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ON THE THREE-SPACES PROBLEM AND EXTENSION OF MLUR NORMS ON BANACH SPACES

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Георги Александров. О ЗАДАЧЕ ТРЕХ ПРОСТРАНСТВ И ПРОДОЛЖЕНИЯХ СИММЕТРИЧНО ЛОКАЛЬНО РАВНОМЕРНО ВЫПУКЛЫХ НОРМ В ПРОСТРАНСТВАХ БАНАХА

Показано, что если X банахово пространство, Y его подпространство, которое имеет эквивалентной симметрично локально равномерно выпуклой (СЛРВ) нормой $\|\cdot\|$ и факторпространство X/Y сепарабельно, тогда норму $\|\cdot\|$ можно продолжить до СЛРВ нормой на всем пространстве X.

George Alexandrov. ON THE THREE-SPACES PROBLEM AND EXTENSION OF MLUR NORMS ON BANACH SPACES

We show that if X is a Banach space, Y is a subspace of X which admits an equivalent midpoint locally uniformly rotund (MLUR) norm $\|\cdot\|$, and if X/Y is separable, then the norm $\|\cdot\|$ has an extension which is a MLUR norm on X.

1. INTRODUCTION

The three-space problem for a property A of Banach space X consists in the question: If two of three spaces X, Y, X/Y (Y is a subspace of X) possess the property A, then does the third space also have the same property A? Also, the following question is close to the three-space problem: If the norm $\|\cdot\|$ on the subspace Y of a Banach space X possesses the property A, then can the norm $\|\cdot\|$ be extended to such a norm $\|\cdot\|_0$ on X (i. e. the restriction of $\|\cdot\|_0$ on Y is equal to $\|\cdot\|$) with the same property A?

These problems are treated in [A1, A2, GTWZ, JZ1, JZ2] for locally uniformly rotund and rotund renorming of Banach spaces. Here we discuss the same problems for the MLUR property.

2. DEFINITIONS AND REMARKS

A norm $\|\cdot\|$ of a Banach space X is called *midpoint locally uniformly rotund* (MLUR) if

$$\lim_n \left(\|x+x_n\|^2+\|x-x_n\|^2-2\|x\|^2\right)=0, \quad x, x_n \in X,$$

implies $\lim_n \|x_n\|=0$.

A norm $\|\cdot\|$ of a Banach space X is called *locally uniformly rotund* (LUR) if

$$\lim_n \left(2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2\right)=0, \quad x, x_n \in X,$$

implies $\lim_n \|x-x_n\|=0$.

Obviously LUR $\Rightarrow$ MLUR.

If Y is a subspace of the Banach space X, then $\hat{x}$ means the element of X/Y given by x.

Lemma. *Let X be a* MLUR *Banach space. Then for each $x \in X$ and $\varepsilon>0$ there is $\delta=\delta(\varepsilon, x)>0$ such that whenever $y \in X$, $\|x-y\|<\delta$ and $z \in X$, $\|y+z\|^2+\|y-z\|^2-2\|y\|^2<\delta$, we have $\|z\|<\varepsilon$.*

3. MAIN RESULTS

Theorem 1 ([A3]). *Let X be a Banach space and let Y be a subspace of X such that Y and X/Y admit, respectively, an equivalent* MLUR *and* LUR *norm. Then X admits an equivalent* MLUR *norm.*

Theorem 2. *Let X be a Banach space and let Y be a subspace of X which admits an equivalent* MLUR *norm $\|\cdot\|$, and let X/Y be separable. Then the norm $\|\cdot\|$ can be extended to an equivalent* MLUR *norm on X.*

Proof. We construct the extension of the norm $\|\cdot\|$ on X following the method of [JZ2].

First, we extend the given MLUR norm $\|\cdot\|$ on Y to an equivalent norm $\|\cdot\|$ on X. (For a simple construction of such a norm see e. g. [JZ2].)

Since X/Y is separable, then, as known ([K]), the space X/Y admits an equivalent MLUR norm $\|\cdot\|_0$.

Let $B: X/Y \to X$ be the Bartle–Graves continuous selection map (i. e. $B\hat{x} \in \hat{x}$) [BP].

Let $\{\hat{a}_n\}_{n=1}^{\infty}$, $\hat{a}_n \neq 0$, be a dense subset of X/Y. We assume that $a_n = B\hat{a}_n$.

