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ON AN EQUATION INVOLVING FRACTIONAL POWERS WITH PRIME NUMBERS OF A SPECIAL TYPE

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We consider the equation $[p_1^c]+[p_2^c]+[p_3^c]=N$, where N is a sufficiently large integer, and $[t]$ denotes the integer part of t. We prove that if $1<c<\frac{17}{16}$, then it has a solution in prime numbers p_1, p_2, p_3 such that each of the numbers p_1+2, p_2+2, p_3+2 has at most $\left[\frac{95}{17-16c}\right]$ prime factors, counted with their multiplicities.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In 1937 I. M. Vinogradov [16] proved that for every sufficiently large odd integer N the equation

$$p_1+p_2+p_3=N \tag{1.1}$$

has a solution in prime numbers p_1, p_2, p_3.

Analogous problem was considered in 1952 by Piatetski-Shapiro [9]. If $H(c)$ denotes the least integer s such that the diophantine inequality

$$|p_1^c+\cdots+p_s^c-N|<\varepsilon,$$

has a solution in primes $p_1,\ldots,p_s$, where $c>1$ is not an integer, $\varepsilon>0$ is small, and N is large real number, then Piatetski-Shapiro proved that

$$\limsup_{c\to\infty}\frac{H(c)}{c\log c}\leq 4.$$

He also proved that if $1 < c < 3/2$, then $H(c) \le 5$. In 1992, Tolev [14] established that if $1 < c < \frac{15}{14}$, then the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < N^{-\kappa}$$

has a solution in prime numbers p_1, p_2, p_3 for certain $\kappa = \kappa(c) > 0$. Several improvements were made and the strongest of them is due to Baker and Weingartner [1], who improved Tolev's result with $1 < c < \frac{10}{9}$.

In 1995, M. B. Laporta and D. I. Tolev [7] considered the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = N, \tag{1.2}$$

where $c \in \mathbb{R}$, $c > 1$, $N \in \mathbb{N}$ and $[t]$ denotes the integer part of t. They showed that if $1 < c < \frac{17}{16}$ and N is a sufficiently large integer, then the equation (1.2) has a solution in prime numbers p_1, p_2, p_3.

For any natural number r, let $\mathcal{P}_r$ denote the set of r-almost primes, i.e. the set of natural numbers having at most r prime factors counted with multiplicities. There are many papers devoted to the study of problems involving primes and almost primes. For example, in 1973 J. R. Chen [4] established that there exist infinitely many primes p such that $p + 2 \in \mathcal{P}_2$. In 2000 Tolev [12] proved that for every sufficiently large integer $N \equiv 3 \pmod 6$ the equation (1.1) has a solution in prime numbers p_1, p_2, p_3 such that $p_1 + 2 \in P_2$, $p_2 + 2 \in P_5$, $p_3 + 2 \in P_7$. Thereafter this result was improved by Matomäki and Shao [8], who showed that for every sufficiently large integer $N \equiv 3 \pmod 6$ the equation (1.1) has a solution in prime numbers p_1, p_2, p_3 such that $p_1 + 2, p_2 + 2, p_3 + 2 \in \mathcal{P}_2$.

Recently Tolev [15] established that if N is sufficiently large, $E > 0$ is an arbitrarily large constant and $1 < c < \frac{15}{14}$, then the inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E}$$

has a solution in prime numbers p_1, p_2, p_3, such that each of the numbers $p_1 + 2$, p_2+2, p_3+2 has at most $\left[\frac{369}{180-168c}\right]$ prime factors, counted with their multiplicities.

In this paper, we prove the following

Theorem 1.1. *Suppose that $1 < c < \frac{17}{16}$. Then for every sufficiently large N the equation* (1.2) *has a solution in prime numbers p_1, p_2, p_3, such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ has at most $\left[\frac{95}{17-16c}\right]$ prime factors, counted with their multiplicities.*

We note that the integer $\left[\frac{95}{17-16c}\right]$ is equal to 95 if c is close to 1 and it is large if c is close to $\frac{17}{16}$.

