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CLASSIFICATION OF DECOMPOSABLE MANIN TRIPLES AND SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

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Тони Пантев, Васил Цанов. КЛАССИФИКАЦИЯ РАЗЛОЖИМЫХ ТРОЙНЫХ СИСТЕМ МАНИНА И РЕШЕНИЯ КЛАССИЧЕСКИХ УРАВНЕНИЙ ЯНГА-БАКСТЕРА

Получена классификация разложимых тройных систем Манина алгебр Ли над произвольной гладкой алгебраической кривой. Дается описание соответсвующих решений классических уравнений Янга-Бакстера. Приводятся примеры (и контрапримеры), выясняющие природу соответствия между общими тройными системами Манина и решениями классических уравнений Янга-Бакстера.

Tony Pantev, Vasil Tsanov. CLASSIFICATION OF DECOMPOSABLE MANIN TRIPLES AND SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

We classify decomposable triple Manin systems over an arbitrary smooth algebraic curve and describe the corresponding solutions of the classical Yang-Baxter equation. We also give some examples and counter-examples clarifying the nature of the correspondence between general triple Manin systems and solutions of the classical Yang–Baxter equation.

1. INTRODUCTION

In the present paper we discuss solutions of the classical Yang-Baxter equation

$$(1)\qquad \langle [r,r]\rangle \stackrel{\text{def}}{=} [r^{1,2}(u_1,u_2),\ r^{1,3}(u_1,u_3)] + [r^{1,3}(u_1,u_3),\ r^{2,3}(u_2,u_3)] + [r^{1,2}(u_1,u_2),\ r^{2,3}(u_2,u_3)] = 0,$$

$$r^{1,2}(u_1, u_2) = -r^{2,1}(u_2, u_1),$$

where $r(u_1,\ u_2)$ is a rational function of the cartesian square $X \times X$ of an algebraic curve X, which takes values in the tensor square of a simple finite dimensional Lie algebra $\mathfrak{p}$, and, e. g., $r^{1,3}(u_1, u_3)$ is the superposition of the functions $r : X \times X \to \mathfrak{p} \otimes \mathfrak{p}$ and $\phi^{1,3} : p \otimes \mathfrak{p} \to U(\mathfrak{p})^{\otimes 3}$, defined by $\phi^{1,3}(a \otimes b) = a \otimes 1 \otimes b$ (see, e. g., [11] for details). After the works of Drinfeld and Cherednik (see [5–8] and [11]) it is general wisdom that there is a certain correspondence between solutions of the Yang-Baxter equations and systems of relevant infinite dimensional Lie algebras called **triple Manin systems** (see the definition bellow). We treat the triple Manin systems which are related to Lie algebras of types $\mathcal{A}(X) \otimes \mathfrak{p}$ and $\mathcal{R}(X) \otimes \mathfrak{p}$, where $\mathcal{A}(X)$, $\mathcal{R}(X)$ are respectively the ring of adels and the field of rational functions on X. The study of the relevant solutions of the equation (1) is extremely important for the classification of completely integrable systems of non-linear equations representable as Lax pairs by the method of Adler-Konstant-Simms (see, e. g., [10]).

Solutions of (1) which are meromorphic functions on $\mathbb{C} \times \mathbb{C}$ of type $r(u_1 - u_2)$ are classified in [1, 2]. It is also proved in [1, 3] that any solution of (1) meromorphic on the cartesian product of two discs is equivalent (on the germ level) to a solution of type $r(u_1 - u_2)$, where the function $r(u)$ can be extended to a meromorphic function on $\mathbb{C}$. *This gives a complete classification of the local solutions of the* equation (1). However, the classification of global solutions of (1) on an arbitrary algebraic curve X is an open problem. We discuss this problem from the viewpoint of triple systems of Manin.

In Sect. 3 we define and study the natural class of **decomposable Manin triple systems** to obtain their complete classification (compare with [5], where a very close class of triple Manin systems is defined and discussed). It turns out that they produce essentially only one solution of the equation (1).

In Sect. 4 we discuss other important examples. The systems of Example 1 are known ([8]), but treated in the present scheme they produce **rational** solutions of the Yang-Baxter equation which seem to have been overlooked. These solutions are **not** of type $r(u_1 - u_2)$, but can be reduced to this type by a local (trigonometric) change of variables to obtain the well-known trigonometric solutions. Example 2 is a Manin triple which does **not correspond** to any solution of the equation (1). Example 3 is known (but Remark 3 might be interesting).

2. BASIC CONCEPTS

Let X be a smooth algebraic curve (over the field of complex numbers $\mathbb{C}$), let $\mathcal{R}(X)$, $\mathcal{A}(X)$ be respectively the field of rational functions and the ring of adels on X (for general information on adels see, e. g., [4, Ch. VII, § 2]), and let $\mathfrak{p}$ be a simple Lie algebra (finite dimensional).

Definition 1. A **triple Manin system** is an ordered triple $(A, B, \mathfrak{g} \otimes \mathfrak{h})$, where A is a Lie algebra, $\mathfrak{g}$, $h \subseteq A$ are Lie subalgebras, B is a non-degenerate, symmetric, bilinear, ad-invariant form on A, $\mathfrak{g}$ and $\mathfrak{h}$ are isotopic for B, i. e.

$$B(\xi, \eta) = 0 \quad \text{for all} \quad \xi, \eta \in \mathfrak{g} \text{ (or } \xi, \eta \in \mathfrak{h}),$$

and $A = \mathfrak{g} \oplus \mathfrak{h}$ as a linear space.

We shall discuss Manin triples for which $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathcal{R}(X) \otimes \mathfrak{p}$. There is a general method due to Cherednik (see [5, 6, 11]) to construct Manin triples of this type, which we summarize briefly here. Denote by $\mathcal{A}^+(X)$ the ring of regular adels on the curve X, let $(\ ,\ )$ denote the Killing form of the algebra $\mathfrak{p}$, let $S \subseteq X$ be a non-empty subset of X and ω be a meromorphic 1-form on X. Let $\chi_S \in \mathcal{A}(X)$ be the characteristic function of the set S. For each subalgebra $A \subseteq \mathcal{A}(X)$ we denote by A_S the subalgebra $\chi_S.A$ of $A(X)$. Let A_S denote the Lie algebra $\mathcal{A}_S(X) \otimes \mathfrak{p}$ and H_S be the bilinear form on A_S defined by

(2) $$H_S(f, g) = \sum_S \operatorname{Res}_x(f, g).\omega.$$

Obviously, H_S satisfies all the conditions required by the definition of a triple system.

