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ADMISSIBLE FUNCTIONALS IN ABSTRACT STRUCTURES WITH ARBITRARY POWER*

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Стела Николова. ДОПУСТИМЫЕ ФУНКЦИОНАЛЫ В АБСТРАКТНОЙ СТРУКТУРЕ ПРОИЗВОЛЬНОЙ МОЩНОСТИ

В работе предлагается один способ обобщения понятия допустимой (∀-рекурсивной [2]) функции для абстрактных структур произвольной мощности. Для этого здесь вводится и изучается понятия допустимого функционала. При его помощи определяется понятие допустимости для частично-многозначной функции в произвольно мощной абстрактной структуре. Устанавливается, что частично-многозначная функция допустима тогда и только тогда, когда она абсолютно поисково вычислима.

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A way for generalizing the notion of admissible (or ∀-recursive [2]) function for the case of arbitrary (not only denumerable) abstract structure is considered. For this purpose a notion of admissible functional is introduced and studied in the paper. Using this notion, a concept of admissibility for partial multiple valued function over arbitrary structure is introduced. It is established that a function is admissible if and only if it is absolutely search computable.

In the present paper we suggest a certain way for generalizing the notion of ∀-recursiveness for the case of abstract structures with arbitrary power. The notion of ∀-recursive predicate, introduced by Lacombe [2], is aimed at describing effectively definable relations on denumerable structures with equality. Later on Moschovakis

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proved that it is equivalent to search computability with constants [3]. In some recent works of Soskov [11–13] a certain modification of the original definition of ∀-recursiveness is suggested. It is used there as a basis for an uniform and very natural classification of several well-known concepts of abstract computability — prime and search computability [4], computability by means of effectively definable schemes (EDS) of Friedman [1] etc. What is more, the equality relation is not already supposed to be among the initial predicates.

Since the notion of ∀-recursiveness incorporates different numberings of the data domain, in each of the above mentioned works the structures are supposed to be at most denumerable. Exactly, to embrace the general case of arbitrary domains, the notion of admissible functional is designed.

There are several versions of the notion in question, depending on the kind of relative recursiveness over the naturals that we have taken as basic and according to our understanding of the expression "function, computable in a given structure $\mathfrak{A}$". Here we study one, in a sense the easiest to examine, of these versions, the other cases being considered in [5, 6]. We shall regard as basic the broadest notion of relative recursiveness over the set of all natural numbers — the partial recursiveness. Moreover, given $\mathfrak{A}$, we shall assume that some "oracle" for the data domain (along with "oracles" for the initial functions and predicates of $\mathfrak{A}$) is available.

1. PRELIMINARIES

Assume that a partial structure $\mathfrak{A} = (B; \varphi_1, \ldots, \varphi_a; R_1, \ldots, R_b)$ of some fixed signature $\sigma = (f_1, \ldots, f_a; P_1, \ldots, P_b)$ is given. Suppose first that B is at most denumerable. An *enumeration* of $\mathfrak{A}$ is any ordered pair $(\varkappa, \mathfrak{B})$, where $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$ is a σ-structure over the set N of all natural numbers and $\varkappa$ is a total mapping from N onto B, the following conditions being satisfied:

$\varkappa(\psi_i(x_1, \ldots, x_{l_i})) \simeq \varphi_i(\varkappa(x_1), \ldots, \varkappa(x_{l_i}))$ for all $x_1, \ldots, x_{l_i}$ in N, $1 \leqq i \leqq a$;

$Q_j(x_1, \ldots, x_{m_j}) \simeq R_j(\varkappa(x_1), \ldots, \varkappa(x_{m_j}))$ for all $x_1, \ldots, x_{m_j}$ in N, $1 \leqq j \leqq b$.

Suppose now that $\mathfrak{A}$ is a structure with arbitrary power. It is clear that in this case we cannot speak about enumerations of $\mathfrak{A}$. Nevertheless a notion of admissibility in $\mathfrak{A}$ is still possible. The key to it is the observation that every computational path is at most countable, hence no more than countably many elements of B can be involved in the course of the computation. Our idea is to break $\mathfrak{A}$ into some suitable denumerable parts, to enumerate them and to combine all these parts in some reasonable way.

We begin with some notational conventions. The elements of the basic set B will be denoted by s, p, r, possibly with indexes; as usual $(s_1, \ldots, s_k)$ will be abbreviated to $\bar{s}$. We shall use small greek letters to denote sequences — β, δ will range over the class of all infinite sequences of elements of B (to be denoted by B^N), while α, γ will denote infinite sequences of natural numbers (which sometimes will be viewed as total functions in N). For any $A \subseteq B$ set $\mathcal{T}(A) = \{p \mid p \simeq \tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_n/s_n)$, where $\tau(X_1, \ldots, X_n)$ is a σ-term with variables among $X_1, \ldots, X_n$ and $s_1, \ldots, s_n$ are elements of $A\}$. In particular, the set $\mathcal{T}(\{s_1, \ldots, s_k\} \cup \{\beta(n) \mid n \in N\})$ will be denoted by $\mathcal{T}(\bar{s}, \beta)$. This set is closed under the initial functions of $\mathfrak{A}$, so there exists a substructure of $\mathfrak{A}$ with domain $\mathcal{T}(\bar{s},\beta)$, which we shall denote by $\mathfrak{A}(\bar{s},\beta)$. Notice that $\mathfrak{A}(\bar{s},\beta)$ is at most denumerable and hence it can be enumerated.

Let $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$ be some structure over the natural numbers (further $\mathfrak{B}$ will always denote such structures). Whenever Γ is an enumeration operator (cf. [8]) of appropriate arity, we shall write $\Gamma(\mathfrak{B})$ to denote the set $\Gamma(\psi_1, \ldots, \psi_a, Q_1^*, \ldots, Q_b^*)$, where Q_i^* is the characteristic function of the predicate Q_i. Further $\Gamma(\mathfrak{B})$ will be looked upon rather as the partial multiple-valued function with a graph $\Gamma(\mathfrak{B})$. If α is a function in N, we shall set also $\mathfrak{B}_\alpha = (N; \alpha, \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$.