To prove Theorem 1.1 we combine ideas developed by Laporta and Tolev [7] and Tolev [15]. First we apply a version of the vector sieve and then the circle method. In section 4 we find an asymptotic formula for the integrals Γ_1' and Γ_4' (defined by

(3.11) and (3.14) respectively). In section 5 we estimate Γ_1'' and Γ_4'' (defined by (3.12) and (3.15) respectively) and we then complete the proof of Theorem 1.1.

2. NOTATION AND SOME LEMMAS

We use the following notations: with $\{t\} = t - [t]$ we denote the fractional part of t. With $||t||$ we denote the distance from t to the nearest integer. As usual with $\mu(n)$, $\varphi(n)$ and $\Lambda(n)$ we denote respectively, Möbius' function, Euler's function and von Mangoldt's function. Also $e(t) = e^{2\pi i t}$.

We use Vinogradov's notation $A \ll B$, which is equivalent to $A = O(B)$. If we have simultaneously $A \ll B$ and $B \ll A$, then we shall write $A \asymp B$.

We reserve p, p_1, p_2, p_3 for prime numbers. By ϵ we denote an arbitrarily small positive number, which is not necessarily the same in the different formulae.

With $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ we will denote respectively the set of natural numbers, the set of integer numbers and the set of real numbers.

Now we quote some lemmas, which shall be used later.

Lemma 2.1. *Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^{\pm}(d)$ (Rosser's functions of level D) with the following properties:*

1. *For any positive integer d we have*

$$|\lambda^{\pm}(d)| \leq 1, \qquad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

2. *If $n \in \mathbb{N}$, then*

$$\sum_{d|n} \lambda^{-}(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^{+}(d).$$

3. *If $z \in \mathbb{R}$ is such that $z^2 \leq D \leq z^3$ and if*

$$\begin{aligned} P(z) &= \prod_{2<p<z} p, \qquad \mathcal{B} = \prod_{2<p<z} \left(1 - \frac{1}{p-1}\right), \\ \mathcal{N}^{\pm} &= \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \qquad s_0 = \frac{\log D}{\log z}, \end{aligned} \tag{2.1}$$

then we have

$$\mathcal{B} \leq \mathcal{N}^{+} \leq \mathcal{B}\left(F(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{2.2}$$

$$\mathcal{B} \geq \mathcal{N}^{-} \geq \mathcal{B}\left(f(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{2.3}$$

where $F(s)$ and $f(s)$ satisfy

$$f(s) = 2e^{\gamma} s^{-1} \log(s-1), \quad F(s) = 2e^{\gamma} s^{-1} \quad \text{for} \quad 2 \leq s \leq 3. \tag{2.4}$$

Here γ is Euler's constant.

Proof. See Greaves [5, Chapter 4, Theorem 3]. □

Lemma 2.2. *Suppose that* $\Lambda_i, \Lambda_i^{\pm}$ *are real numbers satisfying* $\Lambda_i = 0$ *or* 1, $\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+$, $i = 1, 2, 3$. *Then*

$$\Lambda_1\Lambda_2\Lambda_3 \geq \Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+. \tag{2.5}$$

Proof. The proof is similar to the proof of Lemma 13 in [2]. □

Lemma 2.3. *Suppose that* x, $y \in \mathbb{R}$ *and* $M \in \mathbb{N}$, $M \geq 3$. *Then*

$$e(-x\{y\}) = \sum_{|m|\leq M} c_m e(my) + O\left(\min\left(1, \frac{1}{M||y||}\right)\right),$$

where

$$c_m = \frac{1 - e(-x)}{2\pi i(x + m)}. \tag{2.6}$$

Proof. Proof can be find in Buriev [3, Lemma 12]. □

Lemma 2.4. *Consider the integral*

$$I = \int_a^b e(f(x))dx,$$

where $f(x)$ *is real function with continuous second derivative and monotonous first derivative. If* $|f'(x)| \geq h > 0$ *for all* $x \in [a, b]$, *then* $I \ll h^{-1}$.