The problem of constructing triple systems in this context amounts to finding a couple of H_S-isotropic subalgebras $\mathfrak{g}$, $\mathfrak{h}$ of A_S with trivial intersection such that the direct sum $\mathfrak{F} = \mathfrak{g} \oplus \mathfrak{h}$ is a subalgebra and the restriction of H_S is still non-degenerate on $\mathfrak{F}$.

Cherednik proposes that the algebra $\mathfrak{h}$ of the triple is chosen to be the subalgebra $A_S^+(X) \otimes \mathfrak{p}$, which is obviously isotropic, whence the problem reduces to finding a suitable complement $\mathfrak{g}$. It is essential (and convenient) for our purpose of looking for solutions

$$r \in \mathcal{R}(X \times X) \otimes (\mathfrak{p} \otimes \mathfrak{p})$$

of the equation (1) that the algebra $\mathfrak{g}$ be a subalgebra of $\mathcal{R}(X) \otimes \mathfrak{p}$. For each triple Manin system $(A, B, \mathfrak{g} \otimes \mathfrak{h})$ we define a map $\tilde{p} : \mathfrak{g} \to (\Lambda^2 \mathfrak{h})^*$ with the formula

(3) $$\tilde{p}(x)\,(\xi \Lambda \eta) = B(x, [\eta, \xi]).$$

A map $p : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ such that

$$B(p(x), \xi \Lambda \eta) = \tilde{p}(x)\,(\xi \Lambda \eta)$$

is called a **cocommutator** of $\mathfrak{g}$ (if it exists). One can check by a straightforward computation that p is a cocycle of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\Lambda^2 \mathfrak{g}$ and that p satisfies the equation $p_2(p(x)) = 0$ for all $x \in \mathfrak{g}$, where

$$p_2(x \Lambda y) \stackrel{\text{def}}{=} p(x) \Lambda y - x \Lambda p(y)$$

(this is exactly the adjoin of the Jacobi identity, see, e. g., [11]). If it happens that the cocycle p is a coboundary, i. e. if

(4) $$\partial(r) = p$$

for some $r \in C^0(\mathfrak{g}, \Lambda^2 \mathfrak{g}) = \Lambda^2 \mathfrak{g}$, then $\langle [r, r] \rangle$ is ad-variant, and with some luck we may expect that

$$\langle [r, r] \rangle = 0,$$

i. e. that r is a solution of the equation (1).

It is a rare occasion that the equation (4) be satisfied by some $r \in \Lambda^2 \mathfrak{g}$ (this module is "too small"). In the situation we are treating, one looks for a solution of the equation (4) (and hence of the equation (1)) as a 0-chain in the $\mathfrak{g}$-module

$$\mathcal{B}(X) = \mathcal{R}(X \times X) \otimes (\mathfrak{p} \otimes \mathfrak{p})$$

of rational functions on the Cartesian square of X with values in $\mathfrak{p} \otimes \mathfrak{p}$.

3. DECOMPOSABLE MANIN TRIPLES

From this moment on we treat only subalgebras of A_S as described above.

Definition 2. A Manin triple $(A_S, H_S, \mathfrak{g} \otimes \mathfrak{h})$ will be called **decomposable** iff

$$\mathfrak{g} = I \otimes \mathfrak{p}, \quad \mathfrak{h} = J \otimes \mathfrak{p},$$

where I, J are subalgebras of $\mathcal{R}(X)_S$, $\mathcal{A}^+(X)_S$, respectively.

Remark 1. It is easy to see that the classification of **decomposable** Manin triples amounts to the classification of triples (I, J, ω), where I, J are as above, $I \otimes J = A_S(X)$ and ω is a meromorphic differential on X such that the bilinear form

$$B(a, b) = \sum_S \mathrm{Res}_x a.b.\omega \tag{5}$$

is non-degenerate on $A_S(X)$ and vanishes identically on I and J. Indeed, as the Killing form on $\mathfrak{p}$ is non-degenerate, by suitable choice of coefficients the vanishing of (2) is reduced to the vanishing of (5).

Proposition 1. *Let (I, J, ω) be as in Definition 2. Then either*

i) $\mathbb{C} \subseteq J$ *(implying $J = \Pi_S \widehat{\mathcal{O}}_X = \mathcal{A}_S^+(X)$) or*

ii) $\mathbb{C} \subseteq I$ *(implying $J = (\Pi_{S\setminus\{y\}} \widehat{\mathcal{O}}_X) \times \mathfrak{M}_y$ for some $y \in S$), where $\mathfrak{M}_y$ is the maximal ideal of the local ring $\widehat{\mathcal{O}}_y$.*

Proof. We need two lemmas.

Lemma 1. $I \cap \mathcal{A}_S^+(X) \subseteq \mathbb{C}$.

Proof. Assume that there exists an $f \in I \cap A_S^+(X)$ which is not constant. Let $x_1, \ldots, x_n \in X$ be the poles of the differential ω and let k_i be the multiplicity of the pole x_i. The function

$$g \overset{\text{def}}{=} \Pi_i (f^2 - f(x_i) f)$$

belongs to I, because $f \in I$, and I is an algebra over $\mathbb{C}$. Obviously, $g(x_i) = 0$ for each i, but g does not vanish identically, because f is not constant. Thus one can conclude that there exists a $k \in \mathbf{N}$ such that $\mathrm{mult}_{X_i}(g^k) \geqq k_i$ for each i. For any $a \in \mathcal{A}_S^+(X)$ we have

$$B(a, g^k) = \sum_S \mathrm{Res}_x (a \cdot g^k \cdot \omega).$$

For each i $\mathrm{mult}_{X_i}(a \cdot g^k \cdot \omega) \geqq 0$, and hence $B(a, g^k) = 0$, because for each $x \in S \setminus \{x_1, \ldots, x_n\}$ we have $\mathrm{mult}_X(a \cdot g^k \cdot \omega) \geqq 0$ (as a, g and ω have no poles in $S \setminus \{x_1, \ldots, x_n\}$).

