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A NOTE ON INTERSECTION OF MODALITIES

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Тинко Тинчев. ЗАМЕТКА О ПЕРЕСЕЧЕНИЯ МОДАЛЬНОСТЕЙ

Рассматривается метод семантических таблиц для пропозициональной полимодальной логике, когда некоторые модальностей интерпретируются в моделей Крипке как теоретико-множественное пересечение других.

Tinko Tinchev. A NOTE ON INTERSECTION OF MODALITIES

In this paper a variant of the tableaux method for a polymodal language is considered, where some of the modalities are interpreted in the Kripke models as the set theoretic intersection of the interpretations of some of the remaining ones. As a consequence the completeness theorem with respect to the class of finite tree-like frames and decidability are obtained.

0. INTRODUCTION

The development of modal logic applications in theoretical computer science (viz. propositional dynamic logic) motivates many new directions of investigations in the (poly)modal logic. For example, raised in 1979 by Vakarelov, the question of axiomatizing the logic of intersection of modalities has led to enriched modal languages with names and universal modality (cf. [3, 4]). The main reason for this enrichement is to avoid the modal undefinability of intersection known from Gargov (1984, unpublished) and Van der Hoek ([5]). More precisely, there is no set Γ of formulae from the propositional modal language with three modalities $\langle R_1\rangle$, $\langle R_2\rangle$, $\langle R_3\rangle$ such that for each Kripke frame $\mathcal{F} = \langle W, R_1, R_2, R_3\rangle$, $\mathcal{F} \models \Gamma$ iff $R_3 = R_1 \cap R_2$. However, in the simplest case of tri-modal language one can axiomatize the logic of intersection — the set of modal formulae valid in all Kripke frames $\mathcal{F} = \langle W, R_1, R_2, R_3 \rangle$ with $R_3 = R_1 \cap R_2$ — joining the axiom scheme $[R_1]\varphi \vee [R_2]\varphi \Longrightarrow [R_3]\varphi$ with the minimal normal tri-modal logic. This is shown by Tinchev and Vakarelov in an unpublished manuscript (February 1986) and in [1], where the Vakarelov's copying method is used; another proof is contained in [5]. All these proofs make use of the standard canonical construction followed by some p-morphic pre-image.

Here we shall give an elementary proof — without using the axiom of choice — of the above mentioned completeness theorem, finite model property and decidability (the last two not mentioned, but also known). The semantic tableaux method — in some sense "reversed" version of that from [2, 6] — is used. The "prix" is completeness with respect to the smaller class of Kripke frames — the so-called tree-like frames — and an easily obtainable refutation system.

1. SYNTAX AND SEMANTICS

We shall consider the poly-modal language L with a denumerable set of propositional letters Φ and a set $\mathbb{O}$ of unary modal operators. The set of well-formed formulas over Φ and $\mathbb{O}$ is build up using propositional letters $p \in \Phi$ and modal operators $O \in \mathbb{O}$, as usual, according to the following inductive rules:

— every propositional letter is a formula;
— if φ_1 and φ_2 are formulas, then so are $\neg\varphi_1$ and $(\varphi_1 \,\&\, \varphi_2)$;
— if φ is a formula and O is a modal operator, then $[O]\varphi$ is a formula.

We adopt the usual abbreviations: $(\varphi \vee \psi)$, $(\varphi \Longrightarrow \psi)$, $(\varphi \Longleftrightarrow \psi)$ and $\langle O\rangle\varphi$ for $\neg(\neg\varphi \,\&\, \neg\psi)$, $(\neg\varphi \vee \psi)$, $((\varphi \Longrightarrow \psi) \,\&\, (\psi \Longrightarrow \varphi))$ and $\neg[O]\neg\varphi$, respectively.

For the rest of this paper we assume that the set of modal operators $\mathbb{O}$ is structured — there exists a set B containing at least two elements such that $\mathbb{O}$ is a set of finite subsets of B, $\varnothing \notin \mathbb{O}$ and $\{a\} \in \mathbb{O}$ for any $a \in B$. If $O = \{a_1, \ldots, a_n\}$ is a modal operator, we shall write $[a_1, \ldots, a_n]\varphi$ and $\langle a_1, \ldots, a_n\rangle\varphi$ instead of $[\{a_1, \ldots, a_n\}]\varphi$ and $\langle\{a_1, \ldots, a_n\}\rangle\varphi$, respectively.