Proof. See [10, Lemma 4.3]. □

3. BEGINNING OF THE PROOF

Let η, δ, ξ and μ be positive real numbers depending on c. We shall specify them later. Now we only assume that they satisfy the conditions

$$\xi + 3\delta < \frac{12}{25}, \qquad 2 < \frac{\delta}{\eta} < 3, \qquad \mu < 1. \tag{3.1}$$

We denote

$$X = N^{\frac{1}{c}}, \qquad z = X^{\eta}, \qquad D = X^{\delta}, \qquad \Delta = X^{\xi - c} \tag{3.2}$$

and

$$P(z) = \prod_{2<p<z} p. \tag{3.3}$$

Consider the sum

$$\Gamma = \sum_{\substack{\mu X < p_1, p_2, p_3 \le X \\ [p_1^c]+[p_2^c]+[p_3^c]=N \\ (p_i+2, P(z))=1,\ i=1,2,3}} (\log p_1)(\log p_2)(\log p_3). \tag{3.4}$$

If we prove the inequality

$$\Gamma > 0, \tag{3.5}$$

then equation (1.2) would have a solution in primes p_1, p_2, p_3 satisfying conditions in the sum Γ. Suppose that $p_i + 2$ has l prime factors, counted with multiplicities. From (3.2), (3.3) and $(p_i + 2, P(z)) = 1$ we have

$$X + 2 \ge p_i + 2 \ge z^l = X^{\eta l}$$

and then $l \le \frac{1}{\eta}$. This means that $p_i + 2$ has at most $[\eta^{-1}]$ prime factors counted with multiplicities. Therefore, to prove Theorem 1.1 we have to establish (3.5) for an appropriate choice of η.

For $i = 1, 2, 3$ we define

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (p_i + 2, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then we find that

$$\Gamma = \sum_{\substack{\mu X < p_1, p_2, p_3 \le X \\ [p_1^c]+[p_2^c]+[p_3^c]=N}} \Lambda_1\Lambda_2\Lambda_3(\log p_1)(\log p_2)(\log p_3).$$

We can write Γ as

$$\Gamma = \sum_{\mu X < p_1, p_2, p_3 \le X} \Lambda_1\Lambda_2\Lambda_3(\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^c] + [p_2^c] + [p_3^c] - N))d\alpha.$$

Suppose that $\lambda^{\pm}(d)$ are the Rosser functions of level D. Let also denote

$$\Lambda_i^{\pm} = \sum_{d|(p_i+2, P(z))} \lambda^{\pm}(d), \qquad i = 1, 2, 3. \tag{3.7}$$

Then from Lemma 2.1, (3.6) and (3.7) we find that

$$\Lambda_i^- \le \Lambda_i \le \Lambda_i^+.$$

We use Lemma 2.2 and find that

$$\Gamma \ge \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4,$$

where $\Gamma_1, \dots, \Gamma_4$ are the contributions coming from the consecutive terms of the right-hand side of (2.5). We have $\Gamma_1 = \Gamma_2 = \Gamma_3$ and

$$\Gamma_1 = \sum_{\mu X < p_1, p_2, p_3 \le X} \Lambda_1^- \Lambda_2^+ \Lambda_3^+ (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^c] + [p_2^c] + [p_3^c] - N))\, d\alpha,$$

$$\Gamma_4 = \sum_{\mu X < p_1, p_2, p_3 \le X} \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^c] + [p_2^c] + [p_3^c] - N))\, d\alpha.$$