On the other hand, $g^k \in I$ and $g \neq 0$, whence there exists (Remark 1) an element $b \in J$ such that $B(b, g^k) \neq 0$, which is a contradiction. Thus we have $I \cap \mathcal{A}_S^+(X) \subseteq \mathbb{C}$.

Lemma 2. *J is an ideal of the ring $\mathcal{A}_S^+(X)$.*

Proof. Let $a \in \mathcal{A}_S^+(X)$, $b \in J$. By definition $\mathcal{A}_S(X) = I \oplus J$, so $a = f + a_1$ with unique $f \in I$ and $a_1 \in J$. But $a - a_1 = f \in \mathcal{A}_S^+(X)$, because a and a_1 are elements of $\mathcal{A}_S^+(X)$. Thus $f \in I \cap \mathcal{A}_S^+(X)$ and by Lemma 1 f is a constant. As J is an algebra over $\mathbb{C}$, we have $f.b \in J$ and $a.b = f.b + a_1.b$ is an element of J, i. e. J is an ideal of $\mathcal{A}_S^+(X)$.

Obviously, the field $\mathbb{C}$ may belong to only one of the algebras I, J.

i) Let $\mathbb{C} \subseteq J$. Then $J = \mathcal{A}_S^+(X)$, because $J \subseteq \mathcal{A}_S^+(X)$ is an ideal by Lemma 2.

ii) Let $\mathbb{C} \subseteq I$. We have $\mathcal{A}_S^+(X) \subseteq \mathcal{A}_S(X) = I \oplus J$ and by Lemma 1 we get $\mathcal{A}_S^+(X) = \mathbb{C} + J$. Because of $\mathbb{C} \subseteq I$ we have $\mathbb{C} \cap J = \{0\}$ and hence $\mathcal{A}_S^+(X) = \mathbb{C} \oplus J$. By Lemma 2 J is an ideal in $\mathcal{A}_S^+(X)$ and $\mathcal{A}_S^+(X)_{/J} \cong \mathbb{C}$, whence we conclude that J is a maximal ideal. Let $x \in S$, then $J_x \triangleleft \widehat{\mathcal{O}}_x$. But $\widehat{\mathcal{O}}_x$ is a local ring, hence we have two possibilities:

$$\text{either} \quad J_x \subseteq \mathfrak{M}_x \quad \text{or} \quad J_x = (\mathcal{A}_S^+(X))_x = \widehat{\mathcal{O}}_x.$$

If $J_x = \widehat{\mathcal{O}}_x$ for all $x \in S$, then $J = \mathcal{A}_S^+(X)$, which contradicts $\mathcal{A}_S^+(X) = \mathbb{C} \oplus J$. Thus there exists an $y \in S$ such that $J_y \subseteq \mathfrak{M}_y$, whence $J \subseteq \Pi_{S \setminus \{y\}} \widehat{\mathcal{O}}_x \times \mathfrak{M}_y \triangleleft \mathcal{A}_S^+(X)$. But J is a maximal ideal, i.e. $J = \Pi_{S \setminus \{y\}} \widehat{\mathcal{O}}_x \times \mathfrak{M}_y$, which concludes the proof of Proposition 1.

In the following theorem we keep the notation of Proposition 1.

Theorem 1. *Let (I, J, ω) be as in Definition 2. Then:*

i) *If $\mathbb{C} \subseteq I$, then there exists a function $u \in I$ which is injective on the set S with the following properties:*

a) *$\omega = dz$, where $z = u^{-1}$;*

b) *The multiplicity of z at all points of S is 1;*

c) *Let $u_x = (z - z(x))^{-1}$. The algebra I is generated by the functions u_x for all x of S.*

ii) *If $\mathbb{C} \subseteq J$, then there exists a function $u \in I$ which is injective on the set S with a pole at the point y such that:*

a) *$\omega = du$;*

b) *The multiplicity of u at all points of $S \setminus \{y\}$ is 1;*

c) *The algebra I is generated by the functions 1, u, $(u - u(x))^{-1}$ for all $x \in S \setminus \{y\}$.*

Proof. Case ii). The fact that $\mathcal{A}_S(X) = I \oplus J$ yields for each $x \in S$ the existence of a function

$$\phi_x \in I$$

which has one simple pole in S at the point x.

By Lemma 1 each function $f \in I$ has at least one pole in S. Thus each x determines ϕ_x up to a constant. Obviously, the algebra I is generated by the functions 1, ϕ_x for all $x \in S$. By Lemma 1 for a fixed $x \in S$ the function

$$(\phi_x - \phi_x(Y)) \cdot (\phi_y - \phi_y(X)) \in I$$

is a non-zero constant, whence ϕ_x is a non-degenerate Möbius transformation of ϕ_y. We set

$$u \stackrel{\text{def}}{=} \phi_y,$$

so now the algebra I is generated by

$$1,\ u,\ (u - u(x))^{-1}$$

for all $x \in S \setminus \{y\}$. We expand the differential ω at each point $x \in S$ in power series. At each point $x \in S \setminus \{y\}$ the differential ω is regular. Indeed, fix $x \in S \setminus \{y\}$ and denote by z_x the meromorphic function $u - u(x)$. The function z_x is a local parameter in a neighbourhood of the point x.

Let the Taylor expansion of ω in the parameter z_x be

$$\omega = (\alpha_k . z_x^{-k} + \cdots) . \, dz_x, \quad k \geqq 1,\ \alpha_k \neq 0.$$

Define the adels a and b by

$$a_p \stackrel{\text{def}}{=} \chi_x(p) . z_x^{k-1}, \quad b_p \stackrel{\text{def}}{=} \chi_x(p)$$

for a point $p \in S$, where $\chi_x(\cdot)$ is the characteristic function of the set $\{x\}$. As $J_x = \widehat{\mathcal{O}}_x$, we know that $a,\ b \in J$. By Remark 1 the bilinear form B is isotopic on J, i. e. $B(a, b) = \alpha_k = 0$, which is a contradiction.

