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MAPPING THEOREMS FOR COHOMOLOGICALLY TRIVIAL MAPS

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Симеон Стефанов. ТЕОРЕМЫ О СОВПАДЕНИИ ДЛЯ КОГОМОЛОГИЧЕСКИ ТРИВИАЛЬНЫХ ОТОБРАЖЕНИЙ

Получены некоторые теоремы о совпадении для отображений n-сферы S^n. Следствием показано, что каждое когомологически тривиальное отображение $f : S^n \xrightarrow{\text{на}} Y$ сферы S^n на некоторое Y склеивает пару точек $x_1,\ x_2 \in S^n$, расстояние между которыми не меньше диаметра правильного $(n+1)$-симплекса вписанного в S^n:

$$f(x_1) = f(x_2), \quad \|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}.$$

Дальше доказано, что для каждого разложения S^n на n замкнутые подмножества, некоторое из них содержит континуум K с $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Показано также, что каждое понижающее размерность отображение $f : S^n \to Y$ постоянно на континуум K с $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Наконец получена теорема о совпадении для отображений S^n в k-мерные стягиваемые полиэдры (для $k < n$).

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Some mapping theorems for maps of the n-sphere S^n are obtained. As a corollary, it is shown that every cohomologically trivial map $f : S^n \xrightarrow{\text{on}} Y$ of S^n onto some Y identifies a pair of points $x_1,\ x_2 \in S^n$ such that the distance between them is not less than the diameter of

the regular $(n+1)$-simplex inscribed in S^n:

$$f(x_1) = f(x_2), \quad \|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}.$$

Furthermore, it is proved that for any decomposition of S^n into n closed subsets some of them contains a continuum K with $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Also it is shown that every lowering dimension map $f : S^n \to Y$ is constant on a continuum K with $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Finally, a mapping theorem for maps of S^n into k-dimensional contractible polyhedra is obtained (for $k < n$).

The Borsuk-Ulam theorem states that every map of the n-dimensional unit sphere S^n into $\mathbb{R}^n$ identifies a pair of antipodal points. This theorem ceases to be true if we replace $\mathbb{R}^n$ by an arbitrary n-dimensional contractible polyhedron P_n. However, it is easy to see that there exists a positive α such that any map of S^n into some P_n identifies a pair of points $x_1,\ x_2 \in S^n$ with $\|x_1 - x_2\| \geqq \alpha$. We shall find the greatest α with this property and we shall prove the corresponding mapping theorem in a more general situation (Theorem 1). The fact is that the greatest α with the above-mentioned property is the diameter of the regular $(n+1)$-simplex inscibed in S^n. Corollary 1 of Theorem 1 gives a generalization of a theorem due to J. Väisälä [1]. Furthermore, we obtain a theorem for decomposition of S^n into n closed subsets (Theorem 2), a mapping theorem for lowering dimension maps of S^n (Theorem 3), and finally a mapping theorem for maps of S^n into lower dimensional contractible polyhedra (Theorem 4).

Lemma 1. *Let $e_1,\ e_2,\ \ldots,\ e_{n+1}$ be unit vectors in $\mathbb{R}^n$ such that $\sum_1^{n+1} \lambda_i\, e_i = O$ for some $\lambda_i \geqq 0$ with $\sum_1^{n+1} \lambda_i = 1$.*

Then $(e_i,\ e_j) \leqq -\frac{1}{n}$ for some $i,\ j$.

Proof. Suppose the contrary — $(e_i,\ e_j) > -\frac{1}{n}$ for every $i,\ j$. Then

$$1 = \left(\sum_1^{n+1} \lambda_i\right)^2 = \sum_1^{n+1} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j$$

and

$$0 = \left(\sum_1^{n+1} \lambda_i\, e_i,\ \sum_1^{n+1} \lambda_i\, e_i\right) = \sum_1^{n+1} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j (e_i,\ e_j) > \sum_1^{n+1} \lambda_i^2 - \frac{2}{n} \sum_{i<j} \lambda_i \lambda_j,$$

hence

$$0 > \sum_1^{n+1} \lambda_i^2 - \frac{1}{n} \left(1 - \sum_1^{n+1} \lambda_i^2\right),$$

i. e. $\sum_1^{n+1} \lambda_i^2 < \frac{1}{n+1}$. On the other hand, a well-known inequality gives

$$\sum_1^{n+1} \lambda_i^2 \geqq \frac{\left(\sum_1^{n+1} \lambda_i\right)^2}{n+1} = \frac{1}{n+1},$$

which is a contradiction.