Hence, we get

$$\Gamma \ge 3\Gamma_1 - 2\Gamma_4. \tag{3.8}$$

Let us first consider Γ_1. We have

$$\Gamma_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} e(-N\alpha) L^-(\alpha) L^+(\alpha)^2 \, d\alpha, \tag{3.9}$$

where

$$L^\pm(\alpha) = \sum_{\mu X < p \le X} (\log p) e(\alpha [p^c]) \sum_{d | (p+2, P(z))} \lambda^\pm(d).$$

Changing the order of summation we get

$$L^\pm(\alpha) = \sum_{d | P(z)} \lambda^\pm(d) \sum_{\substack{\mu X < p \le X \\ p+2 \equiv 0 (\bmod d)}} (\log p) e(\alpha [p^c]).$$

We divide the integral from (3.9) into two parts:

$$\Gamma_1 = \Gamma_1' + \Gamma_1'', \tag{3.10}$$

where

$$\Gamma_1' = \int_{|\alpha| < \Delta} e(-N\alpha) L^-(\alpha) L^+(\alpha)^2 \, d\alpha, \tag{3.11}$$

$$\Gamma_1'' = \int_{\Delta < |\alpha| < \frac{1}{2}} e(-N\alpha) L^-(\alpha) L^+(\alpha)^2 \, d\alpha, \tag{3.12}$$

with Δ defined by (3.2).

Similarly, for Γ_4 we have

$$\Gamma_4 = \Gamma_4' + \Gamma_4'', \tag{3.13}$$

where

$$\Gamma_4' = \int\limits_{|\alpha|<\Delta} e(-N\alpha)L^+(\alpha)^3 d\alpha, \tag{3.14}$$

$$\Gamma_4'' = \int\limits_{\Delta<|\alpha|<\frac{1}{2}} e(-N\alpha)L^+(\alpha)^3 d\alpha, \tag{3.15}$$

and Δ is defined by (3.2).

4. THE INTEGRALS Γ_1' AND Γ_4'

We shall find an asymptotic formula for the integrals Γ_1' and Γ_4' defined by (3.11) and (3.14), respectively. The arithmetic structure of the Rosser weights $\lambda^{\pm}(d)$ is not important here, so we consider a sum of the form

$$L(\alpha) = \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<p\le X \\ p+2\equiv 0(\bmod d)}} (\log p) e(\alpha[p^c]), \tag{4.1}$$

where $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \le 1, \qquad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0. \tag{4.2}$$

It is easy to see that

$$\begin{aligned} L(\alpha) &= \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<p\le X \\ p+2\equiv 0(\bmod d)}} (\log p) e(\alpha p^c + O(|\alpha|)) \\ &= \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<p\le X \\ p+2\equiv 0(\bmod d)}} (\log p) e(\alpha p^c)(1 + O(|\alpha|)) \\ &= \overline{L}(\alpha) + O(\Delta X(\log X)), \end{aligned} \tag{4.3}$$

where

$$\overline{L}(\alpha) = \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<p\le X \\ p+2\equiv 0(\bmod d)}} (\log p) e(\alpha p^c).$$

For $\overline{L}(\alpha)$ we use the asymptotic formula from Lemma 10 in [15]. From (3.1) and (3.2) we see that, when $|\alpha| < \Delta$, then for every constant $A > 0$, we have

$$\overline{L}(\alpha) = \sum_{d\le D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X(\log X)^{-A}), \tag{4.4}$$

where

$$I(\alpha) = \int\limits_{\mu X}^{X} e(\alpha t^c) dt. \tag{4.5}$$

Hence from (3.2), (4.3) and (4.4) we see that

$$L(\alpha) = \sum_{d \le D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X(\log X)^{-A}). \tag{4.6}$$

From (2.1) and (4.6) we find

$$L^{\pm}(\alpha) = \mathcal{N}^{\pm} I(\alpha) + O(X(\log X)^{-A}), \qquad \text{for } |\alpha| < \Delta. \tag{4.7}$$