Let $\varkappa$ be a total mapping from N into B, $M \subseteq N$, $\bar{x} \in N^k$ and $\alpha \in N^N$. By $\varkappa(M)$, $\varkappa(\bar{x})$ and $\varkappa(\alpha)$ we shall denote $\{\varkappa(x) \mid x \in M\}$, $(\varkappa(x_1), \ldots, \varkappa(x_k))$ and $\{\varkappa(\alpha(n))\}_n$, respectively.

Suppose now that a *computational process* over $\mathfrak{A}$ at some input $\bar{s}$ is initiated. As we have mentioned above, we assume that an oracle $\mathbb{O}$ for the data domain B may also be used during this computation, operating in the following way: whenever some question to $\mathbb{O}$ is asked, it returns (generates) an arbitrary element of B and its answers do not depend on the current configuration. Now it is clear that if the answers of $\mathbb{O}$ are $t_1, \ldots, t_m$ (or $t_1, t_2, \ldots$ if the process is infinite and infinitely many questions are put to $\mathbb{O}$), then each $p \in B$, which appears during the execution, is an element of $\mathcal{T}(\bar{s},\beta)$, where $\beta(n) = t_n$ for $n = 1, 2, \ldots$

We shall use mappings F of the set $B^k \times B^N$ to describe mathematically the behavior of non-deterministic algorithms over $\mathfrak{A}$ with k input variables. For every $(\bar{s},\beta) \in B^k \times B^N$ $F(\bar{s},\beta)$ will be interpreted as the result obtained when such an algorithm is applied at the input $\bar{s}$, provided for every $n = 1, 2, \ldots$ the answer to the n-th question to the oracle has been equal to $\beta(n)$ (see also [10, ch. 2, § 5.1] for additional motivation).

The intuitive remarks just maid justify the introduction of the following definition:

$F : B^k \times B^N \to 2^B$ is said to be *admissible* (*in* $\mathfrak{A}$) iff there exists an enumeration operator Γ such that for every $(\bar{s},\beta) \in B^k \times B^N$ and for every enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(\bar{s},\beta)$ the equality

$$\varkappa(\Gamma(\mathfrak{B}_\alpha)(\bar{x})) = F(\bar{s},\beta)$$

holds for each $(\bar{x},\alpha) \in N^k \times N^N$ such that $\varkappa(\bar{x}) = \bar{s}$ and $\varkappa(\alpha) = \beta$.

Now taking into account the interpretation of F, we come to the following notion of admissible function:

The partial multiple-valued (p.m.v.) function $\varphi : B^k \to 2^B$ is said to be *admissible* (*in* $\mathfrak{A}$) iff there exists an admissible functional F such that for every $(s_1, \ldots, s_k, p) \in B^{k+1}$ the following is true:

$$p \in \varphi(s_1, \ldots, s_k) \Leftrightarrow \exists\beta(p \in F(s_1, \ldots, s_k, \beta)).$$

In order to formulate an explicit characterization of the notions introduced so far, some syntactical constructions will be needed.

Let P_0 be a new unary predicate symbol which is intended to represent the predicate $\lambda x.\mathtt{true}$. Throughout the paper a *σ-formula* will be any finite conjunction of atomic formulas in the extended signature $(f_1, \ldots, f_a; P_0, P_1, \ldots, P_b)$ or their negations. An expression of the form $\Phi \Rightarrow \tau$, where Φ is a σ-formula and τ is a σ-term, *will be called a σ-clause. Every recursively enumerable set of σ-clauses we* shall call, following [9], *recursively enumerable* (r.e.) *scheme.*

Let Ψ be some expression with variables $X_1, \ldots, X_k$ and $s_1, \ldots, s_k$ be arbitrary elements of B. We shall write $\Psi_{\mathfrak{A}}(X_1/s_1, \ldots, X_k/s_k)$ ($\Psi_{\mathfrak{A}}(\bar{s})$ for short) to denote the value (if it exists) of Ψ on $\mathfrak{A}$ when X_i is replaced by s_i, $i = 1, \ldots, k$. If Ψ is a term or a formula, the meaning of $\Psi_{\mathfrak{A}}(\bar{s})$ is the usual one. In the case when Ψ is $\Phi \Rightarrow \tau$, $\Psi_{\mathfrak{A}}(\bar{s})$ is introduced by the equivalence

$$(\Phi \Rightarrow \tau)_{\mathfrak{A}}(\bar{s}) \simeq p \quad \text{iff} \quad \Phi_{\mathfrak{A}}(\bar{s}) \simeq \mathrm{t} \text{ and } \tau_{\mathfrak{A}}(\bar{s}) \simeq p.$$

Here and further "t" stands for "**true**".

Let $\Delta = \{\Phi^w \Rightarrow \tau^w \mid w \in W\}$ be some set of clauses with variables $X_1, \ldots, X_k, Y_0, Y_1, \ldots$ (i.e. for every $w \in W$ the variables of $\Phi^w \Rightarrow \tau^w$ are among $X_1, \ldots, X_k, Y_0, Y_1, \ldots$). Δ determines a mapping $\Delta_{\mathfrak{A}} : B^k \times B^N \to 2^B$ defined by the following condition:

$$p \in \Delta_{\mathfrak{A}}(\bar{s}, \beta) \Leftrightarrow \exists w_{w \in W}((\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(\bar{s}, \beta) \simeq p).$$

Here $(\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(\bar{s}, \beta)$ is an abbreviation for $(\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(X_1/s_1, \ldots, X_k/s_k, Y_{j_1}/\beta(j_1), \ldots, Y_{j_n}/\beta(j_n))$, where $X_1, \ldots, X_k, Y_{j_1}, \ldots, Y_{j_n}$ is a list containing all variables of $\Phi^w \Rightarrow \tau^w$.