Lemma 2. *Let F be a closed subset of S^n with* $\operatorname{diam} F < \sqrt{\frac{2(n+2)}{n+1}}$. *Then F is contained in some open semisphere of S^n.*

Proof. Denote by $\operatorname{co} F$ the convex hull of F in $\mathbb{R}^{n+1}$ and suppose that $\operatorname{co} F$ does not contain the origin O. Then there is a hyperplane T in $\mathbb{R}^{n+1}$ such that $O \in T$ and $F \cap T = \varnothing$, hence one of the components of $S^n \setminus T$ is an open semisphere containing F.

Suppose now that $O \in \operatorname{co} F$, then, according to the theorem of Caratheodory, O is a convex linear combination of $n+2$ points of F : $O = \sum_{i=1}^{n+2} \lambda_i x_i$, where $\lambda_i \geqq 0$, $\sum_{i=1}^{n+2} \lambda_i = 1$ and $x_1, \ldots, x_{n+2} \in F$. Consider the unit vectors $e_i = \overrightarrow{Ox_i}$. According to Lemma 1, we have $(e_i, e_j) \leqq -\frac{1}{n+1}$ for some i, j. Let φ denote the angle between e_i and e_j, then $\cos\varphi = (e_i, e_j) \leqq -\frac{1}{n+1}$ and we have $\|x_i - x_j\| = 2\sin\frac{\varphi}{2} = 2\sqrt{\frac{1-\cos\varphi}{2}} \geqq 2\sqrt{\frac{1}{2}\left(1 + \frac{1}{n+1}\right)} = \sqrt{\frac{2(n+2)}{n+1}}$, which contradicts the condition $\operatorname{diam} F < \sqrt{\frac{2(n+2)}{n+1}}$.

Note that the number $\sqrt{\frac{2(n+2)}{n+1}}$ is exactly the diameter of the regular $(n+1)$-simplex inscribed in S^n.

All cohomologies in this note are Čech cohomologies with integral coefficients.

Lemma 3. *Let ω be a finite open covering of the compact space X and $\pi : X \to N_\omega$ be the canonical projection of X into the nerve of ω. Suppose $f : X \xrightarrow{\text{on}} Y$ is a map of X on Y and there exists $\xi \in H^n(X)$ such that $\xi \in \operatorname{Im}\pi^* \setminus \operatorname{Im} f^*$. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ is not contained in any element of ω.*

This proposition is proved by the author in [4, Lemma 2].

Given a map $f : X \to Y$ we shall say that f is *trivial in dimension n* if $f^*(H^n(Y)) = 0$.

Theorem 1. *Let X be a compact metric space and $\varphi : X \xrightarrow{\text{on}} S^n$ be a map, which is non-trivial in dimension n. Suppose $f : X \longrightarrow Y$ is a map of X on Y and there is $\xi \in H^n(X)$ such that $\xi \in \operatorname{Im}\varphi^* \setminus \operatorname{Im} f^*$. Then there exist x_1, $x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.*

Proof. It is enough to prove that there exists $y_0 \in Y$ such that the set $\varphi(f^{-1}(y_0))$ is not contained in any open semisphere of S^n. Really, in this case Lemma 2 implies $\operatorname{diam} \varphi(f^{-1}(y_0)) \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$ and the theorem is proved.

Suppose that for any $y \in Y$ the set $\varphi(f^{-1}(y))$ is contained in some open semisphere O_y. Choose an open $V_y \ni y$ with the property $\varphi(f^{-1}(V_y)) \subset O_y$. The covering $\{V_y \mid y \in Y\}$ has a finite subcovering $\{V_{y_i} \mid i = 1, 2, \ldots, k\}$. Put $\omega = \{O_{y_i} \mid i = 1, 2, \ldots, k\}$. Since $\varphi(X) = S^n$, ω is a finite open covering of S^n such that for any $y \in Y$ the set $\varphi(f^{-1}(y))$ is contained in some element of ω. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & S^n \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi_0} \\ N_{\varphi^{-1}(\omega)} & \underset{\rho}{\approx} & N_\omega \end{array}$$

where π and π_0 are the canonical projections and ρ is the natural simplicial isomorphism. By the same reasoning as in Lemma 3 the diagram

