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A BORSUK–ULAM TYPE THEOREM FOR $\mathbb{Z}_4$-ACTIONS

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Симеон Стефанов. ОДНА ТЕОРЕМА ТИПА ТЕОРЕМЫ БОРСУКА-УЛАМА ДЛЯ $\mathbb{Z}_4$-ДЕЙСТВИЙ

Пусть $n = 2k + 1$ и сфера S^n представлена в виде

$$S^n = \left\{ z = (z_1, \dots, z_{k+1}) \in \mathbb{C}^{k+1} \,|\, \|z\| = 1 \right\}.$$

Рассмотрим каноническое действие группы $\mathbb{Z}_4 = \{1, i, -1, -i\}$ в S^n, определенное умножением. Основным результатом работы является следующая теорема типа теоремы Борсука–Улама:

Для каждой непрерывной функции $f : S^n \to \mathbb{R}^1$ рассмотрим множество

$$A(f) = \{z \in S^n \,|\, f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Тогда $\dim A(f) \geqq n - 3$.

Основное следствие: Для каждой непрерывной функции $f : S^3 \to \mathbb{R}^1$ существует $z \in S^3$ такое, что

$$f(z) = f(iz) = f(-z) = f(-iz).$$

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Let $n = 2k + 1$ and the sphere S^n be represented as

$$S^n = \left\{ z = (z_1, \dots, z_{k+1}) \in \mathbb{C}^{k+1} \,|\, \|z\| = 1 \right\}.$$

Consider the canonical action of the group $\mathbb{Z}_4 = \{1, i, -1, -i\}$ in S^n defined by multiplication. The main result in the article is the following Borsuk-Ulam type theorem:

For any continuous function $f : S^n \to \mathbb{R}^1$ consider the set

$$A(f) = \{z \in S^n \,|\, f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Then $\dim A(f) \geqq n - 3$.

The main corollary: For any continuous function $f : S^3 \to \mathbb{R}^1$ there exists $z \in S^3$ such that

$$f(z) = f(iz) = f(-z) = f(-iz).$$

There exist various generalizations of the classical Borsuk-Ulam theorem, where the antipodal map in the n-sphere S^n is replaced by a free $\mathbb{Z}_p$-action for p prime. Then some coincidence point theorems are obtained ([3–6, 10], etc.). The case of a composite p is more complicated and no result of this kind is known to us. The purpose of this note is to get some Borsuk-Ulam type result for free $\mathbb{Z}_4$-actions in S^n (n is odd). More precisely, we consider the canonical $\mathbb{Z}_4$-action on S^n defined by $z \to iz$ and prove that for any continuous function $f : S^n \to \mathbb{R}^1$ the covering dimension of the set

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}$$

is $\geqq n-3$. (The proof works for arbitrary $\mathbb{Z}_4$-actions). In particular, for any $f : S^3 \to \mathbb{R}^1$ there exists $z_0 \in S^3$ such that $f(z_0) = f(iz_0) = f(-z_0) = f(-iz_0)$. It is easy to see that the estimate $\dim A(f) \geqq n-3$ cannot be strengthened in general, since the set $A(f)$ is defined by 3 equalities.

1. PRELIMINARIES

Recall first some basic definitions. Let G be a group. A G-action in X is a continuous map $\mu : G \times X \to X$, $\mu(g, x) = gx$ such that: i) $1x = x$ and ii) $g_1(g_2x) = (g_1g_2)x$. Then X is called G-space. The subset $A \subset X$ is invariant if $gA = A$ for any $g \in G$. The orbit of a point $x \in X$ is the set orbit $x = \{gx \mid g \in G\}$. The orbit space is the factor $\widetilde{X} = X/\sim$, where $x \sim y$ iff $x \in$ orbit y.