We shall say that the functional $F : B^k \times B^N \to 2^B$ is *definable* iff there exists a r.e. scheme Δ with variables $X_1, \ldots, X_k, Y_0, Y_1, \ldots$ such that $F = \Delta_{\mathfrak{A}}$.

In order to save space, in the following we shall assume that the initial functions and predicates of $\mathfrak{A}$ are unary.

2. CONSTRUCTING A RECURSIVELY ENUMERABLE SCHEME FOR A GIVEN ADMISSIBLE FUNCTIONAL

Denote by σ^+ the signature $\sigma \cup \{\boldsymbol{S}, \boldsymbol{0}, =\}$ with the commonly accepted semantics of the additional symbols $\boldsymbol{S}$, $\boldsymbol{0}$, $=$ over the naturals. Set also $\sigma_0^+ = \sigma^+ \cup \{f_0\}$, f_0 being a new unary function symbol.

From now on we shall suppose that some admissible functional $F : B^k \times B^N \to 2^B$ is fixed. Without any loss of generality we may assume that $k = 1$. In this section we are going to construct a r.e. scheme Δ, for which we shall establish later that $F = \Delta_{\mathfrak{A}}$.

Indeed, since F is admissible, there exists an enumeration operator Γ such that for every (s, β) and every enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s, \beta)$ the equality

$$\varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s, \beta)$$

is satisfied for each x, α: $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$. In the present paper we shall use an equivalent characterization of enumeration operators from [9, Thm. 7.5]. According to it there exists a r.e. set Δ^0 of σ_0^+-clauses with one variable X, such that for every x, α and $\mathfrak{B}$

$$\Gamma(\mathfrak{B}_\alpha)(x) = \Delta^0_{\mathfrak{B}_\alpha}(x). \tag{2.0}$$

Moreover, each clause of Δ^0 is of the form

$$X = \underline{m} \mathbin{\&} f_{i_1}(\underline{m}_1) = \underline{n}_1 \mathbin{\&} \ldots \mathbin{\&} f_{i_e}(\underline{m}_e) = \underline{n}_e \mathbin{\&} \Phi \Rightarrow \underline{m}_0,$$

where $e \geqq 0$ and Φ is a (possibly empty) conjunction of formulas of the kind $P_j(\underline{n})$ or $\neg P_j(\underline{n})$ (here as usual $\underline{k}$ stands for $\boldsymbol{S}^k(\mathbf{0})$). We may also suppose that in one and the same clause of Δ^0 there are not repeating conjuncts as well as conjuncts of the kind $f_i(\underline{k}) = \underline{n}$ and $f_i(\underline{k}) = \underline{n}'$ with $n \neq n'$.

Our main task in this section will be to remove from Δ^0 function and predicate symbols which are not in σ, preserving at the same time (a part of) the information about F that Δ^0 bears. This will be done in a few steps.

At the first step we eliminate f_0. For that purpose choose some sequence $Y_0, Y_1, \ldots$ of different variables and replace each conjunct of the kind $f_0(\underline{n}) = \underline{n}'$ by $Y_n = \underline{n}'$. This reduces Δ^0 to a r.e. scheme Δ^1 which variables are among $X, Y_0, Y_1, \ldots$ Furthermore, for every x, α, $\mathfrak{B}$

$$\Delta^0_{\mathfrak{B}_\alpha}(x) = \Delta^1_{\mathfrak{B}}(x, \alpha), \tag{2.1}$$

which follows immediately from the appropriate definitions and the equivalences

$$(f_0(\underline{n}) = \underline{n}')_{\mathfrak{B}_\alpha}(x) \simeq \mathrm{t} \Leftrightarrow \alpha(n) = n' \Leftrightarrow (Y_n = \underline{n}')_{\mathfrak{B}}(x, \alpha) \simeq \mathrm{t}.$$

Let us now fix some injective recursive function α such that $N \setminus \mathrm{Range}(\alpha)$ is infinite and decidable. Fix also an arbitrary x which is not in $\mathrm{Range}(\alpha)$. Let set for brevity $y_i = \alpha(i)$, $i = 0, 1, \ldots$, and $M = \{x, y_0, y_1, \ldots\}$. In Δ^1 we make the following transformations: first remove each clause containing conjuncts $X = \underline{m}$ ($Y_i = \underline{n}$), where $m \neq x$ ($n \neq y_i$); then delete every conjunct of the kind $X = \underline{x}$ or $Y_i = \underline{y}_i$, or replace it by $P_0(X)$ if it is the unique conjunct in the clause. This procedure yields a set $\Delta^2 = \{A^{(w)} \Rightarrow a^{(w)} \mid w \in W\}$ which is also r.e. (since α is recursive), the following condition being satisfied for all σ-structures $\mathfrak{B}$ and for already fixed x and α:

$$\Delta^1_{\mathfrak{B}}(x, \alpha) = \Delta^2_{\mathfrak{B}}(x, \alpha). \tag{2.2}$$

Now let us fix some clause $A^{(w)} \Rightarrow a^{(w)}$ from Δ^2. It has the form $f_{i_1}(\underline{m}_1) = \underline{n}_1 \mathbin{\&} \ldots \mathbin{\&} f_{i_e}(\underline{m}_e) = \underline{n}_e \mathbin{\&} \Phi \Rightarrow \underline{m}_0$, where $f_{i_1}, \ldots, f_{i_e}$ are already function symbols from σ.

We shall say that the index w of $A^{(w)} \Rightarrow a^{(w)}$ is *suitable* iff the following conditions hold:

(i) $\{n_1, \ldots, n_e\} \subseteq N \setminus M$;

(ii) $m_j < n_j$ for every $j = 1, \ldots, e$;

(iii) $n_1, \ldots, n_e$ are different natural numbers.