For each $n \in \mathbb{N}$ ($\mathbb{N}$ — positive integers) choose $f_n \in X^*$ such that $f_n(a_n)=1$, $\|f_n\|=\|\hat{a}_n\|^{-1}$, $f_n=0$ on Y and denote by $P_n(x)=f_n(x)a_n$, $Q_n=I-P_n$ (I is the identity map on X) and $T_n=Q_n/(1+\|P_n\|)$.

For every $x \in X$ we put

$$\|x\|_1^2 = (1-b)\|x\|^2 + \|\hat{x}\|_0^2 + \sum_{n=1}^{\infty} \|T_n(x)\|^2/2^n,$$

where

$$b = \sum_{n=1}^{\infty} 1/2^n(1+\|P_n\|)^2, \quad 0 < b < 1.$$

Then $\|\cdot\|_1$ is an equivalent norm on X whose restriction on Y coincides with the MLUR norm $\|\cdot\|$.

We now are going to show that $\|\cdot\|_1$ is a MLUR norm.

For this purpose we assume there are ε, $0 < \varepsilon < 1$, $x \in X$, and sequence $\{y_m\}$, such that

(1) $$\|x + y_m\|_1^2 + \|x - y_m\|_1^2 - 2\|x\|_1^2 \to 0$$

but

(2) $$\|y_m\| > \varepsilon,$$

and shall find a contradiction.

From (1) and a convexity argument we get

(3) $$\|x + y_m\|^2 + \|x - y_m\|^2 - 2\|x\|^2 \to 0,$$

(4) $$\|\hat{x} + \hat{y}_m\|_0^2 + \|\hat{x} - \hat{y}_m\|_0^2 - 2\|\hat{x}\|_0^2 \to 0$$

and

(5) $$\|T_n(x + y_m)\|^2 + \|T_n(x - y_m)\|^2 - 2\|T_n(x)\|^2 \underset{m}{\longrightarrow} 0$$

for each $n \in \mathbb{N}$.

The norm $\|\cdot\|_0$ is MLUR on X/Y and therefore from (4) we have

(6) $$\|\hat{y}_m\|_0 \to 0.$$

Case i) Let $x \in Y$. According to (6) for every m there is $y'_m \in Y$ such that

(7) $$\|y_m - y'_m\| \to 0.$$

From (3) and (7) we receive that

$$\|x + y'_m\|^2 + \|x - y'_m\|^2 - 2\|x\|^2 \to 0,$$

and since the norm $\|\cdot\|$ is MLUR on Y, then

(8) $$\|y'_m\| \to 0.$$

Therefore from (7) and (8) $\|y_m\| \to 0$, which contradicts (2).

Case ii) Let $x \notin Y$, $\hat{x} \neq 0$. Put $x = x_0 + y_0$, $x_0 = B\hat{x}$, $y_0 \in Y$. Choose $\hat{a}_n \in \{\hat{a}_n\}$ such that

(9) $$\hat{a}_n \to \hat{x},$$

and since B is a continuous map, then

(10) $$a_n \to x_0.$$

For each $n \in \mathbb{N}$, let $z_n \in \hat{a}_n$ and

$$z_n \to x. \tag{11}$$

Put $z_n = a_n + v_n$, $v_n \in Y$, and from (10) and (11) we have

$$v_n \to y_0. \tag{12}$$

Since $\|P_n\| = \|a_n\|/\|\hat{a}_n\|$ and $Q_n(x_0) = (x_0 - a_n) + f_n(a_n - x_0)a_n$, then

$$\|P_n\| \to \mathrm{d} \tag{13}$$

and

$$\|Q_n(x_0)\| \to 0, \tag{14}$$

where $d = \|x_0\|/\|\hat{x}\|$.

The assumption that $\|\cdot\|$ is a MLUR norm on Y and the Lemma imply that for our $y_0 \in Y$ and $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon/6$, such that if $y \in Y$, $\|y_0 - y\| < \delta$ and $z \in Y$, $\|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 < \delta$, then

$$\|z\| < \varepsilon/6. \tag{15}$$

Choose δ_1 such that

$$0 < \delta_1 < \delta/[1 + 14(d+2)^2(3K+1)],$$

where $K = \max(\sup \|y_m\|, \|x\|, \|y_0\|)$.