Let

$$\mathcal{M}^{\pm} = \mathcal{N}^{\pm} I(\alpha). \tag{4.8}$$

It is easy to see that

$$\mathcal{N}^{\pm} \ll \log X. \tag{4.9}$$

We use (4.7), (4.8) and the identity

$$L^-(L^+)^2 = (L^- - \mathcal{M}^-)(L^+)^2 + (L^+ - \mathcal{M}^+)\mathcal{M}^- L^+ + (L^+ - \mathcal{M}^+)\mathcal{M}^+\mathcal{M}^- + \mathcal{M}^-(\mathcal{M}^+)^2$$

to find that

$$|L^-(L^+)^2 - \mathcal{M}^-(\mathcal{M}^+)^2| \ll X(\log X)^{-A} \left(|L^+|^2 + |\mathcal{M}^-|^2 + |\mathcal{M}^+|^2\right). \tag{4.10}$$

Let

$$B = \int\limits_{|\alpha|<\Delta} e(-N\alpha)\mathcal{M}^-(\alpha)(\mathcal{M}^+(\alpha))^2 d\alpha. \tag{4.11}$$

From (3.11), (4.9) – (4.11) we have

$$\Gamma_1' - B \ll X(\log X)^{2-A} \left(\int\limits_{|\alpha|<\Delta} |L^+(\alpha)|^2 d\alpha + \int\limits_{|\alpha|<\Delta} |I(\alpha)|^2 d\alpha \right).$$

We need the next lemma, which is an analog of Lemma 11 in [15].

Lemma 4.5. *If $\Delta \le X^{1-c}$, then for the sum $L(\alpha)$ defined by (4.1) and for the integral $I(\alpha)$ defined by (4.5) we have*

$$\int\limits_{|\alpha|<\Delta} |L(\alpha)|^2 d\alpha \ll X^{2-c}(\log X)^6,$$

$$\int\limits_{|\alpha|<\Delta} |I(\alpha)|^2 d\alpha \ll X^{2-c}(\log X)^6,$$

$$\int\limits_{|\alpha|<1} |L(\alpha)|^2 d\alpha \ll X(\log X)^5.$$

Proof. The proof is similar to the proof of Lemma 11 in [15]. □

Hence

$$\Gamma_1' - B \ll X^{3-c}(\log X)^{8-A}. \tag{4.12}$$

Consider now the integral

$$B_1 = \int_{-\infty}^{\infty} e(-N\alpha)I(\alpha)^3 d\alpha. \tag{4.13}$$

Using the method in Lemma 5.6.1 in [11] we find

$$B_1 \gg X^{3-c}. \tag{4.14}$$

For $I(\alpha)$ we apply Lemma 2.4 and see that $I(\alpha) \ll |\alpha|^{-1}X^{1-c}$. Then from (3.2), (4.8), (4.11) and (4.13) we find

$$|\mathcal{N}^-(\mathcal{N}^+)^2 B_1 - B| \ll (\log X)^3 \int_{|\alpha|>\Delta} |I(\alpha)|^3 d\alpha \ll (\log x)^3 X^{3-c-2\xi}. \tag{4.15}$$

If $A = 12$, then using (4.12) and (4.15) we find

$$\Gamma_1' = \mathcal{N}^-(\mathcal{N}^+)^2 B_1 + O(X^{3-c}(\log X)^{-4}). \tag{4.16}$$

We proceed with Γ_4' in the same way and prove that

$$\Gamma_4' = (\mathcal{N}^+)^3 B_1 + O(X^{3-c}(\log X)^{-4}). \tag{4.17}$$

5. ESTIMATION OF INTEGRALS Γ_1'' AND Γ_4'' AND COMPLETION OF THE PROOF

In this section we consider the integrals Γ_1'' and Γ_4'' defined by (3.12) and (3.15) respectively. We shall show that Γ_1'' and Γ_4'' are small enough. Now we assume that

$$\xi = \frac{16c-5}{32}, \qquad \delta = \frac{17-16c}{32}. \tag{5.1}$$

It is obvious that for Γ_1'' defined by (3.12) we have

$$\Gamma_1'' \ll \max_{\Delta \le |\alpha| \le \frac{1}{2}} |L^-(\alpha)| \int_0^1 |L^+(\alpha)|^2 d\alpha.$$