Now set $\Delta^3 = \{A^{(w)} \Rightarrow a^{(w)} \mid w \text{ is suitable}\}$. Each of the conditions (i), (ii), (iii) is decidable, hence Δ^3 is r.e., too. Clearly, for every structure $\mathfrak{B}$ over the naturals we have

$$\Delta^2_{\mathfrak{B}}(x, \alpha) \supseteq \Delta^3_{\mathfrak{B}}(x, \alpha). \tag{2.3}$$

The opposite inclusion of (2.3) is not always true. *In what follows we shall define a non-empty class of enumerations* $(\varkappa, \mathfrak{B})$ *such that* $\Delta^2_{\mathfrak{B}}(x, \alpha) = \Delta^3_{\mathfrak{B}}(x, \alpha)$ *holds for every* $\mathfrak{B}$ *from this class.*

Indeed, for an arbitrary $(s,\beta) \in B \times B^N$ denote by $\mathcal{K}_{s,\beta}$ the class of those enumerations $(\varkappa, \mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b))$ of $\mathfrak{A}(s,\beta)$ which satisfy the conditions:

0) $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$;

1) $\mathrm{Range}(\psi_i) \subseteq N \setminus M$, $1 \leqq i \leqq a$;

2) if $\psi_i(m)$ is defined, then $\psi_i(m) > m$, $1 \leqq i \leqq a$;

3) $\mathrm{Range}(\psi_i) \cap \mathrm{Range}(\psi_j) = \varnothing$, $i, j \in \{1, \ldots, a\}$ and $i \neq j$;

4) ψ_i is injective, $1 \leqq i \leqq a$.

Let us first check the following

Lemma 2.1. *For every s, β the class $\mathcal{K}_{s,\beta}$ is not empty.*

Proof. Clearly, one can choose $\varkappa$ satisfying 0) and the requirement: for every $q \in \mathcal{T}(s,\beta)$ the set $N^{(q)} = \{n \mid n \in N \setminus M \,\&\, \varkappa(n) = q\}$ is infinite. Let $n_1^{(q)}$, $n_2^{(q)}, \ldots$ be the list of all elements of $N^{(q)}$ put in ascending order. Now for every $i = 1, \ldots, a$ define ψ_i as follows: if $\varphi_i(\varkappa(m)) \simeq q$, set $\psi_i(m) = n_k^{(q)}$, where $k = 2^i.3^m$; if $\varphi_i(\varkappa(m))$ is undefined, set $\psi_i(m)$ to be undefined, too. Set also $Q_j(m) \simeq R_j(\varkappa(m))$ for every $m \in N$ and $j \in \{1, \ldots, b\}$.

We have by definition

$$!\psi_i(m) \Leftrightarrow !\varphi_i(\varkappa(m)) \quad \text{and} \quad !\psi_i(m) \Rightarrow \varkappa(\psi_i(m)) = \varkappa(n_k^{(q)}) = q = \varphi_i(\varkappa(m)),$$

which together with the choice of $Q_1, \ldots, Q_b$ shows that the pair

$$(\varkappa, \mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b))$$

is an enumeration of $\mathfrak{A}(s,\beta)$. Now the conditions 0) and 1) are obviously true and the validity of 2) follows from the observation $n_k^{(q)} \geqq k > (k)_1 = m$. We shall simultaneously prove that 3) and 4) are also true. To do this, we assume that $\psi_i(m) = \psi_j(m') = n$. We have to prove that $i = j$ and $m = m'$. Indeed, using the equalities $\varkappa(\psi_i(m)) = \varkappa(\psi_j(m')) = \varkappa(n) = q$ we get $n_k^{(q)} = \psi_i(m) = \psi_j(m') = n_{k'}^{(q)}$, where $k = 2^i.3^m$ and $k' = 2^j.3^{m'}$. Since there are no repetitions in the sequence $n_1^{(q)}, n_2^{(q)}, \ldots$, the equality $n_k^{(q)} = n_{k'}^{(q)}$ implies $k = k'$, i.e. $i = j$ and $m = m'$.

Now define $\mathcal{K}$ as $\bigcup\{\mathcal{K}_{s,\beta} \mid (s,\beta) \in B \times B^N\}$. We are going to check that

$$\Delta^2_{\mathfrak{B}}(x,\alpha) = \Delta^3_{\mathfrak{B}}(x,\alpha) \tag{2.4}$$

for every $\mathfrak{B}$ such that $(\varkappa, \mathfrak{B}) \in \mathcal{K}$. Indeed, take some $(\varkappa, \mathfrak{B})$ from $\mathcal{K}$ and hence from $\mathcal{K}_{s,\beta}$ for some s, β. Clearly, (2.4) will be proven if we succeed in verifying the following statement: whenever $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$, then w is suitable. Assume $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$; then in particular $A^{(w)}_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$, i.e. $(f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \ldots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi)_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$. Hence $(f_{i_j}(\underline{m}_j) = \underline{n}_j)_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$, which means that $\psi_{i_j}(m_j) = n_j$ for every $1 \leqq j \leqq e$. These equalities together with conditions 1) and 2) of the definition of $\mathcal{K}_{s,\beta}$ ensure the validity of (i) and (ii). To see that (iii) is also true, use along with 3) and 4) the supposition made in the beginning of this section, namely that there are no two conjuncts of the kind $f_i(\underline{k}) = \underline{n}$ and $f_i(\underline{k}) = \underline{n}'$ in one and the same clause of Δ^0.

Now we gather up (2.0)–(2.4) to conclude:

Lemma 2.2. *For every σ-structure $\mathfrak{B}$ the following is true:*
a) $\Gamma(\mathfrak{B}_\alpha)(x) \supseteq \Delta^3_{\mathfrak{B}}(x,\alpha)$;
b) *if there exists* $\varkappa$ *such that* $(\varkappa, \mathfrak{B}) \in \mathcal{K}$, *then* $\Gamma(\mathfrak{B}_\alpha)(x) = \Delta^3_{\mathfrak{B}}(x,\alpha)$.