To estimate the order of the pole of ω at the point y, consider the Taylor expansion of ω at y in the local parameter $z = u^{-1}$:

$$\omega = (\alpha_{-k} . z^{-k} + \cdots + \alpha_{-1} . z^{-1} + \alpha_0 + \cdots) . \, dz.$$

The adel a defined by $a_p = \chi_x(p) . z$ obviously belongs to the algebra $J = \mathfrak{M}_y \times \Pi_{S \setminus \{y\}} \widehat{\mathcal{O}}_x$. Also by Remark 1 we obtain $B(a, a^s) = 0$ for each $s \geqq 1$. Direct computation of $B(a, a^s)$ gives

$$B(a, a^s) = \alpha_{-(s+2)} = 0,$$

whence the order k of the pole of ω at y is estimated by $k \leqq 2$.

Assume that $k < 2$. Then the Taylor expansion is

$$\omega = (\alpha_0 + \cdots) . \, dz$$

as $1 \in I$, using Remark 1, we obtain

$$\alpha_{-1} = B(1, 1) = 0.$$

Let α_l be the first non-vanishing coefficient. We compute $B(u^{l+1}, 1)$ and obtain by Remark 1 (note that $1, u \in I$) that

$$\alpha_l = B(u^{l+1}, 1) = 0,$$

which is a contradiction. Thus $\alpha_{-2} \neq 0$ and normalizing u we may presume that $\alpha_{-2} = -1$. The same argument as for α_{-1} gives $\alpha_n = 0$ for all $n \geqq 0$, whence

$$\omega = -z^{-2} . \, dz = du,$$

which settles case ii).

Case i). Let $x \in S$, $\phi_x \in I$, and z_x be as above. The differential ω has no poles in S. Indeed, if the Taylor expansion of ω at some point $x \in S$ were

$$\omega = (\alpha_{-k}.z_x^{-k} + \cdots).\,dz_x \text{ with } k \geqq 1 \text{ and } \alpha_{-k} \neq 0,$$

then for the adel $a \in \mathcal{A}_S^+(X) = J$ defined by $a_y = \chi_x(y).z_x^{k-1}$ we would have $\alpha_{-k} = B(a,1) = 0$, which is a contradiction. So, for each $x \in S$ we have

$$\omega = (\alpha_0^x + \alpha_1^x.z_x + \cdots + \alpha_k^x.z_x^k + \cdots).\,dz_x.$$

But $\phi_x \in I$ and it has no pole in $S \setminus \{x\}$. Hence

$$\alpha_s^x = B(\phi_x, \phi_x^s) = 0$$

for all $s \geqq 1$. Thus in a neighbourhood of x we have

$$\omega = dz_x$$

(normalizing ϕ_x suitably). Let $y \in S$ be an arbitrary point. Obviously, the functions $\phi_x = z_x^{-1}$ $(x \in S)$ generate I. Denote $u = z_y$. The differentials ω and du are meromorphic sections of $\Omega^1(X)$, whence $f = \frac{\omega}{du}$ is a global meromorphic function on X. But ω and du coincide in an open neighbourhood of y, hence $f = 1$ and

$$\omega = du.$$

Similarly, $\omega = dz_x$ for each $x \in S$. So $dz_x = du$ for each x, i. e. $z_x = u + a_x$, where a_x is a constant. But $z_x(x) = 0$, whence $a_x = u(x)$ and the algebra I is generated by $u_x = (u - u(x))^{-1}$ $(x \in S)$.

Remark. Repeating the above arguments at each point $x \in S$, one can prove under natural restrictions that if we assume the differential ω to be adel (not a global meromorphic differential on X), no real generalization is obtained.

Corollary 1. *If $\mathfrak{F} = (A_S, H_S, \mathfrak{g} \oplus \mathfrak{h})$ is a decomposable Manin triple on a curve X, then there exists a decomposable Manin triple $\mathfrak{F}' = (A', H', \mathfrak{g}' \oplus \mathfrak{h}')$ on $\mathbb{P}^1$ such that $\mathfrak{F}$ is the pullback of $\mathfrak{F}'$ by the function $u : X \to \mathbb{P}^1$ defined in Theorem* 1.

Proof. In the notations of Theorem 1 we have: the algebra I is a direct sum $I = \bigoplus_S u_x.(\mathbb{C}.u_x)$ (as a linear space) in the case i), and $I = \mathbb{C} \oplus \mathbb{C}.u \oplus (\bigoplus_{S\setminus\{y\}} \mathbb{C}.(u - u(x))^{-1})$ in the case ii). This remark makes the corollary obvious.

The Killing form on $\mathfrak{p}$ is represented by a symmetric, non-degenerate, ad-invariant element $k \in \mathfrak{p}^* \otimes \mathfrak{p}^*$. Let K be the element of $\mathrm{Hom}(\mathfrak{p}, \mathfrak{p}^*)$, canonically corresponding to k. If $K^{-1} \in \mathrm{Hom}(\mathfrak{p}^*, \mathfrak{p})$ is the inverse linear map of K, then there is a non-degenerate, symmetric, ad-invariant element $k^{-1} \in \mathfrak{p} \otimes \mathfrak{p}$, canonically corresponding to K^{-1}. For an arbitrary function $f \in \mathcal{R}(X)$ we denote by r_f the $\mathfrak{p} \otimes \mathfrak{p}$ valued function on $X \times X$ defined by the formula

$$r_f(x,y) = (f(x) - f(y))^{-1}.k^{-1}. \tag{6}$$

Let ∂ be the differential of the cochain complex $C^*(\mathfrak{g}, \mathcal{B})$. In the following theorem we describe the cocommutators $p : \mathfrak{g} \to \Lambda^2\,\mathfrak{g}$ of the Lie algebras appearing in the current context.

Theorem 2. *Let $(A_S, H_S, \mathfrak{g}\oplus\mathfrak{h})$ be a decomposable Manin triple, and let (I, J, ω) be the corresponding triple as in Remark* 1. *Then:*

i) *In the case* i) *of Theorem* 1 *we have $p(x) = \partial(r_z)(x)$ for all $x \in \mathfrak{g}$;*

ii) *In the case* ii) *of Theorem* 1 *we have $p(x) = -\partial(r_u)(x)$ for all $x \in \mathfrak{g}$, where $u,\ z \in \mathcal{R}(X)$ are determined in Theorem* 1.

Proof. We shall prove only the case i). The case ii) can be proved similarly.