A G-action is said to be free if $gx \neq x$ for any $g \neq 1$, $x \in X$. Let X and Y be G-spaces. A map $\varphi : X \to Y$ is equivariant if $\varphi(gx) = g\,\varphi(x)$. The join of X and Y is the factor $X * Y = X \times Y \times [0, 1]/\sim$, where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for any $x, x' \in X$, $y, y' \in Y$. As usual, we write

$$X * Y = \{(t_1x, t_2y) \mid t_1 + t_2 = 1;\ t_1, t_2 \geqq 0;\ x \in X,\ y \in Y\}.$$

Note that if X and Y are G-spaces, their join is also a G-space with respect to the action

$$g(t_1x, t_2y) = (t_1gx, t_2gy).$$

Proposition 1. *Let the (metric) G-space X be a sum of two closed invariant subsets $X = X_1 \cup X_2$ and there exist equivariant maps $\varphi_i : X_i \to K_i$, $i = 1, 2$ into some polyhedra K_i. Then there exists an equivariant map $\varphi : X \to K_1 * K_2$.*

Proof. Since K_i are equivariant ANR's, there exist equivariant extensions $\widetilde{\varphi}_i : U_i \to K_i$ for some invariant open neighbourhoods $U_i \supset X_i$. Take

$$\lambda_i(x) = \frac{\operatorname{dist}(x, X \setminus U_i)}{\operatorname{dist}(x, X \setminus U_1) + \operatorname{dist}(x, X \setminus U_2)}.$$

Then $\varphi(x) = (\lambda_1(x)\,\widetilde{\varphi}_1(x), \lambda_2(x)\,\widetilde{\varphi}_2(x))$ is an equivariant map $\varphi : X \to K_1 * K_2$.

We shall define and list some properties of the so-called "B–index" introduced for a space with a fixed point free involution (free $\mathbb{Z}_2$-action) by C. T. Yang [11].

Definition. Let X be a compact space with a fixed point free involution $T : X \to X$. We say that the B-index of X (with respect to T) is not greater than n if there exists a map $\varphi : X \to S^n$ such that

$$\varphi(Tx) = -\varphi(x)$$

for any $x \in X$. Then we write $B(X;\, T) \leqq n$. The equality $B(X;\, T) = n$ means that $B(X;\, T) \leqq n$ and $B(X;\, T) \nleqq n-1$.

We shall make use only of the following properties of this index:

i) $B(X;T) \geqq 1$ iff X contains an invariant (with respect to T) continuum;

ii) $B(S^n;T) = n$ for any fixed point free involution T in S^n;

iii) Let U be an open connected T_0-invariant subset of S^n. Then

$$B(S^n \setminus U; T_0) \leqq n-2,$$

where $T_0(x) = -x$.

The properties i) and ii) (and many others) may be found for example in [8, 9, 11]; iii) follows from a theorem of J. W. Jaworowski [2].

We shall give now a definition, which is important for the following.

Definition. Let X be a G-space. A closed invariant subset $F \subset X$ is said to be *equivariant partition* in X if for any $x \in X$, $g \neq 1$, the points x, gx lie in different components of $X \setminus F$.

It is easy to see that if A is an invariant closed subset of X, then every equivariant partition in A may be extended to an equivariant partition in X.

Proposition 2. *Let X be a compact space with free $\mathbb{Z}_4$-action and F be a closed invariant subset with* $\dim F \leqq k$. *Then there exist equivariant partitions in X $\Phi_1, \ldots, \Phi_{k+1}$, such that*

$$\left(\bigcap_{i=1}^{k+1} \Phi_i\right) \cap F = \varnothing.$$

Proof. Apply an induction on k. For $k=0$ take some sufficiently small (finite) invariant covering ω of X with open sets U such that $F \cap Fr\,U = \varnothing$. Then $\Phi_1 = \cup\{Fr\,U \mid U \in \omega\}$ is an equivariant partition in X and $\Phi_1 \cap F = \varnothing$.

Suppose the proposition is valid for $k-1$ and $\dim F \leqq k$. There exists in F an equivariant partition F_0 with $\dim F_0 \leqq k-1$, hence, there are equivariant partitions $F_1, \ldots, F_k$ in F such that $\left(\bigcap_{i=1}^{k} F_i\right) \cap F_0 = \varnothing$. The partitions F_i may be extended to equivariant ones $\widetilde{F}_i$ in X. Write $\Phi_i = \widetilde{F}_i$, $i = 1, \ldots, k$; $\Phi_{k+1} = \widetilde{F}_0$. Then $\left(\bigcap_{i=1}^{k+1} \Phi_i\right) \cap F = \left(\bigcap_{i=1}^{k} F_i\right) \cap F_0 = \varnothing$.