Our next task is to remove the symbols $\boldsymbol{S}$, $\mathbf{0}$ and $=$ from Δ^3. To this end let us fix some suitable w. We are going to introduce certain auxiliary notion of depth of $n \in N$ with respect to this index w. Let us remind that $A^{(w)} \Rightarrow a^{(w)}$ is of the form $f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \ldots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi \Rightarrow \underline{m}_0$. Now set

$$N^0 = N \setminus \{n_1, \ldots, n_e\};$$
$$N^{i+1} = \{n_j \mid \exists j(1 \leqq j \leqq e \,\&\, m_j \in N^i)\} \text{ for } i = 0, 1, \ldots$$

Lemma 2.3. $N^0 \cup N^1 \cup \ldots = N$ *and* $N^i \cap N^k = \varnothing$ *whenever* $i \neq k$.

Proof. Suppose first that for some $i \neq k$ $N^i \cap N^k \neq \varnothing$. We may assume that i is the least number satisfying this condition. Take some $n \in N^i \cap N^k$. We have $k > i \geqq 0$ and therefore $k \geqq 1$, hence there exists some $j \in \{1, \ldots, e\}$ such that $n = n_j$ and $m_j \in N^{k-1}$. Now $n = n_j \in N^i$ gives us $N^i \cap \{n_1, \ldots, n_e\} \neq \varnothing$, i.e. $i > 0$. It means that there exists some $q \in \{1, \ldots, e\}$ such that $n = n_q$ and $m_q \in N^{i-1}$. The equalities $n = n_j = n_q$ imply $j = q$ (w is a suitable index!). Thus we get $m_j \in N^{i-1} \cap N^{k-1}$, which contradicts the choice of i.

Now let us assume that there exists some $n \in N$ such that $n \notin N^0 \cup N^1 \cup \ldots$ and let n be the least with this property. Apparently, $n = n_j$ for some $j \in \{1, \ldots, e\}$. Again from the fact that w is suitable we obtain $m_j < n_j = n$ and therefore $m_j \in N^i$ for some i. This implies $n_j \in N^{i+1}$ — a contradiction with the choice of n.

Lemma 2.3 makes the following definition correct. Let us call a *depth of n with respect to w* (in symbols: $|n|_w$) the unique natural number i such that $n \in N^i$. Let us notice here the obvious observation that there is an effective way for every $n \in N$ and every suitable index w to find $|n|_w$.

Suppose now that some additional list of different variables Z_0, Z_1, … is chosen. Fix some suitable w. By induction on $|n|_w$ we define a sequence $\{\tau^n\}_n$ of terms in the following way:

$$\text{If } |n|_w = 0, \text{ set } \tau^n = \begin{cases} X, & \text{if } n = x, \\ Y_i, & \text{if } n = y_i, \\ Z_n & \text{otherwise.} \end{cases}$$

When $|n|_w = i > 0$, then by definition $n = n_j$, the depth of m_j being $i - 1$. Set in this case $\tau^n = f_{i_j}(\tau^m j)$.

Before explaining the basic property of τ^n we introduce a notational convention. Whenever Ψ is a σ-expression with variables X, Y_{j_1}, …, Y_{j_k}, Z_{l_1}, …, Z_{l_m}, $\mathfrak{M}$ is an arbitrary σ-structure and $(s, \beta, \delta) \in |\mathfrak{M}| \times |\mathfrak{M}|^N \times |\mathfrak{M}|^N$; we shall write $\Psi_{\mathfrak{M}}(s, \beta, \delta)$ as an abbreviation for

$$\Psi_{\mathfrak{M}}(X/s, Y_{j_1}/\beta(j_1), \ldots, Y_{j_k}/\beta(j_k), Z_{l_1}/\delta(l_1), \ldots, Z_{l_m}/\delta(l_m)).$$

From now on γ will denote the identity function on N. Let $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$ be a structure over N. A routine induction on the depth of n convince us that

$$(2.5) \qquad \tau_{\mathfrak{B}}^{(n)}(x, \alpha, \gamma) = n \quad \text{provided} \quad \psi_{i_1}(m_1) = n_1, \ldots, \psi_{i_e}(m_e) = n_e.$$

Now we are in position to remove the symbols $\boldsymbol{S}$, $\mathbf{0}$ and $=$ from $A^{(w)} \Rightarrow a^{(w)}$. Indeed, let Φ' is obtained from Φ by replacing each conjunct $P_i(\underline{n})$ $(\neg P_i(\underline{n}))$ of Φ by $P_i(\tau^n)$ $(\neg P_i(\tau^n))$ and denote by $B^{(w)} \Rightarrow b^{(w)}$ the clause $P_0(\tau^{n_1}) \,\&\, \ldots \,\&\, P_0(\tau^{n_e}) \,\&\, \Phi' \Rightarrow \tau^{m_0}$. Notice that $B^{(w)} \Rightarrow b^{(w)}$ is already a σ-clause. Moreover, $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)$ is equal to $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha)$ under certain condition about $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$, namely:

Lemma 2.4. *Suppose that* $\psi_{i_1}(m_1) = n_1, \ldots, \psi_{i_e}(m_e) = n_e$. *Then*

$$(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma) \simeq (A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha).$$

Proof. It is obvious if we take into account the observation (2.5)

Let us mention here the following almost obvious fact, which will be used in the next section:

Lemma 2.5. *If* Z_n *is a variable of* $B^{(w)} \Rightarrow b^{(w)}$, *then* $n \notin \{x, y_0, y_1, \ldots\} \cup \{n_1, \ldots, n_e\}$.

Proof. If Z_n is a variable of $B^{(w)} \Rightarrow b^{(w)}$, then Z_n is a variable of $\tau^{(m)}$ for some m. Now use a straightforward induction on the depth of m with respect to w.

