\end{aligned}$$

Hence by the definition of iteration $\omega \le v$ and

$$\omega k^+(X_i) \le v k^+(X_i) = \widetilde{X}_i(\overline{\xi}) = \xi_i,$$

which means that $\omega k^+(\overline{X})$ is the least solution of (6) in E. In the same time we have shown that if (6) has a solution in E, then there is $v\in R$ such that $\tau\varkappa + v\rho\varkappa \le v$. Finally, taking $\omega k^+(\overline{X})$ for $\overline{\xi}$ and multiplying the inequality $\omega \le v$ from right by $k^+(q)$, we obtain the reverse inequality of (16).

An important special case is that of orthorings in which the order "$\le$" is just the equality "$=$". In this case the system (6) and the inequality (11) become a formal system of equations

(17) $$P_i(\overline{X}) = X_i, \quad i<n,$$

and a linear equation (which we call iterational)

$$\tau\varkappa + \xi\rho\varkappa = \xi, \tag{18}$$

respectively. In this case from Theorem 1 (or rather by a trivial modification of its proof) we obtain the following

Corollary 1. *Let in the orthoring R $\varphi \leq \psi$ be equivalent to $\varphi = \psi$ for all $\varphi, \psi \in R$ and the conditions of Theorem 1 hold. If the equation*

$$\xi\rho\varkappa = \xi$$

has no non-zero solutions with respect to ξ, then the iterational equation (18) has a solution $\omega \in R$ with respect to ξ iff the system (17) has a solution $\overline{\xi}(\xi_0, \ldots, \xi_{n-1})$ in the set E in Theorem 1, and in the last case this solution of (17) is unique and is given by $\xi_i = \omega k^+(X_i)$, $i < n$, and for all $q \in G(\overline{X})$ it satisfies

$$\widetilde{q}(\omega k^+(\overline{X})) = \omega k^+(q).$$

3. EXAMPLES AND APPLICATIONS

Example 1 (continued). In this example iterations exist always, i.e. for every two elements $\varphi, \psi \in R$ there is an iteration of φ starting from ψ; and the conditions of Theorem 1 hold for every *finite* system of the form (6) (i.e. such that the sets M_i are finite) with respect to arbitrary G and χ. Usually, a multiplicative subsemigroup of R, produced by a subset $B \subseteq R$, is taken for G, and the identical embedding — for χ. In this case the members of the least solutions of finite systems of the form (6) are called *recursive in B* elements of R. The example is actually well-known in the recursion theory and treats the first order recursion theory of multivalued functions in possibly the most general domain; Theorem 1 implies easily all basic results of the last theory for such functions.

Example 2 (continued). Define for any natural number n

$$\mathbf{n}^+ = F_+^n T_+ \quad \text{and} \quad \mathbf{n}^- = T_- F_-^n.$$

It is not hard to see that all series of the form

$$\sum_{m=0}^{\infty} \varphi_m \mathbf{m}^-,$$

where $\varphi_m \in R$ for all natural m, are convergent in this example and have an unique sum. Then every orthogonal system of the form (2) is, obviously, a coding, provided all its members $(k^+(q), k^-(q))$ have the form $(\mathbf{m}^+, \mathbf{m}^-)$ for a suitable natural m. In this way, for every finite semigroup G one can easily construct codings, using suitable numerations, and let us call such codings numerical. Therefore for numerical codings the set E in Theorem 1 will be the whole orthoring R; and by Corollary 1 with the trivial semigroup, consisting only of the unit e for G, we obtain the following proposition:

Suppose the left hand sides of the system (17) are polinomials with real coefficients, and let ρ be the sum of the series (9) with respect to a numerical coding (2). If the equation

$$\xi\rho = \xi$$

has an unique solution $\xi = 0$ in R, then the system (17) *has a solution in R^n iff the iterational equation $\tau\varkappa + \xi\rho = \xi$ has a solution ω with respect to ξ, and in the last case this solution of* (17) *is unique and is given by $\xi_i = \omega k^+(X_i)$, $i < n$.*

Example 3 (continued). Let G be the trivial semigroup $\{e\}$ and let take an isometrical orthogonal quadruple (T_+, T_-, F_+, F_-). The semigroup $G(X)$ of monomials of one variable X consists of powers X^n, especially X^0 is the unit e, and we have the following coding for $G(X)$:

$$k^+(X^n) = \mathbf{n}^+ = F_+^n T_+, \quad k^-(X^n) = \mathbf{n}^- = T_- F_-^n.$$