2. $\mathbb{Z}_4$–ACTIONS IN S^n

Let $n = 2k+1$ and

$$S^n = \{z = (z_1, \ldots, z_{k+1} \mid \|z\| = 1\}.$$

Then the group $\mathbb{Z}_4 = \{1,\ i,\ -1,\ -i\}$ acts freely in S^n as usual:

$$iz = (iz_1,\ \ldots,\ iz_{k+1}).$$

By $2\,S^n$ we denote the space $S^n \times \{-1,\ 1\}$, where $\mathbb{Z}_4$ acts as follows:

$$i(z \times \{\varepsilon\}) = i\,z \times \{-\varepsilon\}.$$

Proposition 3. *Let X be a compact $\mathbb{Z}_4$-space with equivariant partitions $\Phi_1, \ldots, \Phi_{n+1}$ $(n = 2k+1)$ such that $\bigcap_{j=1}^{n+1} \Phi_j = \varnothing$. Then there exists a $\mathbb{Z}_4$-equivariant map $\varphi : X \to S^n$.*

Proof. Apply an induction on n. Let $n = 1$ and $O\Phi_j$ be open invariant neighbourhoods of Φ_j such that $\bigcap_{j=1}^{2} \overline{O\Phi_j} = \varnothing$. Since Φ_j is an equivariant partition, $X \setminus O\Phi_j = A_j \cup (i\,A_j) \cup (-A_j) \cup (-i\,A_j)$, where $\varepsilon\,A_j$ are non-intersecting closed sets ($\varepsilon \in \mathbb{Z}_4$). Define $\varphi : X \setminus O\Phi_1 \to S^1$ by $\varphi(\varepsilon\,A_1) = \varepsilon$. Then φ may be extended to a map $\varphi : (X \setminus O\Phi_1) \cup A_2 \to S^1$ and finally to an equivariant $\varphi : X \to S^1$ by the formula $\varphi(\varepsilon z) = \varepsilon\,\varphi(z)$.

Suppose the proposition is valid for $n-1$ and consider $\Phi_1,\ \ldots,\ \Phi_{n+1}$ with $\bigcap_{j=1}^{n+1} \Phi_j = \varnothing$. Take as above $O\Phi_j$ with $\bigcap_{j=1}^{n+1} O\Phi_j = \varnothing$. Put $X_1 = \bigcup_{j=1}^{n-1} X \setminus O\Phi_j$, $X_2 = \bigcup_{j=n}^{n+1} X \setminus O\Phi_j$, so $X = X_1 \cup X_2$. Then, evidently, the sets $X_1 \cap \Phi_j$, $j = 1, \ldots, n-1$, are equivariant partitions in X_1 with an empty intersection, as well as $X_2 \cap \Phi_j$, $j = n,\ n+1$, are non-intersecting equivariant partitions in X_2. Then by the induction hypothesis we have equivariant maps $\varphi_1 : X_1 \to S^{n-2}$, $\varphi_2 : X_2 \to S^1$ which induce (by Proposition 1) an equivariant $\varphi : X \to S^{n-2} * S^1$. Note, finally, that $S^{n-2} * S^1 = S^n$ as $\mathbb{Z}_4$-spaces.

Lemma. *Let $f : S^n \to \mathbb{R}^1$ be a continuous map (n is odd). Consider the set*

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}$$

and suppose that $\dim A(f) \leqq n-4$. *Then there exists a $\mathbb{Z}_4$-equivariant map*

$$\psi : S^n \to 2\,S^{n-2} * S^{n-2}.$$

Proof. Since $A(f)$ is a closed invariant set with $\dim A(f) \leqq n-4$, by Proposition 2 we can find in S^n equivariant partitions $\Phi_1, \ldots, \Phi_{n-3}$ such that $\left(\bigcap_{i=1}^{n-3} \Phi_i\right) \cap A(f) = \varnothing$. Write $\Phi = \bigcap_{i=1}^{n-3} \Phi_i$, then for any $z \in \Phi$ $f(\text{orbit}\,z) \neq \text{const}$. Consider the sets

$$M = \{z \in \Phi \mid f(z) = f(-z) \text{ or } f(iz) = f(-iz)\},$$
$$N = \{z \in \Phi \mid f(z) = f(-z) \text{ and } f(iz) = f(-iz)\}.$$