Variables of each clause $B^{(w)} \Rightarrow b^{(w)}$ are among $X, Y_0, Y_1, \ldots, Z_0, Z_1, \ldots$ Our final transformation aims to eliminate the variables from the list $Z_0, Z_1, \ldots$ Let us fix some effective enumeration $\rho^0, \rho^1, \ldots$ of all σ-terms with variables among X, $Y_0, Y_1, \ldots$ (i.e. such that the function which assigns to each n the Gödel number of ρ^n is recursive). For every $m > 0$ denote by $C_m^{(w)} \Rightarrow c_m^{(w)}$ the clause which is obtained from $B^{(w)} \Rightarrow b^{(w)}$ by replacing each variable Z_n by $\rho^{(m)_n}$. Now set $\Delta = \{C_m^{(w)} \Rightarrow c_m^{(w)} \mid m > 0 \text{ and } w \text{ is suitable}\}$. Due to the choice of the sequence $\{\rho^n\}_n$, Δ is a r.e. scheme. In the next section we shall establish that $F = \Delta_{\mathfrak{A}}$.

3. EXPLICIT CHARACTERIZATION OF ADMISSIBILITY

Theorem 3.1. *F is admissible in $\mathfrak{A}$ iff F is definable.*

Proof. The converse part is almost obvious. Let $F = \Delta_{\mathfrak{A}}$ for some r.e. scheme $\Delta = \{\Phi^w \Rightarrow \tau^w \mid w \in W\}$. Define $\Gamma(\mathfrak{B}_\alpha)(x)$ as $\{y \mid \exists w_{w \in W}(\Phi^w \Rightarrow \tau^w)_{\mathfrak{B}}(x, \alpha) \simeq y\}$. A straightforward verification convinces us that for every $(s, \beta) \in B \times B^N$ the equality

$$(3.0) \qquad \varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s, \beta)$$

holds for every enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ and for every (x,α) such that $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$.

Now suppose that F is admissible. Then there exists an enumeration operator Γ such that the equality (3.0) is true. Let Δ be the r.e. scheme for F, constructed in the previous section. Fix some (s,β) — an arbitrary element of $B \times B^N$. We are going to prove that

$$(3.1) \qquad p \in F(s,\beta) \iff p \in \Delta_{\mathfrak{A}}(s,\beta).$$

Indeed, assume first that $p \in F(s,\beta)$. Let choose some enumeration $(\varkappa, \mathfrak{B} = (N;\ \psi_1,\dots,\psi_a; Q_1,\dots,Q_b))$ from $\mathcal{K}_{s,\beta}$ which is not empty according to Lemma 2.1. By the choice of $(\varkappa,\mathfrak{B})$ we have $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$. Therefore by (3.0) $\varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s,\beta)$ and hence there exists $y \in N$ such that $y \in \Gamma(\mathfrak{B}_\alpha)(x)$ and $\varkappa(y) = p$. By Lemma 2.2 $y \in \Delta^3_{\mathfrak{B}}(x,\alpha)$ or, equivalently, $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$ for some suitable index w. In particular, $\psi_{i_1}(m_1) = n_1,\ \dots,\ \psi_{i_e}(m_e) = n_e$, hence Lemma 2.4 can be applied. As a result we obtain $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x,\alpha,\gamma) \simeq (A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) = y$, that is

$$(3.2) \qquad B^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma) \simeq \mathrm{t} \quad \text{and} \quad b^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma) \simeq y.$$

Define the sequence δ as follows: $\delta(n) = \varkappa(n)$ for $n = 0,1,\dots$ We have $\varkappa(\gamma) = \delta$ as well as $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$, and therefore $B^{(w)}_{\mathfrak{A}}(s,\beta,\delta) \simeq B^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma)$ and $b^{(w)}_{\mathfrak{A}}(s,\beta,\delta) \simeq \varkappa(b^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma))$. These equalities combined with (3.2) and the fact that $\varkappa(y) = p$ give us

$$(3.3) \qquad (B^{(w)}, \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq p.$$

Let $Z_{k_1},\dots,Z_{k_n}$ be all variables of $B^{(w)} \Rightarrow b^{(w)}$ from the list $Z_0, Z_1, \dots$ For $i = 1,\dots,n$ we set for short $\varkappa(k_i) = r_i$. Each r_i belongs to Range$(\varkappa) = \mathcal{T}(s,\beta)$, hence $r_i = \rho^{m_i}_{\mathfrak{A}}(s,\beta)$ for some term ρ^{m_i} from the sequence $\{\rho^n\}_n$, fixed in the end of the previous section. Now take some $m \in N$ such that $(m)_{k_i} = m_i$ for $1 \leqq i \leqq n$ and consider the clause $C^{(w)}_m \Rightarrow c^{(w)}_m$. By definition it is obtained from $B^{(w)} \Rightarrow b^{(w)}$ by simultaneous replacement of each Z_{k_i} by $\rho^{(m)_{k_i}}$, i.e. by ρ^{m_i} in our case. Notice that

$$(Z_{k_i})_{\mathfrak{A}}(s,\beta,\delta) = \delta(k_i) = r_i = \rho^{m_i}_{\mathfrak{A}}(s,\beta).$$

So we get

$$(C^{(w)}_m \Rightarrow c^{(w)}_m)_{\mathfrak{A}}(s,\beta) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta)$$

and by (3.3) $(C^{(w)}_m \Rightarrow c^{(w)}_m)_{\mathfrak{A}}(s,\beta) \simeq p$. The last, according to our choice of Δ, means that $p \in \Delta_{\mathfrak{A}}(s,\beta)$, which completes the verification of the first direction of (3.1).