We have $I = \bigoplus_S u_x.(\mathbb{C}.u_x)$, hence the functions of the set

$$\{u_x^k \otimes a \,|\, k \geqq 1,\ x \in S,\ a \in \mathfrak{p}\}$$

span $\mathfrak{g}$ as a linear space. Denote by $a.z_x^1$ the adel $(a.z_x^1)_y = \delta_y(x).(a\otimes z_x^1)$. It is obvious that

$$a.z_x^1 \in \mathcal{A}_S^+(X)\otimes\mathfrak{p} = \mathfrak{h}.$$

The form H_S is a non-degenerate on $\mathcal{A}_S(X)$, so it is sufficient to prove that

$$H_S(p(u_x^k\otimes a),\ b.z_x^n\otimes c.z_x^m) = H_S(\partial(r_z)(u_x^k\otimes a),\ b.z_x^n\otimes c.z_x^m) \ \text{ for each } x\in S,$$

$a, b, c \in \mathfrak{p}$, $k \geqq 1$, $n, m \geqq 0$. Let us define for each $x \in S$ the functions v_x, w_x, α_x, $\beta_x \in M(X\times X)$:

$$\begin{array}{ll} v_x : X\times X \to \mathbb{C} & w_x : X\times X\to\mathbb{C} \\ (p,q)\to u_x(p) & (p,q)\to u_x(q) \\ \alpha_x : X\times X\to\mathbb{C} & \beta_x : X\times X\to\mathbb{C} \\ (p,q)\to z_x(p) & (p,q)\to z_x(q). \end{array}$$

In this notations we have $r_z = (\alpha-\beta).k^{-1}$ as a meromorphic function on $X\times X$ and if $k^{-1} = \sum_i a_i\otimes b_i \in \mathfrak{p}\otimes\mathfrak{p}$, then we compute $\partial(r_z)(u_x^k\otimes a)$ as follows:

$$\partial(r_z)(u_x^k\otimes a) = \sum_i \partial((\alpha-\beta)^{-1}.a_i\otimes b_i)(u_x^k\otimes a)$$

$$= \sum_i (v_x^k.(\alpha-\beta)^{-1}.\mathrm{ad}_a a_i\otimes b_i + w_x^k.(\alpha-\beta)^{-1}.a_i\otimes b_i).$$

But k^{-1} is an ad-invariant element of $\mathfrak{p}\otimes\mathfrak{p}$, whence $\mathrm{ad}_a k^{-1} = 0$, and whence $\sum_i \mathrm{ad}_a a_i\otimes b_i = -\sum_i a_i\otimes\mathrm{ad}_a b_i$.

Thus

$$\partial(r_z)(u_x^k\otimes a) = (v_x^k - w_x^k).(\alpha-\beta)^{-1}.\sum_i \mathrm{ad}_a a_i\otimes b_i$$

$$= (v_x - w_x).(v_x^{k-1}+\cdots+w_x^{k-1}).(\alpha-\beta)^{-1}.\sum_i \mathrm{ad}_a a_i\otimes b_i.$$

By definition we have $\alpha = v^{-1}$, $\beta = w^{-1}$, and $z_x = z - z(x)$, hence $u.u_x^{-1} = (z-z(x)).z^{-1} = 1 - z(x).u$, and

$$(v_x - w_x).(\alpha-\beta)^{-1} = (v_x-w_x).(w-v)^{-1}.wv = -v_xw_x.vw.(vw)^{-1} = -v_xw_x.$$

Finally, we obtain

$$\partial(r_z)(u_x^k\otimes a) = -(v_x^k w_x + \cdots + w_x^k v_x).\sum_i \mathrm{ad}_a a_i\otimes b_i.$$

Let us compute

$$H_S(\partial(r_z)(u_x^k\otimes a),\ b.z_x^n\Lambda\, c.z_x^m) = -H_S(\sum_{i,s} v_x^s w_x^{k+1-s}.\mathrm{ad}_a a_i\otimes b_i,\ b.z_x^n\Lambda\, c.z_x^m)$$

$$= -\frac{1}{2}.\sum_{i,\ s}(\mathrm{Res}_x(z_x^{n-s}.dz_x).\mathrm{Res}_x(z_x^{m-(k+1)+s}.dz_x)$$

$$\times(\mathrm{ad}_a a_i, b).(b_i, c) - \mathrm{Res}_x(z_x^{m-s}.dz_x).\mathrm{Res}_x(z_x^{n-(k+1)+s}.dz_x)\times(\mathrm{ad}_a a_i, c).(b_i, b))$$

$$= -\frac{1}{2}.\sum_{i,\ s}(\delta_{n,\ s+1}.\delta_{m,\ k+2-s}.(\mathrm{ad}_a a_i, b).(b_i, c) - \delta_{m,\ s+1}.\delta_{n,\ k+2-s}.(\mathrm{ad}_a a_i, c).(b_i, b))$$

$$= -\frac{1}{2}\delta_{n+m,k+1}.\sum_i((\mathrm{ad}_a a_i, b).(b_i, c) - (\mathrm{ad}_a a_i, c).(b_i, b)).$$

Using that $\mathrm{ad}_a k^{-1} = 0,\ \forall a\in\mathfrak{p}$, we write

$$H_S(\partial(r_z)(u_x^k\otimes a), b.z_x^n\Lambda\, c.z_x^m)$$
$$= -\frac{1}{2}.\delta_{n+m,k+1}.\sum_i((\mathrm{ad}_a a_i, b).(b_i, c) + (a_i, c).(\mathrm{ad}_a b_i, b)).$$

As k^{-1} is a symmetric tensor and the Killing form (,) is an ad-invariant, we can write the identities

$$-\sum_i((\mathrm{ad}_a a_i, b).(b_i, c) + (a_i, c).(\mathrm{ad}_a b_i, b))$$
$$= \sum_i((a_i, \mathrm{ad}_a b).(b_i, c) + (a_i, c).(b_i, \mathrm{ad}_a b))$$
$$= 2.(k^{-1}, \mathrm{ad}_a b\otimes c).$$

In such a way we obtain

$$H_S(\partial(r_z)(u_x^k\otimes a), b.z_x^n\Lambda\, c.z_x^m) = \delta_{n+m,k+1}.(k^{-1}, \mathrm{ad}_a b\otimes c).$$