Evidently, these are closed invariant sets and $N \subset M$. Moreover, M is an equivariant partition in Φ, and N is an equivariant partition in M. Really, suppose, first, that M is not an equivariant partition in Φ. Then for some $z \in \Phi \setminus M$, $g \neq 1$, the points z, gz lie in the same component K of $\Phi \setminus M$. If $g = i$ or $g = -i$, we have $iK = K$; if $y = -1$, then $-K = K$. In both cases $-K = K$, so we must have $f(z_0) = f(-z_0)$ for some $z_0 \in K \subset \Phi \setminus M$, which means $z_0 \in M$ — a contradiction.

Suppose now N is not an equivariant partition in M. Proceeding as above, we find a component K of $M \setminus N$ such that $-K = K$. Set

$$K_+ = \{z \in K \mid f(z) = f(-z)\},$$
$$K_- = \{z \in K \mid f(iz) = f(-iz)\}.$$

Then $K = K_+ \cup K_-$ and $K_+ \cap K_- = \varnothing$, since $K \cap N = \varnothing$. Hence $K \equiv K_+$ (par example). Considering f on iK, we get some $z_0 \in iK$ such that $f(z_0) = f(-z_0)$. On the other hand, iz_0, $-iz_0 \in K$; therefore $f(iz_0) = f(-iz_0)$, which means that $z_0 \in N$ — a contradiction.

Let Φ_{n-2}, Φ_{n-1} be equivariant partitions in S^n such that $\Phi_{n-2} \cap \Phi = M$, $\Phi_{n-1} \cap M = N$. Then we have

$$\bigcap_{i=1}^{n-1} \Phi_i = N.$$

Put $N_+ = \{z \in N \mid f(z) < f(iz)\}$; $N_- = iN_+$. Then $N = N_+ \cup N_-$, $N_+ \cap N_- = \varnothing$ and $-N_+ = N_+$, $-N_- = N_-$. Consider the set N_+ together with the antipodal involution $T_0(z) = -z$. For its B-index we must have

$$B(N_+;\, T_0) \leqq n-2.$$

Indeed, if $B(N_+;\, T_0) \geqq n-1$, then $B(N_+;\, T_0) \geqq 1$; it means (see i)) that N_+ contains a T_0-invariant continuum and hence by iii) $B(N_-;\, T_0) \leqq n-2$, which contradicts $B(N_-;\, T_0) = B(N_+;\, T_0) \geqq n-1$. Therefore, by the definition of B-index we have a T_0-equivariant map $\varphi_+ : N_+ \to S^{n-2}$. Define $\varphi_- : N_- \to S^{n-2}$ by $\varphi_-(iz) = i\varphi_+(z)$. Then we get a $\mathbb{Z}_4$-equivariant map $\varphi : N \to 2\,S^{n-2}$ defined by

$$\varphi(z) = \begin{cases} \varphi_+(z) \times \{1\}, & z \in N_+, \\ \varphi_-(z) \times \{-1\}, & z \in N_-. \end{cases}$$

Extend φ to some closed invariant neighbourhood $\varphi : \overline{ON} \to 2\,S^{n-2}$. The sets $\Phi_i \setminus ON$ are equivariant partitions in $S^n \setminus ON$ with $\bigcap_{i=1}^{n-1} \Phi_i \setminus ON = \varnothing$. Then by Proposition 3 there exists an equivariant map of $S^n \setminus ON$ into S^{n-2}. Therefore, by Proposition 1 we get some equivariant

$$\psi : S^n \to 2\,S^{n-2} * S^{n-2}.$$

The lemma is proved.