$$\begin{array}{ccc} H^n(X) & \xleftarrow{\ \varphi^*\ } & H^n(S^n) \\ {\scriptstyle \pi^*}\big\uparrow & & \big\uparrow{\scriptstyle \pi_0^*} \\ H^n(N_{\varphi^{-1}(\omega)}) & \underset{\rho^*}{\approx} & H^n(N_\omega) \end{array}$$

is commutative. Note that π_0^* is an isomorphism, since the elements of ω are open semispheres and the intersection of each finite system of semispheres is cohomologically trivial. Then $\xi \in \operatorname{Im}\varphi^*$ implies $\xi \in \operatorname{Im}\pi^*$, so that $\xi \in \operatorname{Im}\pi^* \setminus \operatorname{Im} f^*$. Hence, by Lemma 3, there exists $y_0 \in Y$ such that $f^{-1}(y_0)$ is not contained in any element of $\varphi^{-1}(\omega)$. It means that $\varphi f^{-1}(y_0)$ is not contained in any element of ω, but $y_0 \in V_{y_i}$ for some i, thus $\varphi f^{-1}(y_0) \subset \varphi f^{-1}(V_{y_i}) \subset O_{y_i} \in \omega$, which is a contradiction.

The theorem is proved.

Note that in the case $\varphi^* \not\equiv 0$, $f^* \equiv 0$ the existance of $\xi \in \operatorname{Im}\varphi^* \setminus \operatorname{Im} f^*$ is guaranteed and the theorem is valid.

Let X and A be disjoint closed subsets of $\mathbb{R}^N$. We say that X is *n-linked* with A in $\mathbb{R}^N$ if the inclusion map $i : X \to \mathbb{R}^N \setminus A$ is non-trivial in dimension n.

Assume for convenience that $\mathbb{R}^k \subset \mathbb{R}^n$ for $k < n$.

Corollary 1. *Let the compact space X be n-linked with $\mathbb{R}^k$ in $\mathbb{R}^{n+k+1}$ and $\pi : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+1}$ be the projection of $\mathbb{R}^{n+k+1}$ on the orthogonal complement* *of* $\mathbb{R}^k$ $(\mathbb{R}^{n+1} \cap \mathbb{R}^k = \{O\})$. *Put* $\alpha_0 = \inf_{x \in X} \|\pi(x)\|$. *Then for any map* $f : X \xrightarrow{\text{on}} Y$ *trivial in dimension* n *there exist* $x_1, x_2 \in X$ *such that* $f(x_1) = f(x_2)$ *and* $\|x_1 - x_2\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}$.

Proof. Put $\varphi = \frac{\alpha_0 \pi}{\|\pi\|} i$, where $i : X \to \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k$ is the inclusion map. Then $\frac{\alpha_0 \pi}{\|\pi\|} : \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k \to S_0^n$, where $S_0^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = \alpha_0\}$. We have $\varphi^* = i^* \left(\frac{\alpha_0 \pi}{\|\pi\|}\right)^*$, therefore φ is non-trivial in dimension n, since $\left(\frac{\alpha_0 \pi}{\|\pi\|}\right)^*$ is an isomorphism and i is non-trivial in dimension n. Then the conditions of Theorem 1 are fulfiled, hence there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}$ (φ maps X into S_0^n). Finally, we have

$$\|x_1 - x_2\| \geqq \|\pi(x_1) - \pi(x_2)\| \geqq \left\| \alpha_0 \frac{\pi(x_1)}{\|\pi(x_1)\|} - \alpha_0 \frac{\pi(x_2)}{\|\pi(x_2)\|} \right\|$$
$$= \|\varphi(x_1) - \varphi(x_2)\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}.$$

The first inequality is obvious, the second one holds by the definition of α_0.

In his paper [1] J. Väisälä has proved that if X is a partition in $\mathbb{R}^{n+1}$ between O and ∞, then for any map $f : X \xrightarrow{\text{on}} Y$ trivial in dimension n there exists $y_0 \in Y$ such that the set $f^{-1}(y_0)$ is not contained in any open halfspace W with $\partial W \ni O$. Clearly, this theorem may be obtained by the non-metrical variant of Theorem 1 and Corollary 1 — we only have to replace the condition "there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \sqrt{\frac{2(n+2)}{n+1}}$" by "there exists $y_0 \in Y$ such that $\varphi f^{-1}(y_0)$ is not contained in any open semisphere". To obtain now the theorem of Väisälä, it is enough to take Corollary 1 in the case $k = 0$; really, a compact space X is n-linked with O in $\mathbb{R}^{n+1}$ iff X is a partition between O and ∞.