The element $\tau = T_-$ is a starting element for this coding. The series

$$\varkappa = \sum_{m=0}^{\infty} k^+(X^m)k^-(X^m) = \sum_{m=0}^{\infty} \mathbf{m}^+\mathbf{m}^- \tag{19}$$

converges in the strong operator topology, because the operators $\mathbf{m}^+$, $\mathbf{m}^-$ are orthogonal projections upon pairwise orthogonal subspaces of H. The same holds for the series (4), because it may be obtained from (19) by a multiplication from left by a suitable element of the form F_+^k and from right by F_- (all elements are right additive in this example). In this case $G[\![X]\!]$ is the set of all formal power series

$$P(X) = \sum_{n=0}^{\infty} \lambda_n X^n \tag{20}$$

with scalar coefficients λ_n. As a corollary, we have the following

Proposition 1. *Let the sum $f(F_+)$ of the series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{21}$$

(where a_n are complex numbers) exist in the sense of weak operator topology and let the scalar unit 1 not belong to the spectrum of the operator $f(F_+)F_-\varkappa$. Then there is an unique operator $\xi \in R$ such that $\|\xi\| < 1$, $f(\xi)$ exists in the same sense and $f(\xi) = \xi$.

Indeed, if the value $\widetilde{P}(F_+)$ of (20) exists, then it may be seen immediately that the operator $\rho = \widetilde{P}(F_+)F_-\varkappa$ is a governing element with respect to the above coding for the system of one equation

$$P(X) = X. \tag{22}$$

And since 1 does not belong to the spectrum of ρ, the last element has an iteration starting from any element of R. Applying Corollary 1, we conclude that the operator $\omega\mathbf{1}^+$, where ω is the unique solution of the iterational equation

$$\tau + \xi\rho\varkappa = \xi$$

with respect to ξ, is the unique solution of (22) in the set

$$E = \{\xi \in R \mid \sum_{n=0}^{\infty} \xi^n \mathbf{n}^- \text{ converges in the weak operator topology}\}.$$

By Theorem 1 it follows also that $(\omega\mathbf{1}^+)^n = \omega\mathbf{n}^+$ for all natural n. Using the last equality, we see that the operator $\omega\mathbf{1}^+$ belongs to the set

$$E' = \{\xi \in R \mid \sum_{n=0}^{\infty} \mathbf{n}^+(\xi^*)^n \text{ converges in the strong operator topology}\},$$

because

$$\sum_{n=0}^{\infty} \mathbf{n}^+((\omega\mathbf{1}^+)^*)^n = \sum_{n=0}^{\infty} \mathbf{n}^+(\omega\mathbf{n}^+)^* = \sum_{n=0}^{\infty} \mathbf{n}^+\mathbf{n}^-\omega^*.$$

Since $E' \subseteq E$, the operator $\omega\mathbf{1}^+$ is the unique solution of (22) in E'. On the other hand, for every invertible operator $U \in R$ we have $U^{-1}E'U \subseteq E'$, as it follows easily from Lemma 1. Hence we conclude that $U^{-1}\omega\mathbf{1}^+U \in E'$ for any invertible operator U, but since $U^{-1}\omega\mathbf{1}^+U$ is also bound to be a solution of (22), by the uniqueness of the last solution in E' it follows that $U^{-1}\omega\mathbf{1}^+U = \omega\mathbf{1}^+$, i.e. $\omega\mathbf{1}^+$ commutes with all invertible operators and therefore $\omega\mathbf{1}^+ = cI$ for a scalar c. Since $cI \in E'$, i.e. $\sum_{n=0}^{\infty} c^n\mathbf{n}^+$ converges in the strong operator topology, by Lemma 1 the series $\sum_{n=0}^{\infty} |c|^{2n}$ converges and therefore $|c| < 1$, whence $\|\omega\mathbf{1}^+\| < 1$.

Applying Corollary 1 to other codings, we obtain other propositions of that kind. Consider, for instance, the following one.

Proposition 2. *Let l be a natural number and for* (21) *we have the inequality*

$$(l+1)^2 \sum_{n=0}^{\infty} l^{-n}|a_n|^2 < 1.$$

Then the function f has an unique fixed point ξ_0 in the ball

$$\{\xi \in R \mid \|\xi\| < (l+1)^{-1/2}\},$$

which is of the form $\xi_0 = cI$ for a suitable scalar c.