3. INFORMATION ABOUT THE COHOMOLOGIES OF THE LENS SPACES

Let n be an odd number. The lens space L_4^n is the orbit space of the $\mathbb{Z}_4$-space S^n (with the canonical $\mathbb{Z}_4$-action $z \to iz$), so we write $L_4^n = \widetilde{S}^n$. The space $L_4^\infty = \bigcup_{n=1}^{\infty} L_4^n$ is the classifying space for all principal $\mathbb{Z}_4$-bundles. For their cohomology rings with coefficients modulo 2 we have (see for example [1])

$$H^*(L_4^\infty;\, \mathbb{Z}_2) = \Lambda[u] \otimes \mathbb{Z}_2[v],$$

where $\deg u = 1$, $\deg v = 2$,

$$H^*(L_4^n;\, \mathbb{Z}_2) = \Lambda[u_0] \otimes \mathbb{Z}_2[v_0] \;/\; (v_0^{k+1} = 0),$$

where $\deg u_0 = 1$, $\deg v_0 = 2$ and $n = 2k + 1$.

The natural inclusion $i : L_4^n \to L_4^\infty$ induces an epimorphism i^* in the cohomologies, such that $i^*(u) = u_0$, $i^*(v) = v_0$.

Let X be a compact (free) $\mathbb{Z}_4$-space with an orbit space $\widetilde{X}$. Consider the principal $\mathbb{Z}_4$-bundle $\xi = (X,\, \widetilde{X},\, p)$. Let $f : \widetilde{X} \to L_4^\infty$ be a classifying map for ξ. Then $f^*(u)$ and $f^*(v)$ are some characteristic classes in $H^*(\widetilde{X};\, \mathbb{Z}_2)$ that we shall denote by $u(\widetilde{X})$ and $v(\widetilde{X})$, respectively. (So we have $u_0 = u(L_4^n)$, $v_0 = v(L_4^n)$.) Note, that every $\mathbb{Z}_4$-equivariant map $\varphi : X \to Y$ induces a map $\widetilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ and if $f : \widetilde{Y} \to L_4^\infty$ is a classifying map for $\eta = (Y,\, \widetilde{Y},\, \pi)$, then $f\,\widetilde{\varphi}$ is a classifying map for $\xi = (X,\, \widetilde{X},\, p)$, therefore $u(\widetilde{X}) = \widetilde{\varphi}^*\, u(\widetilde{Y})$ and $v(\widetilde{X}) = \widetilde{\varphi}^*\, v(\widetilde{Y})$.

4. THE MAIN RESULT

Theorem. *Let $n = 2k + 1$ and $f : S^n \to \mathbb{R}^1$ be a continuous map. Consider the set*

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Then $\dim A(f) \geqq n - 3$.

Proof. Suppose the contrary. Then by the lemma we have some $\mathbb{Z}_4$-equivariant $\psi : S^n \to 2\,S^{n-2} * S^{n-2}$. We shall show that such a map does not exist.

The orbit space of S^n is L_4^n; denote by M the orbit space of $2\,S^{n-2} * S^{n-2}$. Let $u_1 = u(M)$, $v_1 = v(M)$. It is enough to prove that

$$u_1\, v_1^k = 0$$

in $H^n(M;\, \mathbb{Z}_2)$. Really, then $u_0 = u(L_4^n) = \widetilde{\psi}^*(u_1)$, $v_0 = v(L_4^n) = \widetilde{\psi}^*(v_1)$, hence $u_0 v_0^k = \widetilde{\psi}^*(u_1 v_1^k) = 0$ in $H^n(L_4^n;\, \mathbb{Z}_2)$, which is a contradiction.

Decompose $2\,S^{n-2} * S^{n-2} = A_1 \cup A_2$, where

$$A_1 = \{(t_1 x,\, t_2 y) \mid t_1 \geqq \tfrac{1}{2}\}, \quad A_2 = \{(t_1 x,\, t_2 y) \mid t_1 \leqq \tfrac{1}{2}\}.$$

Let $M_1 = \widetilde{A}_1$, $M_2 = \widetilde{A}_2$ be the corresponding orbit spaces. Then $M = M_1 \cup M_2$. Define the homotopy equivalences $r_1 : A_1 \to 2\,S^{n-2}$, $r_2 : A_2 \to S^{n-2}$

by $r_1(t_1x, t_2y) = x$, $r_2(t_1x, t_2y) = y$. Since r_i are equivariant, they induce maps between the orbit spaces, which are also homotopy equivalences. Hence

$$H^*(M_1) = H^*(\widetilde{2S}^{n-2}), \quad H^*(M_2) = H^*(\widetilde{S}^{n-2}) = H^*(L_4^{n-2}).$$

(All cohomologies are taken with $\mathbb{Z}_2$-coefficients.)