We use Lemma 4.5 and find that

$$\Gamma_1'' \ll X(\log X)^5 \max_{\Delta \le |\alpha| \le \frac{1}{2}} |L^-(\alpha)|. \tag{5.2}$$

From (4.1) we see that

$$L(\alpha) = L_1(\alpha) + O\left(X^{\frac{1}{2}+\varepsilon}\right), \tag{5.3}$$

where

$$L_1(\alpha) = \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<n\le X\\ n+2\equiv 0(\bmod d)}} \Lambda(n) e(\alpha[n^c]).$$

Let $M = X^\kappa$ for some κ, which will be specified later. Now for $L_1(\alpha)$ we apply Lemma 2.3 with parameters $x = \alpha$, $y = n^c$ and M (note that $[t] = t - \{t\}$). We obtain

$$\begin{aligned} L_1(\alpha) = &\sum_{|m|\le M} c_m \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<n\le X\\ n+2\equiv 0(\bmod d)}} \Lambda(n) e((\alpha+m)n^c) \\ &+ O\left(X^\varepsilon \sum_{\mu X<n\le X} \min\left(1, \frac{1}{M||n^c||}\right)\right). \end{aligned} \tag{5.4}$$

We need the following

Lemma 5.6. *Suppose that D, Δ are defined by* (3.2) *and ξ, δ are specified by* (5.1). *Suppose also that $\lambda(d)$ satisfy* (4.2) *and c_m are defined by* (2.6). *Then*

$$\max_{\Delta\le\alpha\le M+1} \left| \sum_{|m|\le M} c_m \sum_{d\le D} \lambda(d) \sum_{\substack{\mu X<n\le X\\ n+2\equiv 0(\bmod d)}} \Lambda(n) e(\alpha n^c) \right|$$
$$\ll x^\varepsilon \left(X^{\frac{1}{3}+\frac{c}{2}} D M^{\frac{1}{2}} + X^{1-\frac{c}{2}} \Delta^{-\frac{1}{2}} + X^{\frac{3}{4}+\frac{c}{6}} D^{\frac{2}{3}} M^{\frac{1}{6}} + X^{\frac{5}{6}} + X^{1-\frac{c}{6}} D^{\frac{1}{3}} \Delta^{-\frac{1}{6}} + X^{1-\frac{c}{4}} \Delta^{-\frac{1}{4}}\right).$$

Proof. See Lemma 15 in [15]. □

We also need the following result.

Lemma 5.7. *One has*

$$\sum_{\mu X<n\le X} \min\left(1, \frac{1}{M||n^c||}\right) \ll X^\varepsilon \left(XM^{-1} + M^{\frac{1}{2}} X^{\frac{c}{2}}\right). \tag{5.5}$$

Proof. From [13, Lemma 5.2.3] we know that the Fourier series

$$\min\left(1, \frac{1}{M||n^c||}\right) = \sum_{k\in\mathbb{N}} b_M(k) e(kn^c), \tag{5.6}$$

has Fourier coefficients satisfying

$$|b_M(k)| \le \begin{cases} \frac{4\log M}{M} & \text{if } k\in\mathbb{Z}, \\ \frac{M}{k^2} & \text{if } k\in\mathbb{Z}, k\ne 0. \end{cases} \tag{5.7}$$

From (5.6) we get

$$\sum_{\mu X<n\le X}\min\left(1,\frac{1}{M||n^c||}\right)=\sum_{\mu X<n\le X}\sum_{k\in\mathbb{N}}b_M(k)e(kn^c). \tag{5.8}$$