Indeed, consider the semigroup G consisting of $l+1$ elements $e_0, \ldots, e_l$ with the following multiplication law: $e_ie_j = e_{\max(i,j)}$, and let $\chi : G \to \{I\}$ be the trivial homomorphism. Suppose

$$(T_+, T_-, F_+(e_0), F_-(e_0), \ldots, F_+(e_l), F_-(e_l))$$

is an isometrical orthogonal system in R, consisting of $l+2$ pairs of operators. To each monomial

$$q = g_0Xg_1 \ldots g_{n-1}Xg_n \tag{23}$$

in $G(X)$ we assign the operator $W^-(q) = F_-(g_0)\ldots F_-(g_n)$ and define $W^+(q) = (W^-(q))^*$, $k^+(q) = W^+(q)T_+$ and $k^-(q) = T_-W^-(q)$. Using Lemma 1, it is easy to show that the orthogonal system $(k^+(q), k^-(q))_{q\in G(X)}$ is a coding for $G(X)$ with respect to χ. Indeed, condition b) in the definition of coding is fulfilled with $\tau = T_- \sum_{i\leq l} F_-(e_i)$. By Lemma 1 the series

$$\sum_{s\in G(X)\setminus G} k^+(s)k^-(qs),$$

which is adjoint to (4), converges in the strong operator topology iff for all $x \in H$ converges the series

$$\sum_{s \in G(X)\setminus G} \|k^-(qs)x\|^2,$$

which, since the operators $k^+(q)$ are isometrical, is the same as

$$\sum_{s \in G(X)\setminus G} \|k^+(qs)k^-(qs)x\|^2,$$

and the sum of the last series is obviously not greater than $l\|x\|^2$. Thus we see that (4) converges in the weak operator topology, i.e. condition a) in the definition of coding holds. A monomial of the form (23) will be called *regular* iff $g_0 = e_l$ and $e_l \notin \{g_1, \ldots, g_n\}$. Denote by M the set of all regular monomials in $G(X)$ and let for every regular monomial $p(X)$ of degree n define $\lambda_p = l^{-n}a_n$. Since the number of regular monomials of degree n is l^n, the value $\widetilde{P}(\xi)$ of the formal series

$$P(X) = \sum_{p \in M} \lambda_p p(X)$$

with respect to χ is just the sum $f(\xi)$ of (21) for the operator ξ in the sense of the weak operator topology. But the system

$$(W^+(p), W^-(p))_{p \in M}$$

is an orthogonal one, whence by Lemma 1 and the inequality, assumed in Proposition 2, we conclude that the series

$$\psi = \sum_{p \in M} \lambda_p W^+(p)$$

converges in the strong operator topology; and for the sum ψ of this series we have

$$\|\psi\|^2 = \sum_{p \in M} |\lambda_p|^2,$$

because the vectors $W^+(p)x$, $p \in M$, are pairwise orthogonal for all $x \in H$. Now define

$$\rho = \psi(T_+T_- + \varkappa)\sum_{g \in G}\sum_{h \in G} F_-(h)F_-(g),$$

where $\varkappa$ is the projection

(24) $$\varkappa = \sum_{q \in G(X)} k^+(q)k^-(q).$$

We shall show that ρ is a governing element for the equation $P(X) = X$ with respect to the coding in question. The equality (7) is obvious for this ρ, and to see (8), we have to consider two cases for the monomial $q \in G(X)$:

Case 1) $q = h \in G$. Then

$$\rho k^+(qXg) = \rho F_+(g)F_+(h)T_+ = \psi(T_+T_- + \varkappa)T_+ = \psi T_+$$
$$= \sum_{p \in M} \lambda_p W^+(p)T_+ = \sum_{p \in M} \lambda_p W^+(hp)T_+ = \sum_{p \in M} \lambda_p k^+(qp),$$

because $p = hp$ for any regular monomial p.