Corollary 2. *For any map* $f : S^n \xrightarrow{\text{on}} Y$ *trivial in dimension* n *there exist* $x_1, x_2 \in S^n$ *such that* $f(x_1) = f(x_2)$ *and* $\|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}$.

This corollary may be immediately obtained by the theorem of Väisälä and Lemma 2.

We may ask whether f identifies a pair of points $x_1, x_2 \in S^n$ with $\|x_1 - x_2\| = \sqrt{\frac{2(n+2)}{n+1}}$. It is not difficult to show that the answer is "no" — there exists a map $f : S^1 \xrightarrow{\text{on}} T$ from S^1 onto the letter T such that $\|x_1 - x_2\| = \sqrt{3}$ implies $f(x_1) \neq f(x_2)$.

Another question is whether $\sqrt{\dfrac{2(n+2)}{n+1}}$ is the greatest number with this property. The answer is "yes" and the corresponding example may be constructed as follows:

Let P be the regular $(n+1)$-simplex inscribed in S^n and $P^{(n-1)}$ be its $(n-1)$-dimensional squeleton. Put $Y = C\,P^{(n-1)}$, where $C\,P^{(n-1)}$ is the cone over $P^{(n-1)}$ with a vertex O. The obvious deformation $f : S^n \to Y$ has the property $\operatorname{diam} f^{-1}(y) \leqq \sqrt{\dfrac{2(n+2)}{n+1}}$ for any $y \in Y$. Moreover, $\operatorname{diam} f^{-1}(y) = \sqrt{\dfrac{2(n+2)}{n+1}}$ iff $y = O$.

Theorem 2. *Let S^n be the union of n closed subsets $S^n = \bigcup_{i=1}^{n} F_i$. Then some F_i contains a continuum K with* $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.

Proof. Suppose the contrary — then there exist $\varepsilon > 0$ such that none of the sets $\overline{O_\varepsilon F_i}$ contains such a continuum. Let ω_i be the finite family of all components of $\overline{O_\varepsilon F_i}$. Set $\omega = \bigcup_{i=1}^{n} \omega_i$. Then ω is a covering of S^n with $\operatorname{ord} \omega \leqq n$, since every ω_i is a disjoint family. Thus, the nerve N_ω is an $(n-1)$-dimensional polyhedron and the cone $C\,N_\omega$ is an n-dimensional contractible one. Consider the map $f = i\pi$, where $\pi : S^n \to N_\omega$ is the canonical projection and $i : N_\omega \to C\,N_\omega$ is the inclusion map. Then $f : S^n \to C\,N_\omega$ is trivial in each dimension and according to Corollary 2 there exist $x_1,\ x_2 \in S^n$ such that $f(x_1) = f(x_2)$ and $\|x_1 - x_2\| \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$. But $f(x_1) = f(x_2)$ implies $\pi(x_1) = \pi(x_2)$, i. e. x_1 and x_2 belong to one and the same element K of ω, therefore $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$, which contradicts the assumption.

Evidently, n is the greatest number with this property. In the case $n = 2$ we may even prove that for arbitrary decomposition $S^2 = F_1 \cup F_2$ some F_i contains a continuum K with $\operatorname{diam} K = 2$ (i. e. containing a pair of antipodal points). Really, if we suppose the contrary, we find as above a map $\pi : S^2 \to N_\omega$, where N_ω is an 1-dimensional polyhedron such that $\pi(x) \neq \pi(-x)$ for any $x \in S^2$, which contradicts a theorem of E. V. Schepin [3]. For $n \geqq 3$ this is not true. Väisälä [1] has constructed a map $f : S^n \to P_k$ of S^n into a k-dimensional polyhedron, where $k = \left[\dfrac{n+1}{2}\right]$, such that $f(x) \neq f(-x)$ for any $x \in S^n$. Let $P_k = \bigcup_{i=1}^{k+1} F_i$ be the representation of P_k from Lemma 4, where $\varepsilon = \dfrac{1}{2} \min_{x \in S^n} \|f(x) - f(-x)\|$.

Then $S^n = \bigcup_{i=1}^{k+1} f^{-1}(F_i)$ and none of the $f^{-1}(F_i)$ contains a continuum K with $\operatorname{diam} K = 2$. Nevertheless, we do not know whether $\sqrt{\dfrac{2(n+2)}{n+1}}$ is the greatest number with this property.