According to (6), (9) and (11)–(14) there is an $n_0 \in \mathbb{N}$ such that for each $n, m \geqq n_0$ we have

$$\|Q_n\| < d + 2, \tag{16}$$

$$\|Q_n(x_0)\| < \delta_1, \tag{17}$$

$$\|x - z_n\| < \delta_1, \tag{18}$$

$$\|y_0 - v_n\| < \delta_1 \tag{19}$$

and

$$\|(\hat{x} + \hat{y}_m) - \hat{a}_n\|_0 < \delta_1/2. \tag{20}$$

We fix $n \geqq n_0$ until the end of the proof.

From (5)

$$\|Q_n(x + y_m)\|^2 + \|Q_n(x - y_m)\|^2 - 2\|Q_n(x)\|^2 \underset{m}{\longrightarrow} 0.$$

Therefore, there is an $m \geqq n$ such that

$$D_m = \|Q_n(x + y_m)\|^2 + \|Q_n(x - y_m)\|^2 - 2\|Q_n(x)\|^2 < \delta_1. \tag{21}$$

Choose $t_n \in \hat{a}_n$ (use (20)) such that

$$\|(x + y_m) - t_n\| < \delta_1. \tag{22}$$

Put $t_n = a_n + y_0 + u_n$, $u_n \in Y$. Obviously,

$$Q_n(z_n) = v_n, \quad Q_n(t_n) = y_0 + u_n \quad \text{and} \quad Q_n(x - x_0) = y_0. \tag{23}$$

Furthermore, we have (use (23), (16), (17), (18), (21) and (22))

$$\|v_n + u_n\|^2 + \|v_n - u_n\|^2 - 2\|v_n\|^2 =$$

$$= \|Q_n(z_n) + Q_n(t_n) - y_0\|^2 + \|Q_n(z_n) - Q_n(t_n) + y_0\|^2 - 2\|Q_n(z_n)\|^2 \leqq$$

$$\leqq D_m + 2\Big(\|Q_n\|(\|x - z_n\| + \|(x + y_m) - t_n\|) + \|Q_n(x_0)\|\Big) \times \tag{24}$$

$$\times \Big(\|Q_n\|(\|z_n\| + \|t_n\| + \|x\| + \|y_m\|) + \|y_0\|\Big) +$$

$$+ 2\|Q_n\|^2\|x - z_n\|(\|x\| + \|z_n\|) <$$

$$< \delta_1 \left[1 + 14(d+2)^2(3K+1)\right] < \delta.$$

Therefore, by (19), (24) and (15) we get $\|u_n\| < \varepsilon/6$.

Then

$$\|z_n - t_n\| \leqq \|y_0 - v_n\| + \|u_n\| < \delta_1 + \varepsilon/6 < \varepsilon/3. \tag{25}$$

Thus, by (18), (22) and (25)

$$\|y_m\| \leqq \|(x + y_m) - t_n\| + \|t_n - z_n\| + \|z_n - x\| < 2\delta_1 + \varepsilon/3 < 2\varepsilon/3 < \varepsilon,$$

which contradicts (2).

The theorem is proved.

Remark. Let $E(X)$ be the metric space of all equivalent norms on the Banach space X, endowed with the metric of an uniform convergence on unit ball. If there exists at least one equivalent MLUR norm p on the space X, then the set of all equivalent MLUR norms $M(X)$ is dense in $E(X)$. Really, the set

$$R(X) = \left\{r = \sqrt{q^2 + \varepsilon^2 p^2} \;:\; q \in E(X),\ \varepsilon > 0\right\}$$

is subset of $M(X)$ and dense in $E(X)$. In this case, if Y is a subspace of X, obviously "almost all" equivalent MLUR norms on Y can be extended to such norms on X. Indeed, the set of all restrictions of norms from $R(X)$ on Y is dense in $E(X)$.

We finish the paper with the following

Questions. Let X be a Banach space and let Y be a subspace of X.

1) If both Y and X/Y admit equivalent MLUR norms, does X admit an equivalent MLUR norm too?

2) What are the conditions which the space Y has to satisfy, so that the equivalent MLUR norm on Y could be extended to an equivalent MLUR norm on X?

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