Changing the order of summation in last formula we obtain

$$\sum_{\mu X<n\le X}\min\left(1,\frac{1}{M||n^c||}\right)=\sum_{k\in\mathbb{N}}b_M(k)H(k),$$

where

$$H(k)=\sum_{\mu X<n\le X}e(kn^c).$$

Now using (5.7) and (5.8) and the identity $|H(k)|=|H(-k)|$ we find

$$\sum_{\mu X<n\le X}\min\left(1,\frac{1}{M||n^c||}\right)\ll\frac{X\log M}{M}+\frac{\log M}{M}\sum_{1\le k\le M}|H(k)|+M\sum_{k>M}\frac{|H(k)|}{k^2}. \tag{5.9}$$

If $\theta(x)=kx^c$, then $\theta''(x)=c(c-1)kx^{c-2}\asymp kX^{c-2}$ uniformly for $x\in[\mu X,X]$. Hence, we can apply Van der Corput's theorem (see [6, Chapt. 1, Theorem 5] to obtain

$$H(k)\ll k^{\frac12}X^{\frac{c}{2}}+k^{-\frac12}X^{1-\frac{c}{2}}. \tag{5.10}$$

Hence from (5.9) and (5.10) we prove (5.5). □

When combining Lemma 5.6, Lemma 5.7 and (5.3) – (5.4) we find that

$$\max_{\Delta\le\alpha\le M+1}|L(\alpha)|\ll x^{\varepsilon}\Big(X^{\frac13+\frac{c}{2}}DM^{\frac12}+X^{1-\frac{c}{2}}\Delta^{-\frac12}+X^{\frac34+\frac{c}{6}}D^{\frac23}M^{\frac16}+$$
$$+X^{\frac56}+X^{1-\frac{c}{6}}D^{\frac13}\Delta^{-\frac16}+X^{1-\frac{c}{4}}\Delta^{-\frac14}+XM^{-1}\Big).$$

Then from last formula, (3.2) and (5.2) we find

$$\Gamma_1''\ll x^{\varepsilon}\left(X^{\frac43+\frac{c}{2}+\delta+\frac{\kappa}{2}}+X^{\frac74+\frac{c}{6}+\frac{2\delta}{3}+\frac{\kappa}{6}}+X^{\frac{11}{6}}+X^{2+\frac{\delta}{3}-\frac{\xi}{6}}+X^{2-\kappa}\right). \tag{5.11}$$

If we choose $\kappa=\frac{8c-5}{56}$, then from (5.1) and (5.11) we conclude that if $1<c<\frac{17}{16}$ then

$$\Gamma_1''\ll X^{3-c-\varepsilon}.$$

From (3.8), (3.10), (3.13) and (4.14) – (4.17) we conclude that

$$\Gamma\ge|3\mathcal{N}^--2\mathcal{N}^+|(N^+)^3B_1+O(X^{3-c}(\log x)^{-4}). \tag{5.12}$$

Now we shall find a lower bound for the difference $3\mathcal{N}^--2\mathcal{N}^+$. It is easy to see that

$$\mathcal{B}\asymp(\log X)^{-1}. \tag{5.13}$$

From (2.2) and (2.3) we see that

$$3\mathcal{N}^- - 2\mathcal{N}^+ \geq \mathcal{B}(3f(s_0) - F(s_0)) + O\left((\log X)^{-\frac{4}{3}}\right),$$

where s_0 is defined by (2.1) and $F(s)$ and $f(s)$ are defined by (2.4). If we choose $s_0 = 2.95$, then from (2.1), (3.2) and (5.1) we find

$$\eta = \frac{\delta}{2.95} = \frac{17 - 16c}{94.4}$$

and also from (2.4) we find $3f(s_0) - F(s_0) > 0$.

Now from (2.2), (4.14), (5.12) and (5.13) we obtain

$$\Gamma \gg X^{3-c}(\log X)^{-3}.$$

Therefore $\Gamma > 0$ and this proves Theorem 1.1. □

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