For $p(u_x^k\otimes a)$ we have by definition

$$H_S(p(u_x^k\otimes a),\ b.z_x^n\Lambda\, c.z_x^m) = H_S([u_x^k\otimes a, b.z_x^n], c.z_x^m) = H_S(z_x^{n-k}\otimes[a,b], c.z_x^m)$$
$$= \delta_{n+m,k+1}.(\mathrm{ad}_a b, c).$$

But $(x\otimes y, a\otimes b)\overset{\mathrm{def}}{=}(x,a).(y,b) = k\,(x,a).k\,(y,b)$, consequently

$$(k^{-1},\ \mathrm{ad}_a b\otimes c) = \langle k^{-1}\llcorner(k\llcorner\mathrm{ad}_a b),\ K(c)\rangle$$
$$= \langle K^{-1}\circ K(\mathrm{ad}_a b), K(c)\rangle = \langle\mathrm{ad}_a b,\ K(c)\rangle = k\,(\mathrm{ad}_a b, c) = (\mathrm{ad}_a b, c)$$

(here $\langle\ ,\ \rangle$ is the natural pairing of $\mathfrak{p}$ and $\mathfrak{p}^*$).
Thus

$$H_S(\partial(r_z)(u_x^k\otimes a), b.z_x^n\Lambda\, c.z_x^m) = \delta_{n+m,k+1}.(k^{-1},\ \mathrm{ad}_a b\otimes c)$$
$$= \delta_{n+m,k+1}.(\mathrm{ad}_a b, c) = H_S(p(u_x^k\otimes a),\ b.z_x^n\Lambda\, c.z_x^m),$$

i. e. $p(x) = \partial(r_z)(x)$ for each $x\in\mathfrak{g}$.

It is obvious that any two solutions of the equation (4) with a fixed cocommutator p differ by an element c of the module $\mathcal{B}(X)$ such that $\partial(c) = 0$, i. e. by elements of $H^0(\mathfrak{g}, \mathcal{B})$. Thus the problem of uniqueness of solutions for the equation (4) is solved by the next proposition.

Proposition 2. *In the notations of Theorem* 1 *we have*

$$H^0(\mathfrak{p}, \mathcal{B}(X)) = 0. \tag{7}$$

Proof. Let $\mathfrak{p}_0$ be a Cartan subalgebra of $\mathfrak{p}$ and let $\Pi = \{\alpha_1, \ldots, \alpha_\Gamma\}$ be a fixed system of simple roots for $\mathfrak{p}_0$. Let $\{e_\alpha^-\}_{\alpha\in\Delta^-}$, $\{e_i^0\}_{i=1,\ldots,\Gamma}$, $\{e_\beta^+\}_{\beta\in\Delta^+}$ be the corresponding Weyl base. We know that it satisfies the following relations:

$$(*)\qquad \begin{gathered}
[e_i^0, e_\alpha^-] = \alpha(e_i^0).e_\alpha^-, \quad \forall\alpha\in\Delta^-,\\
[e_i^0, e_\mu^+] = \mu(e_i^0).e_\mu^+, \quad \forall\mu\in\Delta^+,\\
[e_i^0, e_j^0] = 0, \quad \forall i, j = 1, \ldots, r,\\
[e_{-\alpha}^-, e_\alpha^+] = \sum_i \alpha(e_i^0).e_i^0, \quad \forall\alpha\in\Delta^+,\\
[e_\alpha^\pm, e_\beta^\pm] = \begin{cases} N_{\alpha\beta}.e_{\alpha+\beta}^\pm, & \alpha+\beta\neq 0\in\Delta,\\ 0, & \alpha+\beta\neq 0\in\Delta.\end{cases}
\end{gathered}$$

Hence the base of $\mathcal{B}(X)$ over the field $\mathcal{R}(X\times X)$ is

$$\begin{matrix}
e_\alpha^-\otimes e_\beta^-, & e_\alpha^-\otimes e_i^0, & e_\alpha^-\otimes e_\mu^+,\\
e_i^0\otimes e_\alpha^-, & e_i^0\otimes e_j^0, & e_i^0\otimes e_\mu^+,\\
e_\lambda^+\otimes e_\alpha^-, & e_\lambda^+\otimes e_j^0, & e_\lambda^+\otimes e_\mu^+
\end{matrix}$$

for each α, $\beta\in\Delta^-$, $i, j = 1, \ldots, r$, λ, $\mu\in\Delta^+$.

Let $A\in\mathcal{B}(X)$ be an arbitrary element and $f_{\alpha\beta}^{--}, \ldots, f_{\lambda\mu}^{++}\in\mathcal{R}(X\times X)$ be the coefficients of A of the corresponding base element (e. g., $e_\alpha^-\otimes e_\beta^-$ and so on). Assume that $A\in Z^0(\mathfrak{g}, \mathcal{B}(X)) = H^0(\mathfrak{g}, \mathcal{B}(X))\subseteq\mathcal{B}(X)$. Let $u\in\mathfrak{g}$ be non-constant (there exists such an element u by Theorem 1). We have

$$\partial(A) = 0, \quad \partial(A)(u^k\otimes a) = 0.$$

By the relations $(*)$ we obtain for the coefficients of $\partial(A)(u^k\otimes e_t^0)$ the following expressions:

— for the coefficient in front of $e_\alpha^-\otimes e_\beta^-$:

$$\tilde{f}_{\alpha\beta}^{--}(p,q) = \alpha(e_t^0).f_{\alpha\beta}^{--}(p,q).u^k(p) + \beta(e_t^0).f_{\alpha\beta}^{--}(p,q).u^k(q), \quad \forall\alpha,\beta\in\Delta^-;$$

— for the coefficient in front of $e_\alpha^-\otimes e_i^0$:

$$\tilde{f}_{\alpha i}^{-0}(p,q) = \alpha(e_t^0).f_{\alpha i}^{-0}(p,q).u^k(p), \quad \forall\alpha\in\Delta^-,\ \forall i = 1, \ldots, r;$$

— the coefficient in front of $e_i^0\otimes e_j^0$ vanishes identically, and so on.