Lemma 4. *Given $\varepsilon > 0$, every n-dimensional polyhedron P_n may be represented as the union $P_n = \bigcup_{i=1}^{n+1} F_i$ of $n+1$ closed subsets such that the components of each F_i have a diameter $< \varepsilon$.*

To prove it, one has to carry out induction on n taking some sufficiently small subdivision of P_n and considering its $(n-1)$-squeleton.

Theorem 3. *Let $f : S^n \to Y$ be a lowering dimension map. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ contains a continuum K with* $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.

Proof. Assume that $Y = f(X)$, so that Y is a compact space with $\dim Y \leqq n-1$. Suppose the contrary. Then we can find a closed finite covering ω of Y with ord $\omega \leqq n$ such that for any $\Phi \in \omega$ the set $f^{-1}(\Phi)$ does not contain such a continuum. Since $\dim Y \leqq n-1$, there is an ω-map $h : Y \to P_{n-1}$ of Y into some $(n-1)$-dimensional polyhedron. Let γ be a closed covering of P_{n-1} such that $h^{-1}(\gamma)$ is inscribed in ω. According to Lemma 4, P_{n-1} may be represented as $P_{n-1} = \bigcup_{i=1}^{n} F_i$, where each component of the sets F_i is contained in some element of γ. Consider the representation of S^n

$$S^n = \bigcup_{i=1}^{n} f^{-1}\, h^{-1}(F_i).$$

According to Theorem 2, some $f^{-1}\, h^{-1}(F_i)$ contains a continuum K with $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$. Then $h\, f(K)$ is contained in some component of F_i and thus in some element of γ. Since $f(K) \subset h^{-1}\, h\, f(K)$, then $f(K)$ is contained in an element of $h^{-1}(\gamma)$ and hence in some element Φ of ω. But then we have $f^{-1}(\Phi) \supset K$, which contradicts the definition of ω.

Our remarks on Theorem 2 remain valid here — for any lowering dimension map $f : S^2 \to Y$ some $f^{-1}(y_0)$ contains a continuum K with $\operatorname{diam} K = 2$ (this is proved by the author in [2] for arbitrary maps $f : S^{2k} \to Y_k$, where $\dim Y_k \leqq k$). On the other hand, for $n \geqq 3$ it is not true, as following from the Väisälä's example.

Let $\varphi : X \to S^n$ be a map non-trivial in dimension n and $f : X \to P_n$ maps X into an n-dimensional contractible polyhedron P_n. We shall prove that for some $x_0 \in X$ the set $\varphi f^{-1} f(x_0)$ is not contained in any open semisphere of S^n. It is enough to show that $H^n(f(X)) = \{0\}$. Really, consider in this case f as a map

$f : X \to f(X)$, then $f^* \equiv 0$ and the existence of $x_0 \in X$ with this property follows by the proof of Theorem 1. Form the exact sequence

$$H^n(P_n) \xrightarrow{i^*} H^n(f(X)) \xrightarrow{\partial^*} H^{n+1}(P_n,\, f(X)).$$

Here $H^n(P_n) = \{0\}$, since P_n is contractible and $H^{n+1}(P_n,\, f(X)) = \{0\}$, consequently $H^n(f(X)) = \{0\}$.

Then Lemma 2 implies diam $\varphi f^{-1} f(x_0) \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$, i. e. there exists $x_1 \in X$ such that $f(x_0) = f(x_1)$ and $\|\varphi(x_0) - \varphi(x_1)\| \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.

Suppose now $f : X \to P_k$ maps X into a k-dimensional polyhedron and $k < n$. At the close of this note we shall answer the question how many $x_0 \in X$ do there exist with the above-mentioned property.

Let $\varphi : X \to S^n$ maps the compact space X into the n-sphere S^n. We shall write

$$\gamma(X,\, \varphi) \leqq k$$

if there exists a map $f : X \to P_{k+1}$ of X into a $(k+1)$-dimensional contractible polyhedron, such that for any $x \in X$ the set $\varphi f^{-1} f(X)$ is contained in some open semisphere of S^n.

The previous reasoning shows, that if φ is non-trivial in dimension n, then $\gamma(X,\, \varphi) \geqq n$.

Lemma 5. $\gamma(X,\, \varphi) \leqq \dim X$ *for any compact space* X.