Case 2) $q = q_0 X h$ for some $q_0 \in G(X)$ and $h \in G$. Then, as in the first case,

$$\rho k^+(qXg) = \rho F_+(g)F_+(h)W^+(q_0)T_+ = \psi(T_+T_- + \varkappa)W^+(q_0)T_+$$
$$= \sum_{p\in M} \lambda_p W^+(p)W^+(q_0)T_+ = \sum_{p\in M} \lambda_p W^+(hp)W^+(q_0)T_+ = \sum_{p\in M} \lambda_p k^+(qp).$$

So ρ is indeed a governing element. On the other hand, since the operator $T_+T_- + \varkappa$ is an orthogonal projection, we have

$$\|\rho\|^2 \leq \|\psi\|^2 \|(T_+T_- + \varkappa)\|^2 \Big\| \sum_{g\in G} \sum_{h\in G} F_-(h)F_-(g) \Big\|^2 = (l+1)^2 \|\psi\|^2$$
$$= (l+1)^2 \sum_{p\in M} |\lambda_p|^2 = (l+1)^2 \sum_{n=0}^{\infty} \sum_{p\in M,\, \deg p = n} |\lambda_p|^2 = (l+1)^2 \sum_{n=0}^{\infty} l^{-n} |a_n|^2,$$

and by the assumptions of the Proposition 2 $\|\rho\| < 1$. Therefore the iterational equation has an unique solution ω and by Corollary 1 the operator $\omega k^+(X)$ is the unique solution of $P(X) = X$ in the set E of all operators $\xi \in R$ for which $\sum_{q\in G(X)} \widetilde{q}(\xi)k^-(q)$ is conventionally convergent in R, i.e. in the weak operator topology. We see as above that this set E contains the set E' of those operators ξ for which the adjoint of the last series converges in the strong operator topology. Using again Theorem 1 and the strong convergence of the series in (24), we conclude that $\omega k^+(X) \in E'$. By Lemma 1 the set E' coincides with the set of all operators ξ for which the series

$$\sum_{q\in G(X)} \|(\widetilde{q}(\xi))^* x\|^2$$

converges for all vectors x. Hence it follows that $U^{-1}E'U \subseteq E'$ for every invertible operator U and therefore $\omega k^+(X)$ commutes with every such operator, which shows that $\omega k^+(X) = cI$ for a suitable scalar c. By the convergence of the last series with cI for ξ it follows that $\sum_{i=0}^{\infty}(l+1)^{i+1}|c|^{2i}$ also converges, whence $|c| < (l+1)^{-1/2}$. The last characterization of the set E' shows also that it contains the ball $\{\xi \in R \mid \|\xi\| < (l+1)^{-1/2}\}$, whence we get all the conclusions of Proposition 2.

An n-dimensional variant of the last proposition holds as well. Its proof differs from the above one only in trivial details and we shall give here its formulation only. For that purpose we use the following notations: for every n-tuple $w = (i_0, \ldots, i_{n-1})$ of natural numbers and every n-dimensional vector $z = (z_0, \ldots, z_{n-1})$ ($z_0, \ldots, z_{n-1}$ being complex numbers) define $z^w = z_0^{i_0} z_1^{i_1} \ldots z_{n-1}^{i_{n-1}}$, $|w| = i_0 + \cdots + i_{n-1}$ and $(w)! = i_0! \ldots i_{n-1}!$.

Proposition 2′. *Let for a system of n series of the form*

$$f_i(z) = \sum_{w\in\mathbb{N}^n} a_{iw} z^w, \quad i < n,$$

we have the inequality

$$(l+1)^2 \sum_{i<n} \sum_{w \in \mathbb{N}^n} \left(l^{|w|}|w|!\right)^{-1} (w)!|a_{iw}|^2 < 1.$$

Then the system of equations

$$f_i(\xi_0, \ldots, \xi_{n-1}) = \xi_i, \quad i < n,$$

has an unique solution in operators $(\xi_0, \ldots, \xi_{n-1}) \in R^n$ *for which*

$$\sum_{i<n} \|\xi_i\|^2 < (l+1)^{-1}$$

and this solution is of the form $(c_0 I, \ldots, c_{n-1} I)$, *where* $c_0, \ldots, c_{n-1}$ *are scalars.*

Some other propositions of this kind can be obtained by using suitable infinite semigroups for G.

Finally, let us note that the method used in the present paper is applicable in various situations. For instance, it holds quite well, promising interesting applications, for orthorings with an additional binary operation having the properties of tensor product. The last orthorings are for the combinatory spaces [2] approximately the same what the orthorings in the present paper are for the operative spaces [1]. It is a more complicated question whether the code evaluation theorem holds under not very restrictive suppositions for orthorings with involution. Here the technique of iterative extensions in the algebraic recursion theory may provide the necessary tool. Perhaps the further investigations will throw more light over this situation.

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