If $\partial(A) = 0$, then all the coefficients of $\partial(A)(u^k\otimes e_t^0)$ vanish identically on $X\times X$. Consider for instance $\alpha(e_t^0).u^k(p).f_{\alpha\beta}^{-0}(p,q) = 0$ for each $k\geqq 1$, $t = 1, \ldots, r$.

For each α, $\exists t : \alpha(e_t^0) \neq 0$ (because α is a non-zero root of $\mathfrak{p}_0$ and $e_1^0, \dots, e_r^0$ are a base of $\mathfrak{p}_0$). Then for a suitable t we have $u(p).f_{\alpha i}^{-0}(p,q) = 0$, $\forall (p,q) \in X \times X$, i. e. $f_{\alpha i}^{-0} \equiv 0$. Similarly, $f_{i\beta}^{0-} = f_{i\mu}^{0+} = f_{\lambda i}^{+0} = 0$, $\forall i = 1, \dots, r$; $\forall \alpha \in \Delta^-$; $\forall \lambda, \mu \in \Delta^+$. Also $f_{\alpha\beta}^{--}(p,q).(\alpha(e_t^0).u^k(p) + \beta(e_t^0).u^k(q)) = 0$, $\forall k \geqq 1$, $t = 1, \dots, r$; $\forall (p,q) \in X \times X$.

Let

$$D(u) = \{(p,q) \in X \times X \mid u(p) = u(q)\} \cup \operatorname{supp}(\operatorname{div}_{X \times X}(u)).$$

If we assume that $f_{\alpha\beta}^{--}(p,q) \neq 0$ for some point, then $f_{\alpha\beta}^{--}(p,q) \neq 0$ in an open subset of $X \times X$. The set $D(u)$ is closed, hence $f_{\alpha\beta}^{--}(p,q) \neq 0$ in an open subset of $X \times X \setminus D(u)$. Setting $k = 1, 2$, we obtain for each (p,q) in an open subset of $X \times X \setminus D(u)$:

$$\left\| \begin{array}{l} \alpha(e_t^0).u(p) + \beta(e_t^0).u(q) = 0, \\ \alpha(e_t^0).u^2(p) + \beta(e_t^0).u^2(q) = 0, \quad \forall t = 1, \dots, r. \end{array} \right.$$

The determinant of this linear system is $u(p).u(q).(u(p) - u(q))$ and it does not vanish for $(p,q) \in X \times X \setminus D(u)$. Hence this system has only the zero solution, i. e. $\alpha(e_t^0) = 0$, $\forall t = 1, \dots, r$, which is a contradiction, because α is a non-zero root and $e_1^0, \dots, e_r^0$ are a base of $\mathfrak{p}_0$. So we have $f_{\alpha\beta}^{--} = f_{\alpha\beta}^{-+} = f_{\lambda\beta}^{+-} = f_{\lambda\mu}^{++} = 0$. The above argument implies that for $A \in \operatorname{Ker}\partial$ we have

$$A = \sum_{i,j} e_i^0 \otimes e_j^0 \otimes f_{ij}^{00}.$$

Consequently, for $\alpha \in \Delta^-$, using the fact that $\partial(A)(u^k \otimes e_\alpha^-) = 0$, $\forall k \geqq 1$, we derive $\alpha(e_i^0).f_{ij}^{00}(p,q).u^k(p) = 0$ for each $i, j = 1, \dots, r$, $\alpha \in \Delta^-$ and $k \geqq 1$.

But for each $i = 1, \dots, r$ there exists $\alpha \in \Delta^-$ such that $\alpha(e_i^0) \neq 0$, hence setting $k = 1$ we obtain $u(p).f_{ij}^{00}(p,q) = 0$, $\forall (p,q) \in X \times X$, i. e. $f_{ij}^{00} = 0$.

It turns out that for all triple systems described in Theorem 1 there exists a solution r_u of the equation (4) which coincides (up to pullback from $\mathbb{P}^1$ to the curve X) with the well-known "rational" solution (6) of the classical Yang-Baxter equation (compare with [8, 11]).

Remark 2. It is easy to construct Manin triple systems on an arbitrary curve X, which are "subtriples" of those described in Theorem 1 and which determine the respective curve X (by taking the intersections of A_S and $\mathfrak{h}$ with $\mathcal{R}(X) \otimes \mathfrak{p}$). However, one can check that we get no essentially new solutions of the Yang-Baxter equation in this way.

4. OTHER EXAMPLES

Apart from the solution (6) there are two well-known classes of solutions of the Yang-Baxter equation — the elliptic and the trigonometric solutions, described, e. g., in [1, 2, 8, 10]. While (as known) the elliptic solutions are directly obtained from (**non-decomposable**) triple Manin systems which we shall describe briefly later, we show here that the trigonometric solutions are obtained from rational solutions of the equation (1) (and (4)) corresponding to suitable triple Manin systems, which we proceed to describe.

Example 1. Let $X = \mathbb{P}^1$, $S = \{0, \infty\}$, and let t be the co-ordinate function on $\mathbb{P}^1$. Let $\mathfrak{u}(\mathbb{P}^1)$ be the ring of polynomial adels on $\mathbb{P}^1$. Denote $A = \mathfrak{u}(\mathbb{P}^1)_S \otimes \mathfrak{p}$. One may interpret A as the cartesian square $L(\mathfrak{p}) \times L(\mathfrak{p})$, where

$$L(\mathfrak{p}) \stackrel{\text{def}}{=} \mathbb{C}[t, t^{-1}] \otimes \mathfrak{p}$$

is the non-twisted affine (Kac-Moody without central extensions) Lie algebra of $\mathfrak{p}$ (see, e. g., [9]). Indeed,

$$A = \mathfrak{u}(\mathbb{P}^1)_S \otimes \mathfrak{p} = (\mathbb{C}[t, t^{-1}])_{\{0\}} \otimes \mathfrak{p} \times (\mathbb{C}[t, t^{-1}])_{\{\infty\}} \otimes \mathfrak{p}.$$

Thus the first factor $L(\mathfrak{p})$ is the polynomial algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{p}$ interpreted as a subalgebra of the adels at the point $0 \in \mathbb{P}^1$, likewise the second factor at ∞. Denote

$$\mathfrak{g} = (\mathbb{C}[t, t^{-1}])_S \otimes \mathfrak{p} \cong L(\mathfrak{p}).$$