Assume now that $p \in \Delta_{\mathfrak{A}}(s,\beta)$. The only way to force $p \in F(s,\beta)$ is to show that there is some enumeration $(\varkappa,\mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ such that

$$(3.4) \qquad \varkappa(x) = s, \quad \varkappa(\alpha) = \beta \quad \text{and} \quad p \in \varkappa(\Gamma(\mathfrak{B}_\alpha)(x)),$$

where, of course, x and α are again those already fixed in Sec. 2, because Δ convey certain information about Γ at the point (x,α) only.

The assumption $p \in \Delta_{\mathfrak{A}}(s,\beta)$ is equal to $(C_m^{(w)} \Rightarrow c_m^{(w)})_{\mathfrak{A}}(s,\beta) \simeq p$ for some suitable index w, whence in particular each of the expressions $C_{m\mathfrak{A}}^{(w)}(s,\beta)$ and $c_{m\mathfrak{A}}^{(w)}(s,\beta)$ is defined. Now having in mind the construction of $C_m^{(w)} \Rightarrow c_m^{(w)}$, we may conclude that for each n, such that Z_n is a variable of $B^{(w)} \Rightarrow b^{(w)}$, $\rho_{\mathfrak{A}}^{(m)_n}(s,\beta)$ is defined. Let us define a sequence $\delta \in B^N$ in the following way:

$$\delta(n) = \begin{cases} \rho_{\mathfrak{A}}^{(m)_n}(s,\beta), & \text{if } Z_n \text{ is a variable of } B^{(w)} \Rightarrow b^{(w)}, \\ s & \text{otherwise.} \end{cases}$$

It is clear that $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq (C_m^{(w)} \Rightarrow c_m^{(w)})_{\mathfrak{A}}(s,\beta)$ and hence

$$(3.5) \qquad (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq p.$$

We have in particular that $B_{\mathfrak{A}}^{(w)}(s,\beta,\delta)$ is defined. From here, since each $P_0(\tau^{n_i})$, $1 \leqq i \leqq e$, is a conjunct of $B^{(w)}$,

$$(3.6) \qquad \tau_{\mathfrak{A}}^{n_i}(s,\beta,\delta) \text{ is defined for every } i = 1,\ldots,e.$$

Now we can explain how to construct an enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ satisfying the requirement (3.4). We begin with the definition of $\varkappa$. Set $\varkappa(x) = s$, $\varkappa(y_i) = \beta(i)$ for $i = 0,1,\ldots$, $\varkappa(n) = \delta(n)$ for every n such that n is a variable of $B^{(w)} \Rightarrow b^{(w)}$ and $\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$ for $j = 1,\ldots,e$ (notice that by (3.6) each $\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$ is defined). To see that this settings are correct, recall our choice of α (α is injective and $x \notin \text{Range}(\alpha)$), take into account the fact that w is a suitable index (and hence $n_1, \ldots, n_e$ are different natural numbers which do not belong to $M = \{x, y_0, y_1, \ldots\}$) and consider Lemma 2.5.

Now we have to extend the definition of $\varkappa$ onto the whole N. Since $N \setminus \text{Range}(\alpha)$ is infinite, this can be done in such a way that $\{\varkappa(n) \mid n \in N\} = \mathfrak{T}(s,\beta)$, i.e. so that $\varkappa$ is a mapping onto $\mathfrak{T}(s,\beta)$. In addition, $\varkappa$ has one very important property, namely

$$(3.7) \qquad \varkappa(n_j) = \varphi_{i_j}(\varkappa(m_j)) \text{ for every } j = 1,\ldots,e.$$

We shall separately consider the two cases for the depth of m_j (with respect to w): it is 0; and it is positive. In the second case by definition $m_j \in \{n_1,\ldots,n_e\}$ and hence $\varkappa(m_j) = \tau_{\mathfrak{A}}^{m_j}(s,\beta,\delta)$. We have also $\tau^{n_j} = f_{i_j}(\tau^{m_j})$ and therefore $\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(\tau^{m_j}))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(\tau_{\mathfrak{A}}^{m_j}(s,\beta,\delta)) = \varphi_{i_j}(\varkappa(m_j))$. This, combined with $\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$, completes the verification of (3.7) for the case when $|m_j|_w > 0$. If $|m_j|_w = 0$, there are three possibilities: $m_j = x$, $m_j \in \{y_0, y_1, \ldots\}$, and $m_j \in N \setminus \{x, y_0, y_1, \ldots\} \cup \{n_1,\ldots,n_e\}$. If $m_j = x$, then by definition $\tau^{m_j} = X$ and $\tau^{n_j} = f_{i_j}(X)$. Further

$$\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(X))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(s) = \varphi_{i_j}(\varkappa(x)) = \varphi_{i_j}(\varkappa(m_j)).$$

In the case $m_j \in \{y_0, y_1, \ldots\}$ we proceed analogously. In the last case we have $\tau^{m_j} = Z_{m_j}$ and $\tau^{n_j} = f_{i_j}(Z_{m_j})$, respectively. Let us notice that by construction $P_0(\tau_{n_j})$ is a conjunct of $B^{(w)}$ and Z_{m_j} is a variable of $B^{(w)}$. Therefore by definition $\varkappa(m_j) = \delta(m_j)$. Now, similarly to the previous case, we get

$$\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(Z_{m_j}))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(\delta(m_j)) = \varphi_{i_j}(\varkappa(m_j)).$$

Now define the structure $\mathfrak{B}$ as follows: $\psi_{i_j}(m_j) = n_j$ for every $j = 1, \ldots, e$; $\psi_i(m) \simeq \mu n[\varkappa(n) \simeq \varphi_i(\varkappa(m))]$ in the remaining cases, i.e. when $(i, m) \neq (i_j, m_j)$ for each $1 \leqq j \leqq e$; $Q_i(m) \simeq R_i(\varkappa(m))$ for every $m \in N$ and $1 \leqq i \leqq b$.