The *modal depth* of a formula φ — the number of nested modalities in φ — we define, as usual, inductively:

$\mathrm{depth}(p) = 0$ for any propositional letter p,
$\mathrm{depth}(\neg\varphi) = \mathrm{depth}(\varphi)$, $\mathrm{depth}(\varphi_1 \,\&\, \varphi_2) = \max(\mathrm{depth}(\varphi_1), \mathrm{depth}(\varphi_2))$,
$\mathrm{depth}([O]\varphi) = \mathrm{depth}(\varphi) + 1$.

The semantics of the language L is based on the *Kripke structures* $\mathcal{F} = \langle W, R\rangle$, where $W \neq \varnothing$, $R : \mathbb{O} \to 2^{W \times W}$ and $R(O) = \bigcap_{a \in O} R(\{a\})$ for any $O \in \mathbb{O}$, which are called *frames*. A *model* $\mathfrak{M}$ *over a frame* $\mathcal{F}$ is a tuple $\langle \mathcal{F}, V\rangle$, where V is an evaluation assigning subsets of W to the propositional letters in Φ, i.e. $V : \Phi \to 2^W$. The truth conditions are

$\mathfrak{M}, x \models p$ iff $x \in V(p)$,
$\mathfrak{M}, x \models \neg\varphi$ iff $\mathfrak{M}, x \not\models \varphi$,
$\mathfrak{M}, x \models \varphi \,\&\, \psi$ iff $\mathfrak{M}, x \models \varphi$ and $\mathfrak{M}, x \models \psi$,

$\mathfrak{M}, x \models [O]\varphi$ iff $\forall y(xR(O)y \rightarrow \mathfrak{M}, y \models \varphi)$.

One can immediately verify $\mathfrak{M}, x \models \varphi \vee \psi$ iff $\mathfrak{M}, x \models \varphi$ or $\mathfrak{M}, x \models \psi$, and $\mathfrak{M}, x \models \langle O\rangle\varphi$ iff $\exists y(xR(O)y$ and $\mathfrak{M}, y \models \varphi)$.

A formula φ is *valid on* $\mathfrak{M}$, $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, x \models \varphi$ for all $x \in W$. A formula φ is *valid on* a frame $\mathfrak{F}$, $\mathfrak{F} \models \varphi$, if φ is valid on any model over $\mathfrak{F}$. If φ is valid on every frame, then it is called *valid*. A formula φ is *refutable* if $\neg\varphi$ is not valid, i.e. if there are a model $\mathfrak{M}$ and a world x in $\mathfrak{M}$ such that $\mathfrak{M}, x \models \neg\varphi$. A formula φ is called *satisfiable* if $\neg\varphi$ is refutable, i.e. if there are a model $\mathfrak{M}$ and a world x in $\mathfrak{M}$ such that $\mathfrak{M}, x \models \varphi$.

The *logic of intersection*, $K^{\cap}(\mathbb{O})$, is the set of all valid formulas.

A frame $\mathfrak{F} = \langle W, R\rangle$ is called *tree-like* if $\langle W, \bigcup_{O\in\mathbb{O}} R(O)\rangle$ is a tree, the root of this tree is called the root of $\mathfrak{F}$.

If a frame $\mathfrak{F} = \langle W, R\rangle$ is tree-like and $R^{\mathrm{ref}}(O) = R(O) \cup \{\langle x, x\rangle \mid x \in \mathrm{dom}R(O) \cup \mathrm{range}R(O)\}$ for all $O \in \mathbb{O}$, then $\langle W, R^{\mathrm{ref}}\rangle$ is called *reflexive tree-like* frame. A frame $\langle W, R^{\mathrm{tr}}\rangle$ is called *transitive tree-like* frame if there is a frame $\langle W, R\rangle$ such that $R^{\mathrm{tr}}(O)$ is the transitive closure of $R(O)$ for all $O \in \mathbb{O}$.

2. THE LOGIC OF INTERSECTION

It is well-known that the logic of intersection $K^{\cap}(\mathbb{O})$ is axiomatizible by the following

Axioms: Ax0. All (or enough) boolean tautologies;

Ax1. $[O](\varphi \Longrightarrow \psi) \Longrightarrow ([O]\varphi \Longrightarrow [O]\psi)$;

Ax2. $[O_1]\varphi \Longrightarrow [O]\varphi$, $O_1 \subseteq O$, $O, O_1 \in \mathbb{O}$; and

Rules: (MP) If $\vdash \varphi$ and $\vdash \varphi \Longrightarrow \psi$, then $\vdash \psi$;

(Nec) If $\vdash \varphi$, then $\vdash [O]\varphi$.