We choose a Cartan decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_+,$$

and hence of $L(\mathfrak{p})$:

$$L(\mathfrak{p}) = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{n}}_0 \oplus \tilde{\mathfrak{n}}_+,$$

where

$$\tilde{\mathfrak{n}}_- = \mathfrak{n}_- \oplus (t^{-1}.\mathbb{C}[t^{-1}] \oplus \mathfrak{p}), \quad \tilde{\mathfrak{n}}_0 = \mathfrak{n}_0, \quad \tilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (t.\mathbb{C}[t] \oplus \mathfrak{p})$$

(see [9, p. 78]. For each $a \in A$ denote by a^0 the projection of $(a)_0$ on $\tilde{\mathfrak{n}}_0$ and with a^∞ the projection of $(a)_\infty$ on $\tilde{\mathfrak{n}}_0$ (note that $A_{\{0\}} = A_{\{\infty\}} = L(\mathfrak{p})$). Define

$$\mathfrak{h} = \{a \in (\tilde{\mathfrak{b}}_-)_{\{0\}} \times (\tilde{\mathfrak{b}}_+)_{\{\infty\}} \mid a^0 + a^\infty = 0\}.$$

Let ω be the meromorphic differential $t^{-1}.dt$ on $\mathbb{P}^1$. Then $(\mathcal{A}, H_S, \mathfrak{g} \otimes \mathfrak{h})$ is a triple Manin system. The **cocommutator** p of $\mathfrak{g}$ is described as follows:

$$p(H_i) = 0,$$
$$p(X_i^{\pm}) = \pm X_i^{\pm} \wedge H_i,$$

where $X_i^{\pm}$, H_i are the canonic generators of the Kac-Moody algebra $\mathfrak{g} \cong L(\mathfrak{p})$ (see [11]). A straightforward computation gives an element $r \in \mathcal{B}(\mathbb{P}^1)$ such that r is a solution to both equations (1) and (4). Thus we get a **rational** solution of the classical Yang-Baxter equation, which corresponds to the Manin triple $(\mathcal{A}, H_S, \mathfrak{g} \oplus \mathfrak{h})$. If we substitute $\exp(u)$ for the co-ordinate function t, we obtain the so-called trigonometric solution of the classical Yang-Baxter equation, which is studied in detail in [1, 2].

The solution r for the case $\mathfrak{p} = \mathfrak{sl}(2, \mathbb{C})$ is the following:

$$r = (t - s)^{-1}.(t.e_- \otimes e_+ + \frac{1}{4}(t + s).e_0 \otimes e_0 + s.e \otimes e_-),$$

where t, s are co-ordinate functions of the first and the second copy of $\mathbb{P}^1$ in $\mathcal{B}(\mathbb{P}^1) = \mathcal{R}(\mathbb{P}^1 \times \mathbb{P}^1) \otimes (\mathfrak{p} \otimes \mathfrak{p})$, and $\{e_-, e_0, e_+)$ is the cànonical Cartan-Wayl basis of the algebra $\mathfrak{p} = \mathfrak{sl}(2, \mathbb{C})$.

Example 2. We preserve the notations of Example 1. Define

$$\tilde{\mathfrak{h}} = \{a \in (b_- \otimes \mathbb{C}[t, t^{-1}])_{\{0\}} \times (b_+ \otimes \mathbb{C}[t, t^{-1}])_{\{\infty\}} \mid$$

$$\pi_0((a)_0) + \pi_0((a)_\infty) = 0\},$$

where π_0 is the projection on the subalgebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{n}_0 \subseteq L(\mathfrak{p})$. Again $(A, H_S, \mathfrak{g} \oplus \tilde{\mathfrak{h}})$ is a triple Manin system, but this time there exists **no** cocommutator of the algebra $\mathfrak{g}$ determined by it. More explicitly, the functional defined by formula (3) is an infinite series and does not correspond to any function $p : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$. Presumably, this pathology is due to the choice of the "bad" definition of the "Cartan subalgebra" of $L(\mathfrak{p})$, contrary to the "good" definition in Example 1. Observe that this drastic change in the cocommutator situation of $\mathfrak{g}$ is achieved by changing only the "$\mathfrak{h}$" part of the triple system.

Example 3. For completeness we include a brief description of the triple Manin system generating the elliptic solutions of the classical Yang-Baxter equation. For more details see [1, 10]. Let I_1, I_2 be the internal automorphisms of $\mathfrak{sl}(n, \mathbb{C})$, defined by the matrices

$$M_1 = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & \dots & . \\ 1 & 0 & \dots & 1 \end{Vmatrix}; \quad M_2 = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{Vmatrix},$$

where ε is a primitive n-th root of unity. Let $X = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau\}$ be an elliptic curve. We choose $S = \{0\}$; $\omega = dz$ (z being the co-ordinate on $\mathbb{C}$); $\mathfrak{p} = \mathfrak{sl}(n, \mathbb{C})$. Define

$$A_S = \mathcal{A}_S(X) \otimes \mathfrak{p}, \quad \mathfrak{h} = \mathcal{A}_S^+(X) = \mathbb{C}[z] \otimes \mathfrak{p},$$

$$\mathfrak{g} = \{f \in \mathcal{R}(X) \otimes \mathfrak{p} \mid f\left(z + \frac{k + l.\tau}{n}\right) = I_1^k(I_2^l(f(z))),$$

and f has no poles outside the set $\left\{\frac{k + l.\tau}{n}\right\}_S$.

The well-known elliptic solutions of Belavin (see, e. g., [10]) correspond to the cocommutator of the algebra $\mathfrak{g}$ determined by the Manin triple $(A_S, H_S, \mathfrak{g} \oplus \mathfrak{h})$.

Remark 3. The elliptic solutions of Belavin are the only essentially non-rational solutions known (to us). It is curious to observe that the triple of Example 3 generates also a (local) solution $\tilde{r}$ of the equations (1) and (4) determined by the cocommutator of the algebra $\mathfrak{h}$. As a function of the local parameter z, $\tilde{r}$ coincides exactly with the "rational" solution r_z (see formula (6)).

Remark 4. One can check that the analog of Proposition 2 is valid for all examples treated above.

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