Proof. Suppose $\dim X = k$. There is a finite open covering ω of X such that for every $U \in \omega$ the set $\varphi(U)$ is contained in some open semisphere of S^n. There exists an ω-map $f : X \to P_k$ of X into a k-dimensional polyhedron P_k. Then for any $x \in X$ the set $\varphi f^{-1} f(x)$ is contained in some open semisphere. Denote by $C P_k$ the cone over P_k. It is clear that $C P_k$ is a contractible $(k+1)$-dimensional polyhedron, and if we consider f as a map $f : X \to C P_k$, then f has the required property.

Consequently, $\gamma(X,\, \varphi) \leqq k$.

Lemma 6. *Let* $\gamma(X,\, \varphi) \geqq n$ *and* $f : X \to P_k$ *maps* X *into a contractible* k*-dimensional polyhedron. Consider the set*

(1) $A(f) = \{x \in X \mid \varphi f^{-1} f(x)$ *is not contained in any open semisphere of* $S^n\}$.

Then $\gamma(A(f),\, \varphi|_{A(f)}) \geqq n - k$.

Proof. Suppose that $\gamma(A(f),\, \varphi|_{A(f)}) \leqq n - k - 1$, i. e. that there exists a map $g : A(f) \to Q$ of $A(f)$ into the $(n-k)$-dimensional contractible polyhedron Q such that for every $x \in A(f)$ the set $\varphi g^{-1} g(x)$ is contained in some open semisphere of S^n. Since Q is contractible, g has an extension $\tilde{g} : X \to Q$. Form the map

$$h = f \times \tilde{g} : X \to P_k \times Q.$$

Clearly, $P_k \times Q$ is an n-dimensional contractible polyhedron. We shall prove that for any $x \in X$ the set $\varphi h^{-1} h(x)$ is contained in some open semisphere. Note, that

$$\varphi h^{-1} h(x) \subset \varphi f^{-1} f(x) \cap \varphi \tilde{g}^{-1} \tilde{g}(x).$$

In the case $x \notin A(f)$ the set $\varphi f^{-1} f(x)$ is contained in some open semisphere, thus $\varphi h^{-1} h(x)$ is contained in the same semisphere.

Suppose now that $x \in A(f)$. Then $f^{-1} f(x) \subset A(f)$ and $h^{-1} h(x) = f^{-1} f(x) \cap \tilde{g}^{-1} \tilde{g}(x) \subset A(f) \cap \tilde{g}^{-1} \tilde{g}(x)$. But $A(f) \cap \tilde{g}^{-1} \tilde{g}(x) \subset g^{-1} g(x)$, really, if $y \in A(f) \cap \tilde{g}^{-1} \tilde{g}(x)$, then $\tilde{g}(y) = \tilde{g}(x)$ and $\tilde{g}(y) = g(y)$, $\tilde{g}(x) = g(x)$, so that $g(y) = g(x)$, thus $y \in g^{-1} g(x)$. Consequently, $h^{-1} h(x) \subset g^{-1} g(x)$, hence $\varphi h^{-1} h(x) \subset \varphi g^{-1} g(x)$. The set $\varphi g^{-1} g(x)$ is contained in some open semisphere, therefore $\varphi h^{-1} h(x)$ is contained in the same one.

All this reasoning implies that $\gamma(X, \varphi) \leqq n-1$, which contradicts the condition $\gamma(X, \varphi) \geqq n$.

Theorem 4. *Let X be a compact metric space and the map $\varphi : X \to S^n$ be non-trivial in dimension n. Let $f : X \to P_k$ map X into a contractible k-dimensional polyhedron. Consider the set*

$$B(f) = \left\{ x \in X \mid \operatorname{diam} \varphi f^{-1} f(x) \geqq \sqrt{\frac{2(n+2)}{n+1}} \right\}.$$

Then $\dim B(f) \geqq n - k$.

Proof. As it is shown above, $\gamma(X, \varphi) \geqq n$. If we consider the set $A(f)$ defined by (1), then the inequality $\gamma(A(f), \varphi|_{A(f)}) \geqq n - k$ holds by Lemma 6, hence $\dim A(f) \geqq n - k$ by Lemma 5. Obviously, $A(f) \subset B(f)$, therefore $\dim B(f) \geqq n - k$.

Corollary. *Let $f : S^n \to P_k$ map S^n into a k-dimensional contractible polyhedron P_k. Consider the set*

$$B(f) = \left\{ x \in S^n \mid \operatorname{diam} f^{-1} f(x) \geqq \sqrt{\frac{2(n+2)}{n+1}} \right\}.$$

Then $\dim B(f) \geqq n - k$.

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