We say that a formula φ is in *normal form* if φ is a disjunction of *basic conjunctions*, i.e. conjunctions of the form

$$(*)\qquad \begin{aligned}&\lambda_1 p_1 \,\&\, \ldots \,\&\, \lambda_s p_s\\ &\&\, \langle O_1\rangle\varphi_1^1 \,\&\, \ldots \,\&\, \langle O_1\rangle\varphi_{n_1}^1 \,\&\, \langle O_2\rangle\varphi_1^2 \,\&\, \ldots \,\&\, \langle O_k\rangle\varphi_{n_k}^k\\ &\&\, [O_1]\psi_1 \,\&\, \ldots \,\&\, [O_l]\psi_l,\end{aligned}$$

where λ_i is $\neg$ or the empty word, and $O_1, \ldots, O_l$ are different modal operators, $l \geqq k$.

Proposition. *For any formula φ one can effectively find a formula φ' which is in normal form, $\vdash \varphi \Longleftrightarrow \varphi'$ and* $\mathrm{depth}(\chi) \leqq \mathrm{depth}(\varphi)$ *for any basic conjunction χ from φ'.*

Proof. The proof is carried out by an easy induction on the construction of φ.

Throughout we assume obvious conditions for effectiveness of $\mathbb{O}$.

Theorem 1. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a finite tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $K^{\cap}(\mathbb{O})$.*

Proof. We construct $\mathfrak{A}$ by induction on the modal depth of formulas. Let we assume that $\mathfrak{A}$ is defined for any formula θ with $\text{depth}(\theta) < \text{depth}(\varphi)$.

First of all we assume without a loss of generality that φ is in normal form. If any basic conjunction from φ contains conjunctive terms $\lambda_i p_i$ and $\lambda_j p_j$ such that one of them is the negation of the other, then we obtain a proof of $\neg\varphi$ by boolean arguments. This proof is $\mathfrak{A}(\varphi)$. Therefore we can suppose that every disjunctive term from φ has a satisfiable boolean part.

If there is a basic conjunction from φ which does not contain a term of the form $\langle O\rangle\theta$, then we set $\mathfrak{A}(\varphi)$ to be the model $\mathfrak{M}$ with $W = \{w\}$, $R(O) = \varnothing$ and $V(p) = \{w\}$ iff the occurrance of p in this term is positive, and $V(p) = \varnothing$ otherwise.

Therefore we have to consider only the case when every basic conjunction from φ contains a term of the form $\langle O\rangle\theta$.

Let χ be the basic conjunction $(*)$ and define $n_1+\cdots+n_k$ formulas χ_{ij}, $1 \leqq i \leqq k$, $1 \leqq j \leqq n_i$, in the following way: If $O_{x_1}, \ldots, O_{x_{\alpha(i)}}$ are all modal operators among $O_1, \ldots, O_l$, which are subsets of O_i, then

$$\chi_{ij} = \varphi^i_{n_j} \,\&\, \psi_{x_1} \,\&\, \ldots \,\&\, \psi_{x_{\alpha(i)}}.$$

From $\text{depth}(\chi_{ij}) < \text{depth}(\chi) \leqq \text{depth}(\varphi)$ for $1 \leqq i \leqq k$ and $1 \leqq j \leqq n_i$ we conclude that the algorithm $\mathfrak{A}$ is defined for them.

Let we consider the set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$. If one of its elements $\mathfrak{A}(\chi_{ij})$ is a proof (of $\neg\chi_{ij}$), then this proof can be extended to the proof of $\neg\chi$ in the following way:

$$\vdash \psi_{x_1} \,\&\, \ldots \,\&\, \psi_{x_{\alpha(i)}} \Longrightarrow \neg\varphi^i_{n_j},$$

$$\vdash [O_i]\psi_{x_1} \,\&\, \ldots \,\&\, [O_i]\psi_{x_{\alpha(i)}} \Longrightarrow [O_i]\neg\varphi^i_{n_j} \quad \text{(by Ax0, (Nec), Ax1, (MP), Ax0)},$$

$$\vdash [O_{x_1}]\psi_{x_1} \Longrightarrow [O_i]\psi_{x_1}, \quad \ldots, \quad \vdash [O_{x_{\alpha(i)}}]\psi_{x_{\alpha(i)}} \Longrightarrow [O_i]\psi_{x_{\alpha(i)}} \qquad \text{(by Ax2)}.$$

From here we obtain $\vdash \neg\chi$ by boolean arguments.

Case 1. Let for every disjunctive term χ of φ the corresponding set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$ contain a proof. Then we have a proof of $\neg\chi$ by the above considerations. Then one can easily obtain a proof of $\neg\varphi$ and it is $\mathfrak{A}(\varphi)$.