In order to show that $(\varkappa, \mathfrak{B})$ is an enumeration of $\mathfrak{A}(s, \beta)$ it suffices to check that $\varkappa(\psi_i(m)) \simeq \varphi_i(\varkappa(m))$. Indeed, whenever $(i, m) = (i_j, m_j)$ for some $j \in \{1, \ldots, e\}$, this equality is true on the grounds of (3.7). Suppose now that $(i, m) \neq (i_j, m_j)$ for every $j \in \{1, \ldots, e\}$ and assume, first, that $\varphi_i(\varkappa(m))$ is defined. We have $\varkappa(m) \in \mathcal{T}(s, \beta)$ and next $\varphi_i(\varkappa(m)) \in \mathcal{T}(s, \beta)$. Since $\varkappa$ is a mapping onto $\mathcal{T}(s, \beta)$, there exists n such that $\varkappa(n) = \varphi_i(\varkappa(m))$ and hence $\psi_i(m)$ is defined. Further

$$\varkappa(\psi_i(m)) = \varkappa(\mu n[\varkappa(n) = \varphi_i(\varkappa(m))]) = \varphi_i(\varkappa(m)).$$

Whenever $\varphi_i(\varkappa(m))$ is undefined, $\psi_i(m)$ is undefined by definition.

A straightforward verification convinces us that $\varkappa(V_{\mathfrak{B}}(x, \alpha, \gamma)) = V_{\mathfrak{A}}(s, \beta, \delta)$ for every variable V of $B^{(w)} \Rightarrow b^{(w)}$. Thus we may conclude that

$$\varkappa((B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s, \beta, \delta)$$

and by (3.5) $\varkappa((B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)) \simeq p$. It means that there exists $y \in N$ such that

$$(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma) \simeq y \text{ and } \varkappa(y) = p. \tag{3.8}$$

On the other hand, owing to the special construction of $\mathfrak{B}$, Lemma 2.4 can be applied. So we get $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)$ and therefore by (3.8) $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha) \simeq y$. Since $A^{(w)} \Rightarrow a^{(w)}$ is a clause of Δ^3, $y \in \Delta^3_{\mathfrak{B}}(x, \alpha)$. Finally, using Lemma 2.2 we come to the conclusion that $y \in \Gamma(\mathfrak{B}_\alpha)(x)$. We have also $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$ and hence by (3.0) $\varkappa(y) \in F(s, \beta)$, that is $p \in F(s, \beta)$ by (3.8). This completes the verification of the first direction of the theorem. As we noticed above, the opposite is straightforward. Thus the proof of the theorem is completed.

As a consequence of the above theorem we get the following characterizations of the admissible functions, which agree with the corresponding result from [11, Thm. 4], obtained for the case of denumerable $\mathfrak{A}$:

Theorem 3.2. *$\varphi : B \to 2^B$ is admissible in $\mathfrak{A}$ iff it is absolutely search computable over $\mathfrak{A}$.*

Proof. We shall use the following normal form theorem for absolutely search computable (ASC) functions [7, Cor. 3]. A k-ary p.m.v. function φ is ASC in $\mathfrak{A}$ iff there exists r.e. scheme $\Delta = \{\exists Y_{j_1} \ldots \exists Y_{j_{w_n}} (\Phi^w \Rightarrow \tau^w) \mid w \in W\}$ with free variables $X_1, \ldots, X_k$ such that for every $\overline{s} \in B^k$ the following is true: $\varphi(\overline{s}) \ni p$ iff

$$\exists p_{j_1} \ldots \exists p_{j_{w_n}} (\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(X_1/s_1, \ldots, X_k/s_k, Y_{j_1}/p_{j_1}, \ldots, Y_{j_{w_n}}/p_{j_{w_n}}) \simeq p.$$

Now the proof of the theorem is straightforward if we take into account Theorem 3.1.

Admissible functionals were introduced as a mathematical description of algorithms over $\mathfrak{A}$, which use an additional input from the data domain B. A certain way to separate amongst them those F which do not depend on B is to postulate

$$F(\bar{s},\beta) = F(\bar{s},\beta') \text{ for every } \bar{s} \in B^k \text{ and } \beta,\beta' \in B^N. \tag{4.0}$$

We shall say that F is stable if it is admissible and satisfy (4.0).

The next characterization of the stable functionals follows directly from Theorem 3.1.

Proposition 4.1. *$F : B^k \times B^N \to 2^B$ is stable iff there is a r.e. scheme Δ with variables $X_1, \ldots, X_k$ such that*

$$F(\bar{s},\beta) = \Delta_{\mathfrak{A}}(\bar{s}) \text{ for every } (\bar{s},\beta) \in B^k \times B^N.$$

Proof. If F is stable, then F is admissible and according to Theorem 3.1 it is definable by some r.e. scheme Θ with variables $X_1, \ldots, X_k, Y_0, Y_1, \ldots$ Replace in Θ each variable Y_i by X_1 and denote the scheme obtained in this way by Δ.

Now let us fix some $(s_1, \ldots, s_k)$ and define $\beta_0 \in B^N$ as $\beta_0(n) = s_1$ for every n. Clearly, $\Theta(\bar{s},\beta_0) = \Delta_{\mathfrak{A}}(\bar{s})$. We have $F(\bar{s},\beta) = F(\bar{s},\beta_0) = \Theta(\bar{s},\beta_0) = \Delta_{\mathfrak{A}}(\bar{s})$. If the right hand side of the proposition holds, then F is definable and therefore admissible. Obviously, $F(\bar{s},\beta) = F(\bar{s},\beta')$ for every β,β' in B^N, which means that F is stable.

For any stable F we define φ_F by setting

$$\varphi_F(\bar{s}) = F(\bar{s},\beta) \text{ for any } \beta \in B^N.$$

φ_F may be thought as the function, computable by F, so it is reasonable to expect that the following proposition will be true.

Proposition 4.2. *If F is stable, then φ_F is computable by means of some recursively enumerable definitional scheme* (REDS) *of Shepherdson* [9].

Proof. It is a straightforward consequence of Proposition 4.2.

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