Consider the Meyer–Vietoris sequence

$$\longrightarrow H^{n-2}(M_1 \cap M_2) \xrightarrow{\delta} H^{n-1}(M) \xrightarrow{(i_1^*, i_2^*)} H^{n-1}(M_1) \oplus H^{n-1}(M_2) \longrightarrow .$$

It is clear that $i_1^*(v_1^k) = i_2^*(v_1^k) = 0$, since $H^{n-1}(M_1) = H^{n-1}(M_2) = \{0\}$. Therefore $v_1^k = \delta\omega$ for some $\omega \in H^{n-2}(M_1 \cap M_2)$. Then $u_1 v_1^k = u_1\,\delta\omega = \delta(i^*(u_1)\omega)$, where $i : M_1 \cap M_2 \to M$ is the inclusion map (see for example [7]). To prove $u_1 v_1^k = 0$ it is enough to show that $i^*(u_1) = 0$. Since $i = i_1 j_1$, where $j_1 : M_1 \cap M_2 \to M_1$, $i_1 : M_1 \to M$ are the inclusions, we have $i^*(u_1) = j_1^* i_1^*(u_1) = j_1^*(u(M_1))$. We shall prove that $u(M_1) = 0$. Note that $\widetilde{2S}^{n-2}$ is a deformation retract of M_1, therefore $u(M_1) = u(\widetilde{2S}^{n-2})$. We have to show that

$$u(\widetilde{2S}^{n-2}) = 0.$$

Suppose, first, $n > 3$. Clearly, $\widetilde{2S}^{n-2} = \mathbb{R}P^{n-2}$. Let $f : \mathbb{R}P^{n-2} \to L_4^\infty$ be a classifying map for $\xi = (2\,S^{n-2}, \mathbb{R}P^{n-2}, p)$, then $u(\widetilde{2S}^{n-2}) = f^*(u)$. Suppose that $f^*(u) \neq 0$. Then in the cohomology ring of $\mathbb{R}P^{n-2}$ we have $[f^*(u)]^2 \neq 0$ (see [7]). On the other hand, $[f^*(u)]^2 = f^*(u^2) = f^*(0) = 0$, which is a contradiction.

Let now $n = 3$. Then we directly see that a classifying map for the $\mathbb{Z}_4$-bundle $\xi = (2S^1, \widetilde{2S}^1, p)$ is the map $f : \widetilde{2S}^1 \to L_4^1 \subset L_4^\infty$ which is a double covering (both $\widetilde{2S}^1$ and L_4^1 are homeomorphic to S^1). Therefore $u(\widetilde{2S}^1) = f^*(u) = 0$.

The theorem is proved.

Corollary 1. *For any continuous function $f : S^3 \to \mathbb{R}^1$ there exists $z \in S^3$ such that*

$$f(z) = f(iz) = f(-z) = f(-iz).$$

Corollary 2. *Let S^3 be a sum of two closed (non-invariant) subsets $S^3 = A \cup B$. Then some of them contains a whole* orbit $z = \{z, iz, -z, -iz\}$.

Proof. Take $f(z) = \operatorname{dist}(z, A)$ in Corollary 1.

The 3-sphere is a group with respect to the multiplication induced by the quaternion structure in $\mathbb{C}^2$.

Corollary 3. *Let F and A be closed subsets of S^3. Then there exists $z \in S^3$ such that*

$$\operatorname{vol}(F \cap zA) = \operatorname{vol}(F \cap i\,zA) = \operatorname{vol}(F \cap -zA) = \operatorname{vol}(F \cap -i\,zA),$$

where "vol" is the 3-volume in S^3.

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