Case 2. Let there exist a disjunctive term χ of φ such that the corresponding set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$ does not contain a proof. Then $\mathfrak{A}(\chi_{ij})$ is a finite tree-like model $\mathfrak{M}_{ij}$, in which root w_{ij} the formula χ_{ij} is true. Let $\mathfrak{M}_{ij} = \langle \mathfrak{F}_{ij}, V_{ij}\rangle$, where $\mathfrak{F}_{ij} = \langle W_{ij}, R_{ij}\rangle$, and w_{ij} be the root of $\mathfrak{M}_{ij}$. Without a loss of generality we can suppose that the sets W_{ij} are disjoint. We are ready to define a finite tree-like model $\mathfrak{M}$ with a root w such that $\mathfrak{M}, w \vDash \varphi$, and we set $\mathfrak{A}(\varphi) = \mathfrak{M}$. Let $W' = \bigcup\{W_{ij} \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$, $w \notin W'$ and $W = W' \cup \{w\}$. We set

$$R(\{a\}) = \bigcup\{R_{ij}(\{a\}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\} \cup \{\langle w, w_{ij}\rangle \mid a \in O_i\} \quad \text{for } a \in B,$$

$$R(O) = \bigcap_{a \in O} R(\{a\}) \quad \text{for } O \in \mathbb{O},$$

$$V'(p) = \bigcup \{V_{ij}(p) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\},$$

and

$$V(p) = \begin{cases} V'(p) & \text{if } p \text{ has not a positive occurrance in the boolean part of } \chi, \\ V'(p) \cup \{w\} & \text{otherwise.} \end{cases}$$

One can immediately verify $\mathfrak{M}, w \models \varphi$.

Thus $\mathfrak{A}(\varphi)$ is defined and has the desired properties.

Theorem 1 is proven.

Corollary 1. *The logic of intersection is decidable and complete with respect to the finite tree-like frames.*

3. SOME SIMPLE EXTENSIONS OF THE LOGIC OF INTERSECTION

If we put some additional axioms to the considered in the previous section formal system, we obtain the so-called *simple extensions* of $K^{\cap}(\mathbb{O})$. For the sake of notational simplicity let assume that $\mathbb{O} = \{\{a\}, \{b\}, \{a, b\}\}$. In many cases adding axiom schemes only for the modalities $[a]$ and $[b]$, which have corresponding first order conditions, leads to the obvious modification of the tableaux used in the proof of Theorem 1. For example, we shall mention only the seriality axiom – $\langle a \rangle$**true**, $\langle b \rangle$**true**, $\langle a, b \rangle$**true**. We can add an arbitrary subset of these axioms and find an appropriate modification of the construction to obtain decidability and completeness with respect to the class of finite frames much like to the tree-like frames (some leafs must be reflexive).

A bit more complicated case is when we add axioms that guarantee the reflexivity:

(Ref^a)	$[a]\varphi \Longrightarrow \varphi,$
(Ref^b)	$[b]\varphi \Longrightarrow \varphi,$
($\mathrm{Ref}^{a,b}$)	$[a, b]\varphi \Longrightarrow \varphi.$

If we add only (Ref^a) and (Ref^b), the obtained system is incomplete as one can easily see. The reason is that the reflexivity of $R(a)$ and $R(b)$ implies the reflexivity of $R(a) \cap R(b) = R(\{a, b\})$. Let $T^{\cap}(\mathbb{O})$ be the logic $K^{\cap}(\mathbb{O}) + (\mathrm{Ref}^{a,b})$.

Theorem 2. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a finite reflexive tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $T^{\cap}(\mathbb{O})$.*

Proof. It is enough to make small modifications in the proof of Theorem 1. When the case χ does not contain a term of the form $\langle O \rangle \theta$ is considered, we take the formula which is the conjunction of the boolean part of χ and the formulas after $[O]$. And when χ_{ij} is formed, we have to consider a formula obtained as just we mentioned, write it in a normal form and so on.

More complicated case is when we add axioms for transitivity:

$$[O]\varphi \Longrightarrow [O][O]\varphi \quad \text{for any } O \in \mathbb{O}.$$

Let denote this simple extension by $K4^{\cap}(\mathbb{O})$.

Theorem 3. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a transitive tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $K4^{\cap}(\mathbb{O})$.*

Sketch of a proof. It is enough to destroy a formula in a systematic way guaranteeing transitivity. The condition for finishing and starting the construction of the model is some periodicity of the considered finite number of potential conjunctive terms.

Let we mention that we can produce finite models in the last theorem, but we lose its property to be a tree.

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